Binary Classification with Linear Models

CMSC 422

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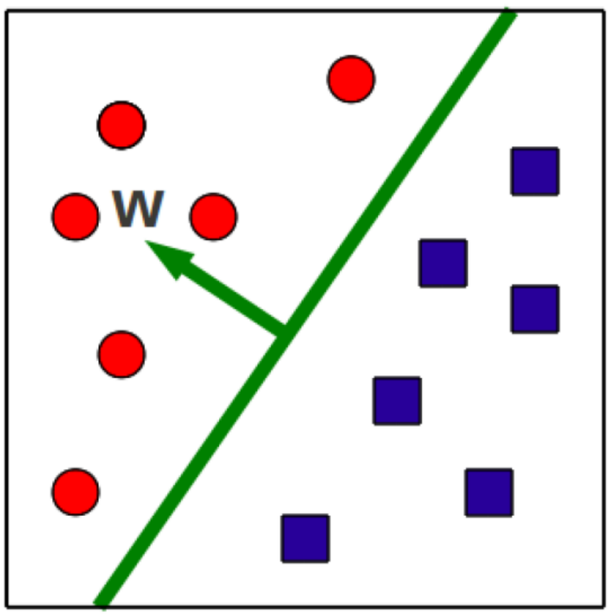
Slides adapted from MARINE CARPUAT

Binary classification via hyperplanes



- A classifier is a hyperplane (w,b)
- At test time, we check on what side of the hyperplane examples fall

$$\hat{y} = sign(w^T x + b)$$

- This is a **linear classifier**
  - Because the prediction is a linear combination of feature values x

Task: Binary Classification

Given:

1. An input space $\mathcal{X}$

2. An unknown distribution $\mathcal{D}$ over $\mathcal{X} \times \{-1, +1\}$

Compute: A function f minimizing: $\mathbb{E}_{(\boldsymbol{x},y)\sim\mathcal{D}}\left[f(\boldsymbol{x}) \neq y\right]$

Learning a Linear Classifier as an Optimization Problem

Objective function

Loss function measures how well classifier fits training data

Regularizer prefers solutions that generalize well

$$\min_{\mathbf{w}, b} L(\mathbf{w}, b)$$

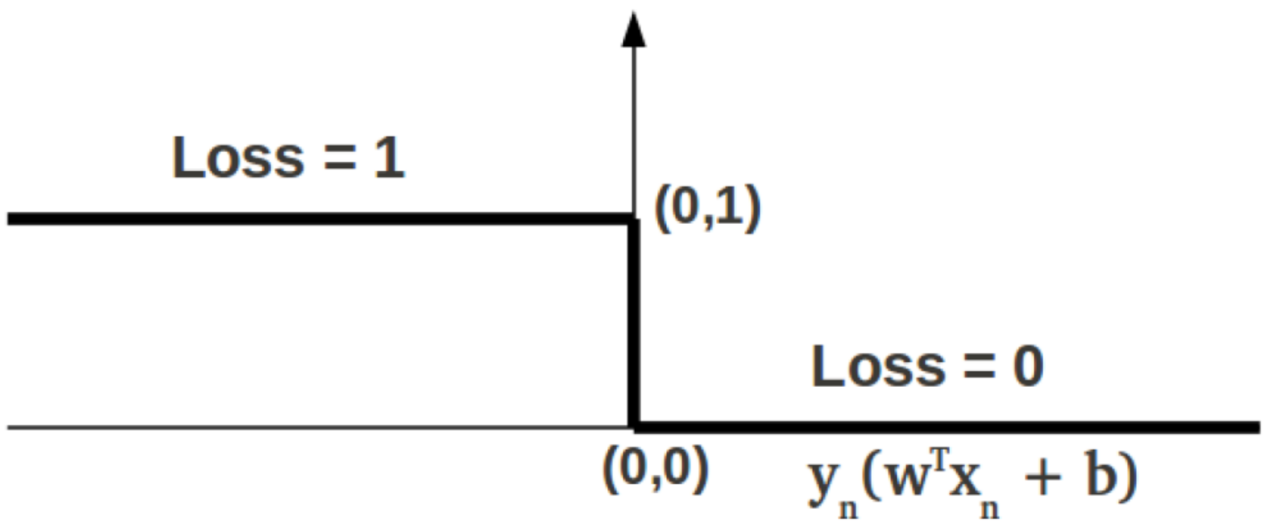
Learning a Linear Classifier as an Optimization Problem

$$\min_{\mathbf{w},b} L(\mathbf{w}, b) = \min_{\mathbf{w},b} \sum_{n=1}^{N} \mathbb{I}(y_n(\mathbf{w}^T\mathbf{x}_n + b) < 0) + \lambda R(\mathbf{w}, b)$$

- **Problem:** The 0-1 loss above is NP-hard to optimize exactly/approximately in general

- **Solution:** Different loss function approximations and regularizers lead to specific algorithms
  (e.g., perceptron, support vector machines, logistic regression, etc.)

The 0-1 Loss

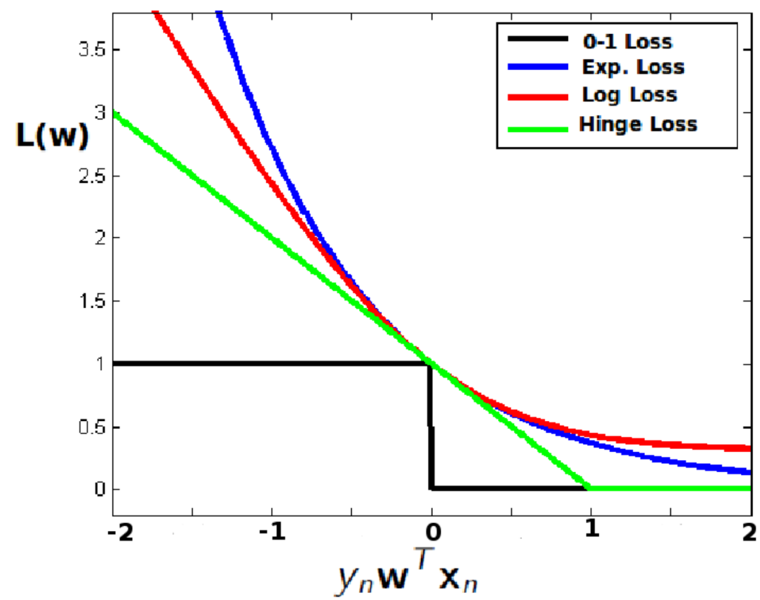


- Small changes in w,b can lead to big changes in the loss value
- 0-1 loss is non-smooth, non-convex

Approximating the 0-1 loss with surrogate loss functions

- Examples (with b = 0)
  - Hinge loss $[1 - y_n \mathbf{w}^T \mathbf{x}_n]_+ = \max\{0, 1 - y_n \mathbf{w}^T \mathbf{x}_n\}$
  - Log loss $\log[1 + \exp(-y_n \mathbf{w}^T \mathbf{x}_n)]$
  - Exponential loss $\exp(-y_n \mathbf{w}^T \mathbf{x}_n)$

- All are convex upper-bounds on the 0-1 loss



Approximating the 0-1 loss with surrogate loss functions

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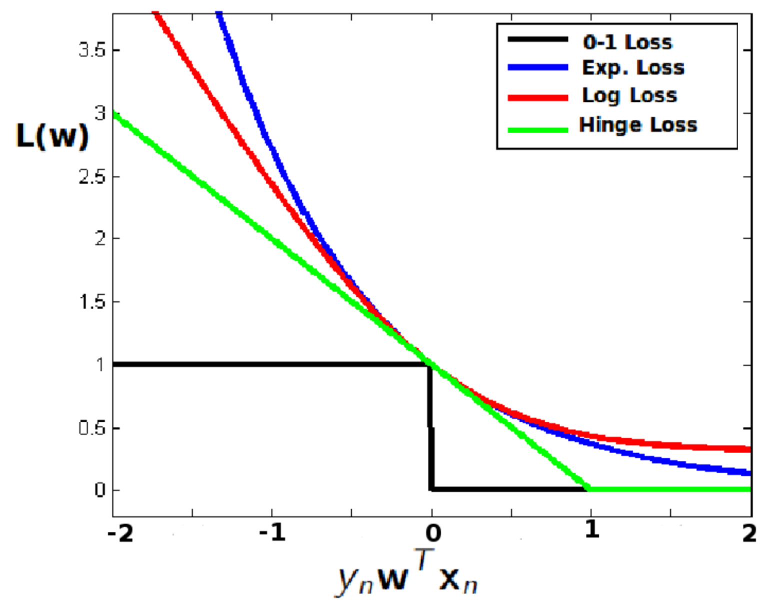
- **Q: Which of these loss functions is not smooth?**



Approximating the 0-1 loss with surrogate loss functions

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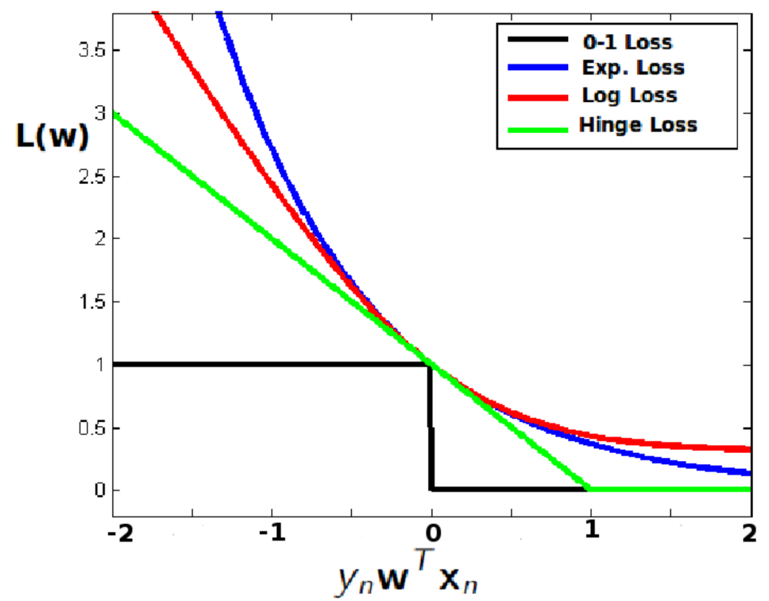
- **Q: Which of these loss functions is most sensitive to outliers?**



Casting Linear Classification as an Optimization Problem

Objective function

Loss function measures how well classifier fits training data

Regularizer prefers solutions that generalize well

$$\min_{\mathbf{w},b} L(\mathbf{w},b) = \min_{\mathbf{w},b} \sum_{n=1}^{N} \mathbb{I}(y_n(\mathbf{w}^T\mathbf{x}_n + b) < 0) + \lambda R(\mathbf{w},b)$$

$\mathbb{I}(.)$ Indicator function: 1 if (.) is true, 0 otherwise

The loss function above is called the 0-1 loss

The regularizer term

- Goal: find simple solutions (inductive bias)

- Ideally, we want most entries of w to be zero, so prediction depends only on a small number of features.

- Formally, we want to minimize:

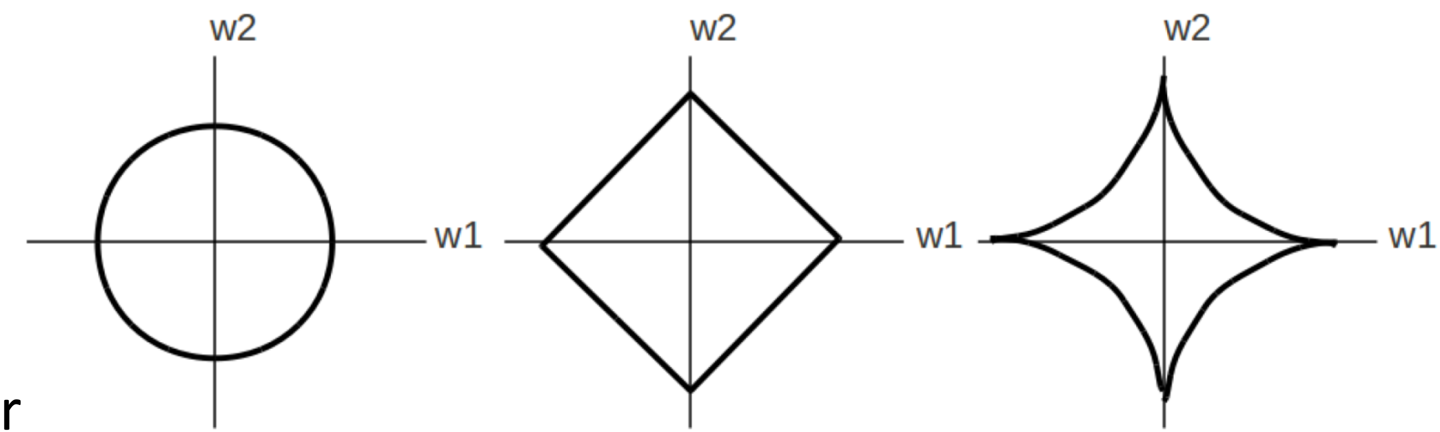
$$R^{cnt}(\mathbf{w}, b) = \sum_{d=1}^{D} \mathbb{I}(w_d \neq 0)$$

- That's NP-hard, so we use approximations instead.
  - E.g., we encourage w_d's to be small

Norm-based Regularizers

- l_p norms can be used as regularizers

$$||\mathbf{w}||_2^2 = \sum_{d=1}^{D} w_d^2$$
$$||\mathbf{w}||_1 = \sum_{d=1}^{D} |w_d|$$
$$||\mathbf{w}||_p = (\sum_{d=1}^{D} w_d^p)^{1/p}$$



Contour plots for p = 2 p = 1 p < 1

Norm-based Regularizers

- l_p norms can be used as regularizers
- Smaller p favors sparse vectors w
  - i.e. most entries of w are close or equal to 0

- l_2 norm: convex, smooth, easy to optimize
- l_1 norm: encourages sparse w, convex, but not smooth at axis points
- $p < 1$: norm becomes non convex and hard to optimize

Casting Linear Classification as an Optimization Problem

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$$\min_{\mathbf{w},b} L(\mathbf{w},b) = \min_{\mathbf{w},b} \sum_{n=1}^{N} \mathbb{I}(y_n(\mathbf{w}^T\mathbf{x}_n + b) < 0) + \lambda R(\mathbf{w},b)$$

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What is the perceptron optimizing?

Algorithm 5 PerceptronTrain($\mathbf{D}$, $MaxIter$)

```
w_d ← 0, for all d = 1 ... D                          // initialize weights
b ← 0                                                  // initialize bias
for iter = 1 ... MaxIter do
    for all (x,y) ∈ D do
        a ← Σ_{d=1}^{D} w_d x_d + b                    // compute activation for this example
        if ya ≤ 0 then
            w_d ← w_d + y x_d, for all d = 1 ... D     // update weights
            b ← b + y                                  // update bias
        end if
    end for
end for
return w_0, w_1, ..., w_D, b
```

- Loss function is a variant of the hinge loss

$$\max\{0, -y_n(\mathbf{w}^T\mathbf{x}_n + b)\}$$

Gradient descent

- A general solution for our optimization problem

$$\min_{\mathbf{w},b} L(\mathbf{w},b) = \min_{\mathbf{w},b} \sum_{n=1}^{N} \mathbb{I}(y_n(\mathbf{w}^T\mathbf{x}_n + b) < 0) + \lambda R(\mathbf{w},b)$$

Idea: take iterative steps to update parameters in the direction of the gradient

Gradient descent algorithm

Objective function to minimize

Number of steps

Step size

Algorithm 22 $\textsc{GradientDescent}(\mathcal{F}, K, \eta_1, \ldots)$

```
z^(0) ← ⟨0, 0, ..., 0⟩                      // initialize variable we are optimizing
for k = 1 ... K do
    g^(k) ← ∇_z F |_{z^(k-1)}                // compute gradient at current location
    z^(k) ← z^(k-1) − η^(k) g^(k)            // take a step down the gradient
end for
return z^(K)
```

$z^{(0)} \leftarrow \langle 0, 0, \ldots, 0 \rangle$ // initialize variable we are optimizing

for $k = 1 \ldots K$ **do**

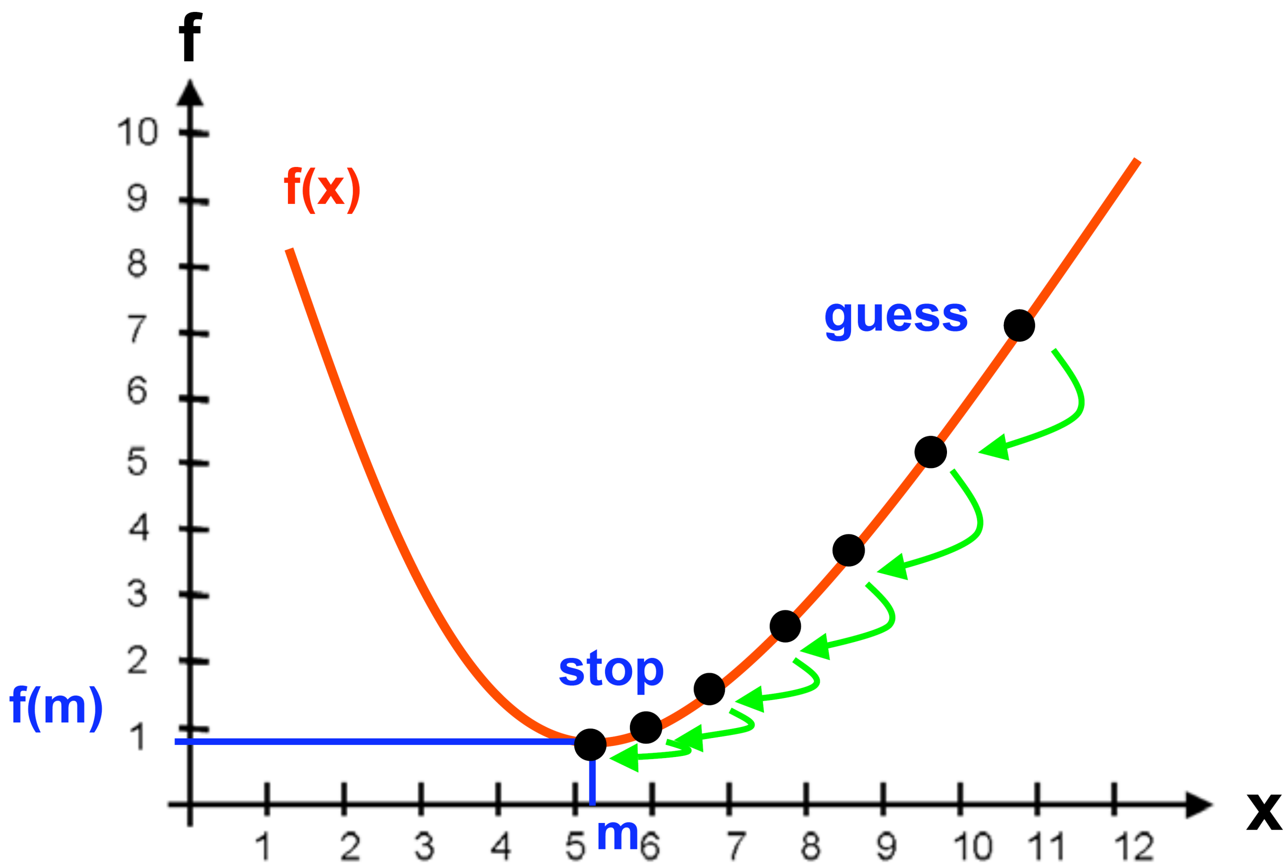
$\quad g^{(k)} \leftarrow \nabla_z \mathcal{F}|_{z^{(k-1)}}$ // compute gradient at current location

$\quad z^{(k)} \leftarrow z^{(k-1)} - \eta^{(k)} g^{(k)}$ // take a step down the gradient

end for

return $z^{(K)}$

Illustrating gradient descent in 1-dimensional case



Recap: Linear Models

- General framework for binary classification
- Cast learning as optimization problem
- Optimization objective combines 2 terms
  - loss function: measures how well classifier fits training data
  - Regularizer: measures how simple classifier is
- Does not assume data is linearly separable
- Lets us separate model definition from training algorithm **(Gradient Descent)**