MATH299M/CMSC389W – *Visualization Through Mathematica*
Spring 2019 – Ajeet Gary, Devan Tamot, Vlad Dobrin
Project 2b: Dot Product and Cross Product Visualization
Assigned: Friday March 15th
Due: Monday April 15th, 11:59PM

Now that you have plenty of visualization tools, in both two dimensions and three dimensions (or at least you will soon) under your belt for drawing things, you're equipped to start doing some awesome math stuff. For the second project you need only complete one of Project 2a, Project 2b (this one), or Project 2c. If you like linear algebra, this is one is a good choice.

For this project I want you to make a series of models for visualizing the dot product and cross product – a model for visualizing matrix transformations of two-vectors and three-vectors in general would also be cool (this are so-called *linear transformations* as they respect addition and scalar multiplication; this is the "Linear" in "Linear Algebra").

The *dot product* (also called sometimes called the *inner product* in linear algebra classes – an inner product is really a more general idea, the dot product is an instance of an inner product) is an operator between two vectors that returns a scalar, define as:

$$v = \left(v_x, v_y, v_z\right) \;\; u = \left(u_x, u_y, u_z\right)$$

$$v \cdot u \equiv v_x u_x + v_y u_y + v_z u_z$$

Great! Okay, now, what does this mean? Why on earth would it every be useful to us to sum up the product of the respective components of two vectors? Can you make a visualization that gives some sort of idea of what's going on? I think this one will actually be harder than the cross product one – I've personally been struggling for years to get an idea of a dot product really means. At this point I think of inner products as tools for creating *norms*:

$$\langle v, v \rangle = \|v\|^2$$

that is, the inner product of a vector with itself gives its magnitude. Generally, a choice of inner product defines for you a norm – but we only need to worry about the inner product above. Perhaps exploiting the fact that the dot product of vector with itself being equal to the square of the Euclidian norm (the magnitude as derived with the Baudhayana theorem). What does this mean for two different vectors though? Recall that the distance formula between two vectors looks an awful lot like the Baudhayana Theorem – in fact it's magnitude of the difference vector between them. There are connections here to investigate!

The *cross product* of two vectors is defined using a determinate:

$$v \times u \equiv \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = \left(v_y u_z - v_z u_y, v_z u_x - v_x u_z, v_x u_y - v_y u_x\right)$$

The cross product of two vectors returns another vector with two important properties:

(1) It's perpendicular to the plane spanned by those two vectors

(2) Its magnitude is the area of the parallelogram created by those two vectors

Both of these would be awesome to see!

What happens to (1) if u and v are parallel? What happens to (2)? And if they're perpendicular? What if we permute the vectors, is the cross product commutative? What's the interpretation of the sign of the resulting vector (hint: think orientation, right-hand-rule). To clarify, what I mean by the area of the parallelogram is that if you draw the vectors u and v starting at the origin, then draw the vector u added to v tip-to-tail, and v added to u tip-to-tail, u+v and v+u will meet up, and now we have a parallelogram made of the four points in the plane {*O*,u,v,u+v} where *O* is the origin. Can you show why this works?

In general, the dot product is a measure of parallel two vectors are, and the cross product is a measure of how perpendicular to vectors are. Here are two useful formulas (you can take them as definitions of the dot and cross products):

$$u \cdot v = \|u\|\|v\| \, Cos(\theta)$$
$$u \times v = \|u\|\|v\| \, Sin(\theta)$$

where θ is the angle between the two vectors u and v. Explore this, understand it! Note: the function ArcTan[x,y] in Mathematica takes a vector (x,y) and returns the angle from the positive x-axis, whereas if you give it ArcTan[y/x] you'll run into problems in the 2nd and 3rd quadrants since arctangent is only defined from $-\pi/2$ to $\pi/2$, this also handles the whole can't-divide-by-zero-thing.

Here are some other cool things you could do:

Projection

This is the *projection* of the vector u onto the vector v:

$$Pr_v u = \frac{u \cdot v}{\|u\|\|v\|} v$$

I'll help you out with the intuition: When you project one vector onto another, you want the resulting vector to be in the direction of the vector you projected onto, and to have a magnitude equal to the "amount of the first vector that's in that direction" – okay, say you're projecting onto the x unit vector for simplicity, then all you're doing is taking the x component of the vector, which is easy to visualize. For any projection, turn the paper so that the vector you're projecting onto is on the x-axis, then your projection amounts to putting a light source up above in the y direction and recording the shadow of the vector down onto the x-axis. This formula makes sense because the dot product of u and v gives the product of their magnitudes times the cosine of the angel between them, then divides by the magnitudes to just leave the cosine of the angle, and then multiplies that by v, giving you vCosθ, which should look eerily familiar – it looks the like x-component of the vector v, except here we've set it up so it's actually the u-component (imagine a change of basis from $\hat{x}$ and $\hat{y}$ to $\hat{u}$ and $\widehat{u'}$ where u' is the unit vector perpendicular to u).

Parallelepipeds and Pseudoscalars

The *triple product* of three vectors u, v and w is:

$$(u \times v) \cdot w$$

This returns a scalar… almost, it's actually something called a *pseudoscalar*. If you switch around the u, v and w, you will always get either the same result, or the same result with a negative sign – this amount to a change in orientation in the geometric interpretation I'm about to explain – this property is why it's a "pseudo"scalar, as an honest scalar is invariant of orientation. It turns out that the triple product of u, v and w is the volume of the *parallelepiped* that they construct, that is, the parallelogram that the three vectors draw in the same way that we draw a parallelogram with the cross product of two vectors. This is so interesting! Let's investigate.

Questions:

- Why is this the volume of the parallelepiped? Can you visualize that?
- Why does their order not matter? Under which re-orderings does the sign not even change?
- What's the geometric interpretation of the negative sign, that is, can we visualize the orientation difference?
- The determinate definition of the cross product above is actually the volume of a certain parallelepiped – can you see why?

Good luck! Make something awesome that they could (nay, should) use in a Linear Algebra class. I realize pictures would have helped explain this stuff – but that's what I want you to make! I can draw on the board a bit in class if any of these concepts aren't clear. The final product should be cleaned up, that is, there should be a title, your name and the date on it, text in Text cells, and no extraneous code or outputs floating around. After making your model, you should play around with it, and maybe in a Text cell describe some interesting behavior you found.