ASSIGNMENT 1 CMSC/PHYS 457 (Spring 2019)

Due by 12:30 pm on Thursday, February 7. Submit your solutions in PDF via Gradescope. Please include a list of students in the class with whom you discussed the problems, or else state that you did not discuss the assignment with your classmates.

1. *Linear algebra review: Hermitian and unitary matrices.*

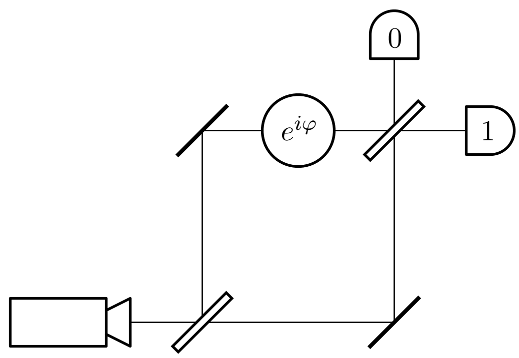
   Recall that a matrix H is called *Hermitian* if $H = H^\dagger$ (where $\dagger$ denotes the conjugate transpose), and a matrix U is called *unitary* if $U^\dagger = U^{-1}$, the matrix inverse of U. The *matrix exponential* is defined by its Taylor series as $\exp(A) = \sum_{j=0}^{\infty} A^j/j!$.

   (a) [3 points] Prove that if H is Hermitian, then its eigenvalues are real.

   (b) [7 points] Prove that if H is Hermitian, then $\exp(iH)$ is unitary.

   (Hint: Use the spectral theorem.)

2. *Mach-Zehnder interferometer with a phase shift.*



   Analyze the experiment depicted above using the mathematical model described in class. (Note that the model from class differs slightly from the model described in the textbook; in particular, you should use the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to model the beamsplitters.)

   (a) [6 points] Compute the quantum state of the system just before reaching the detectors. Express your answer using Dirac notation.

   (b) [4 points] Compute the probability that the "0" detector clicks as a function of φ, and plot your result for $\varphi \in [0, 2\pi]$.

3. *Universality of reversible logic gates.*

   The CCCNOT (triple-controlled NOT) gate is a four-bit reversible gate that flips its fourth bit if and only if the first three bits are all in the state 1.

   (a) [5 points] Show how to compute CCCNOT using AND, OR, NOT and FANOUT gates.

   (b) [5 points] Show how to implement a CCCNOT gate using Toffoli gates. You may use additional workspace as needed. You may assume that bits in the workspace start with a particular value, either 0 or 1, provided you return them to that value.

4. *Computing reversibly.*

The function EQ: $\{0,1\}^3 \to \{0,1\}$ determines whether its three input bits are equal, namely

$$\text{EQ}(x,y,z) = \begin{cases} 1 & \text{if } x = y = z \\ 0 & \text{otherwise.} \end{cases}$$

(a) [5 points] Show how to compute the function EQ using AND, OR, NOT, and FANOUT gates.

(b) [5 points] Show how to compute the function EQ reversibly using Toffoli gates. You may use ancilla bits initialized to either 0 or 1 provided you return them to that value. You may use gates other than Toffoli gates provided you explain how to implement any such gates using Toffoli gates.

5. *Pauli operators.*

(a) [3 points] Express each of the three Pauli operators,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

using Dirac notation in the computational basis.

(b) [3 points] Find the eigenvalues and the corresponding eigenvectors of each Pauli operator. Express the eigenvectors using Dirac notation.

(c) [2 points] Write the operator $X \otimes Z$ as a matrix and using Dirac notation (in both cases using the computational basis).

(d) [2 points] What are the eigenspaces of the operator $X \otimes Z$? Express them using Dirac notation.