CMSC425 Notes

Building a rotation matrix directly

Building a rotation matrix from Euler angles, or even angle-axis, can be tricky. One frequently used trick is to build a matrix from two 3D coordinate frames.

Assume you want to rotate the standard coordinate frame vectors i, j and k to or from a new coordinate frame u, v, n so u -> i, v -> j and n -> k. You can build the orthonormal rotation matrix directly by putting the u, v and n vectors in as rows:

$$M_r = \begin{bmatrix} ux & uy & uz \\ vx & vx & vz \\ nx & ny & nz \end{bmatrix} = \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{n} \end{bmatrix}$$

Now when we put in the vectors u, v and n, we get the standard vectors as output. This uses the fact that u, v and n are orthonormal – dot(u,u) =1 but dot(u,v)=dot(u,n)=0.

$$M_r\vec{u} = \begin{bmatrix} ux & uy & uz \\ vx & vy & vz \\ nx & ny & nz \end{bmatrix} \begin{bmatrix} ux \\ uy \\ uz \end{bmatrix} = \begin{bmatrix} \vec{u} \cdot \vec{u} \\ \vec{v} \cdot \vec{u} \\ \vec{n} \cdot \vec{u} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This trick works in reverse. If we want to rotate the i, j and k vectors to become u, v and n, we can build the matrix with u, v and n for columns.

$$M_r^{-1} = M_r^{T} = \begin{bmatrix} ux & uy & uz \\ vx & vy & vz \\ nx & ny & nz \end{bmatrix}^T = \begin{bmatrix} ux & vx & nx \\ uy & vy & ny \\ uz & vz & nz \end{bmatrix}$$

Now we have i -> u as in

$$M_r^{-1}\vec{\imath} = \begin{bmatrix} ux & vx & nx \\ uy & vy & ny \\ uz & vz & nz \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ux \\ uy \\ uz \end{bmatrix}$$

Finally, we can observe that the product of M and M inverse is the identity matrix.

$$M_r M_r^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$