ASSIGNMENT 2 CMSC/PHYS 457 (Spring 2019)

Due by 12:30 pm on Thursday, February 21. Submit your solutions in PDF via Gradescope. Please include a list of students in the class with whom you discussed the problems, or else state that you did not discuss the assignment with your classmates.

1. *Product and entangled states.* Determine which of the following states are entangled. If the state is not entangled, show how to write it as a tensor product; if it is entangled, prove this.

   (a) [3 points] $\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle$

   (b) [3 points] $\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)$

   (c) [3 points] $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$

2. *Unitary operations and measurements.* Consider the state

$$|\psi\rangle = \frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle.$$

   (a) [2 points] Let $|\phi\rangle = (I \otimes H)|\psi\rangle$, where H denotes the Hadamard gate. Write $|\phi\rangle$ in the computational basis.

   (b) [3 points] Suppose the first qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the second qubit?

   (c) [3 points] Suppose the second qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the first qubit?

   (d) [2 points] Suppose $|\phi\rangle$ is measured in the computational basis. What are the probabilities of the four possible outcomes? Show that they are consistent with the marginal probabilities you computed in the previous two parts.

3. *Distinguishing quantum states.*

   Let θ be a fixed, known angle. Suppose someone flips a fair coin and, depending on the outcome, either gives you the state

$$|0\rangle \qquad \text{or} \qquad \cos\theta|0\rangle + \sin\theta|1\rangle$$

   (but does not tell you which).

   (a) [4 points] Suppose you measure the state in the basis consisting of the two states

$$|\phi\rangle = \cos\phi|0\rangle + \sin\phi|1\rangle, \qquad |\phi^\perp\rangle = \sin\phi|0\rangle - \cos\phi|1\rangle.$$

   Find the probabilities of the two possible measurement outcomes for each of the two states.

   (b) [3 points] Calculate the probability of correctly distinguishing the states as a function of θ and ϕ, assuming we associate the outcome $|\phi\rangle$ with the state $|0\rangle$ and the outcome $|\phi^\perp\rangle$ with the other state.

   (c) [3 points] Find the value of ϕ that distinguishes the states with the highest possible probability, and compute the success probability with this optimal choice of ϕ.

4. *Distinguishing states with local operations and classical communication.*

   Suppose Alice and Bob are each given one qubit of a two-qubit state that is promised to be either $|\psi\rangle$ or $|\phi\rangle$, for some fixed states $|\psi\rangle, |\phi\rangle$. Working together, their goal is to tell which state they have using only local measurements. They can each apply some one-qubit gate to their part of the state and then measure in the computational basis. They can then compare their measurement results by exchanging classical information, but they cannot send qubits or perform any two-qubit gates. For each pair of states, either give such a procedure that Alice and Bob can use to distinguish the states perfectly, or explain why this is not possible.

   (a) [3 points] $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

   (b) [3 points] $|\psi\rangle = |00\rangle$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

   (c) [4 points] $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

5. *Teleporting through a Hadamard gate.*

   (a) [2 points] Write the state $(I \otimes H)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in the computational basis.

   (b) [5 points] Suppose Alice has a qubit in the state $|\psi\rangle$ and also, Alice and Bob share a copy of the state $(I \otimes H)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. If Alice measures her two qubits in the Bell basis, what are the probabilities of the four possible outcomes, and in each case, what is the post-measurement state for Bob?

   (c) [3 points] Suppose Alice sends her measurement result to Bob. In each possible case, what operation should Bob perform in order to have the state $H|\psi\rangle$?