Image Processing

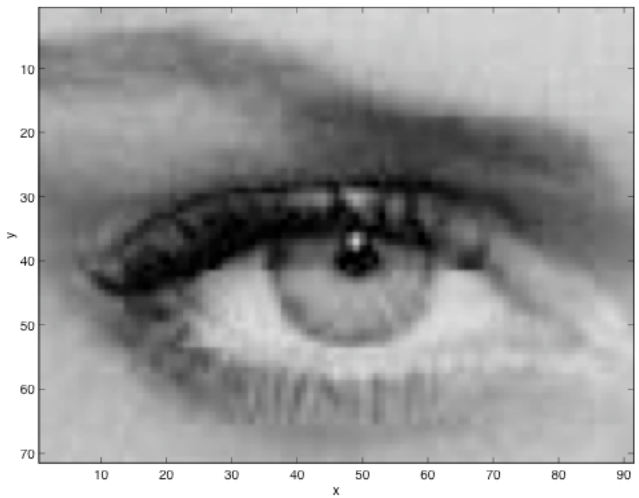
What is an image?

- We can think of an **image** as a function, f, from R^2 to R:
  - $f(x, y)$ gives the **intensity** at position (x, y)
  - Realistically, we expect the image only to be defined over a rectangle, with a finite range:
    - f: $[a,b]\times[c,d] \rightarrow [0,1]$

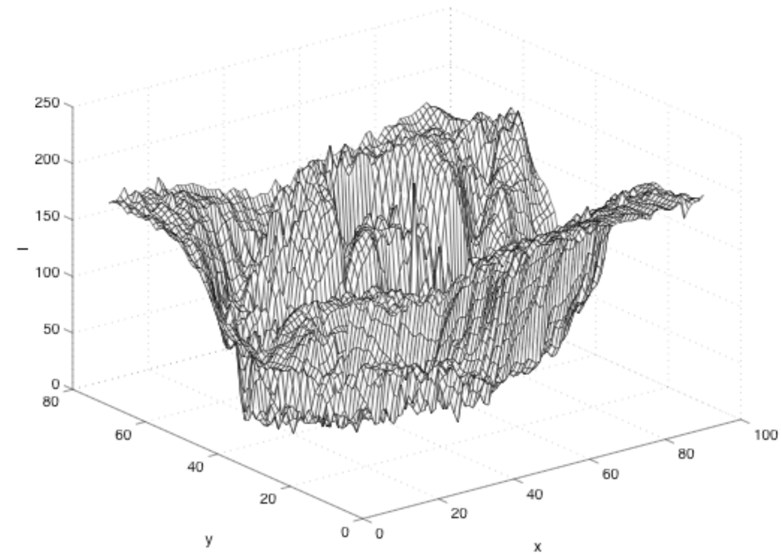
- A color image is just three functions pasted together. We can write this as a "vector-valued" function:

$$f(x, y) = \begin{bmatrix} r(x, y) \\ g(x, y) \\ b(x, y) \end{bmatrix}$$

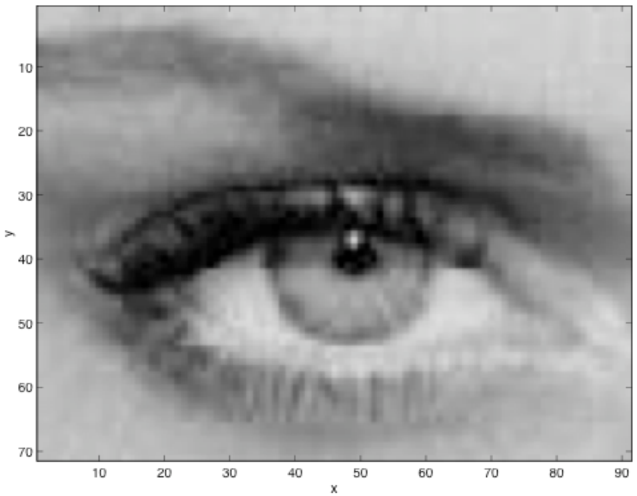
Image

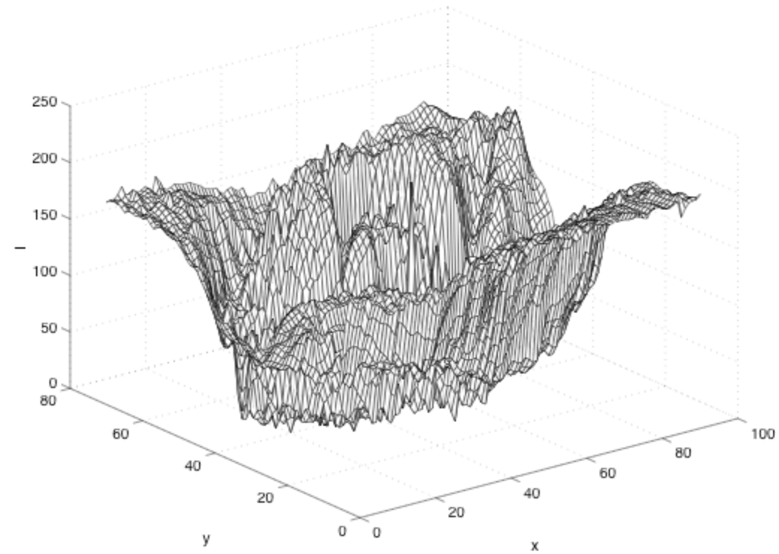


Brightness values



I(x,y)





62	79	23	119	120	105	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
0	89	144	147	187	102	62	208
255	252	0	166	123	62	0	31
166	63	127	17	1	0	99	30

What is a digital image?

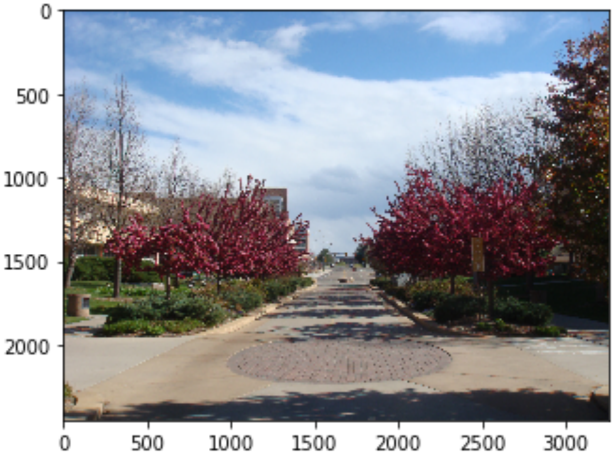
- In computer vision we usually operate on **digital** (**discrete**) images:
  - **Sample** the 2D space on a regular grid
  - **Quantize** each sample (round to nearest integer)
- If our samples are Δ apart, we can write this as:
- $f[i\ ,j] = \text{Quantize}\{\ f(i\ \Delta,\ j\ \Delta)\ \}$
- The image can now be represented as a matrix of integer values

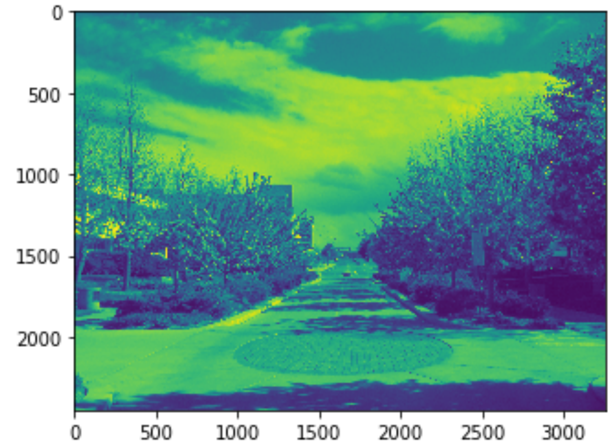
j

i

62	79	23	119	120	105	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
0	89	144	147	187	102	62	208
255	252	0	166	123	62	0	31
166	63	127	17	1	0	99	30

2448,3264,3





im[1:10,1:10,0]

```
array([[88, 89, 90, 90, 91, 91, 92, 92, 91],
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       [90, 92, 92, 92, 91, 92, 92, 93, 93],
       [91, 93, 93, 92, 91, 92, 92, 93, 93],
       [91, 92, 92, 92, 92, 92, 93, 93, 94],
       [91, 91, 92, 92, 93, 93, 94, 95, 94],
       [92, 92, 92, 93, 94, 94, 96, 96, 96],
       [93, 93, 93, 93, 92, 92, 91, 94, 94],
       [94, 94, 94, 94, 93, 92, 92, 94, 94]], dtype=uint8)
```
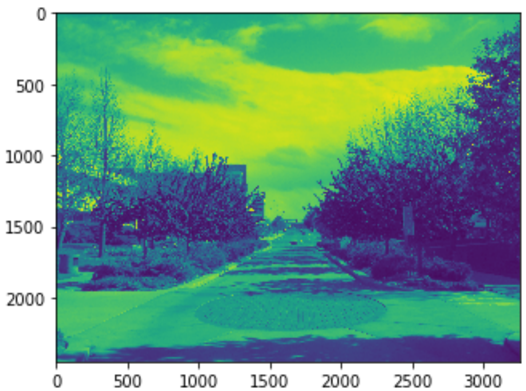
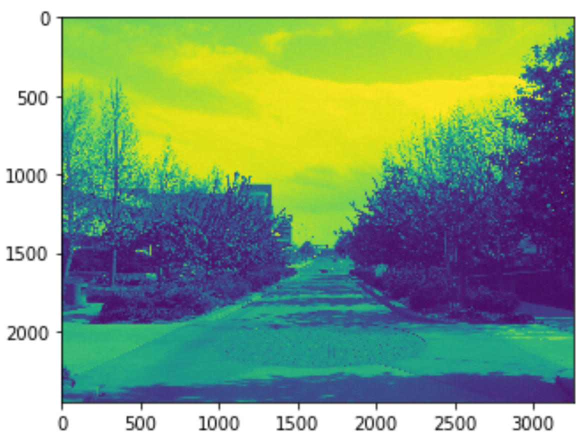



im[1:10,1:10,1]

```
array([[134, 135, 136, 136, 137, 137, 138, 138, 137],
       [135, 136, 137, 137, 137, 138, 138, 138, 138],
       [136, 138, 138, 138, 137, 138, 138, 139, 139],
       [137, 139, 139, 138, 137, 138, 138, 139, 139],
       [137, 138, 138, 138, 138, 138, 139, 139, 139],
       [137, 137, 138, 138, 139, 139, 140, 140, 139],
       [138, 138, 138, 139, 140, 140, 141, 141, 141],
       [137, 137, 137, 137, 138, 138, 137, 140, 140],
       [138, 138, 138, 138, 139, 138, 138, 140, 140]], dtype=uint8)
```



im[1:10,1:10,2]

```
array([[194, 195, 196, 196, 197, 197, 198, 198, 196],
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       [196, 198, 198, 197, 196, 197, 197, 198, 198],
       [196, 197, 197, 197, 197, 197, 198, 198, 198],
       [196, 196, 197, 197, 198, 198, 199, 199, 198],
       [197, 197, 197, 198, 199, 199, 200, 200, 200],
       [198, 198, 198, 198, 197, 197, 196, 199, 199],
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```
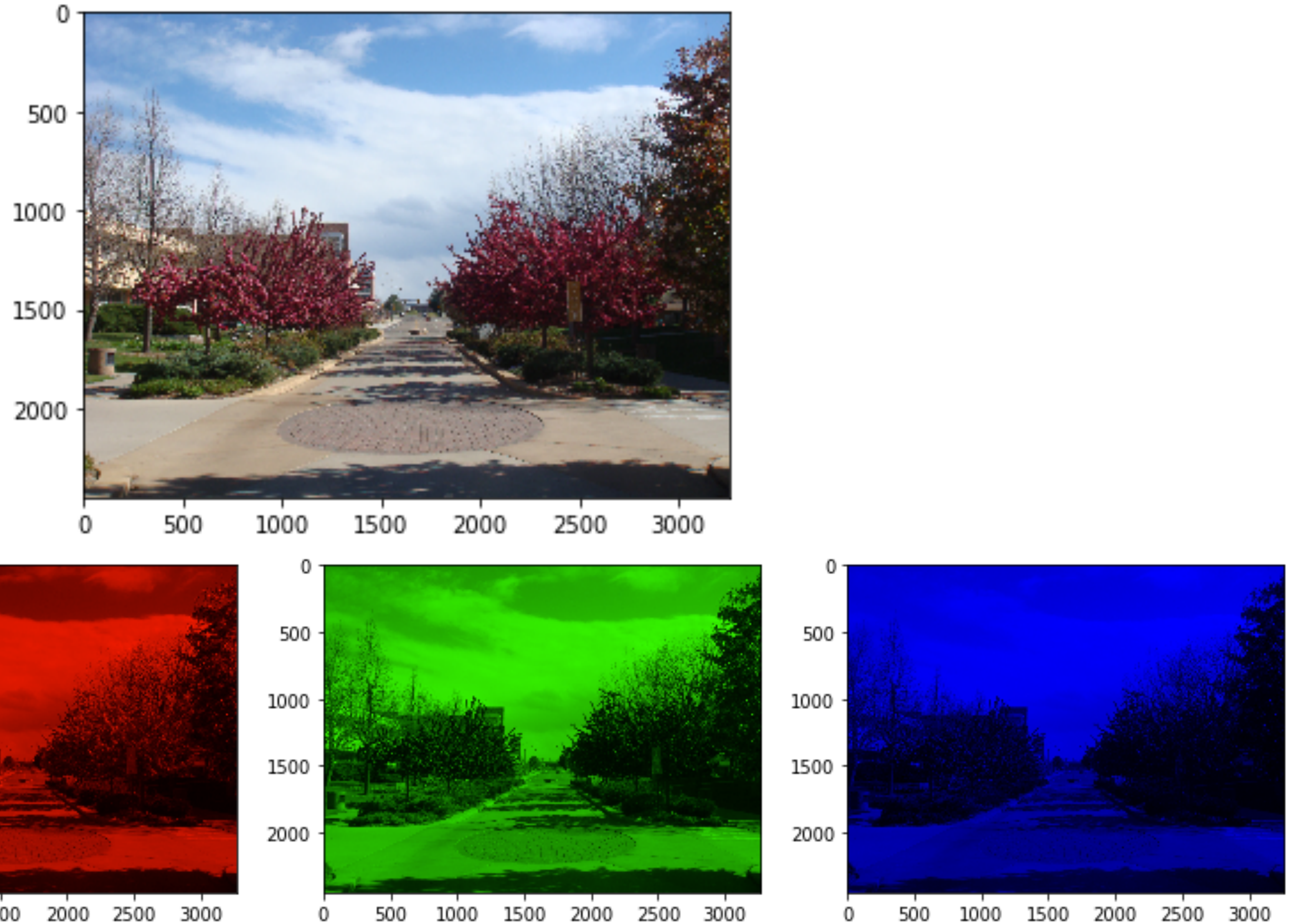
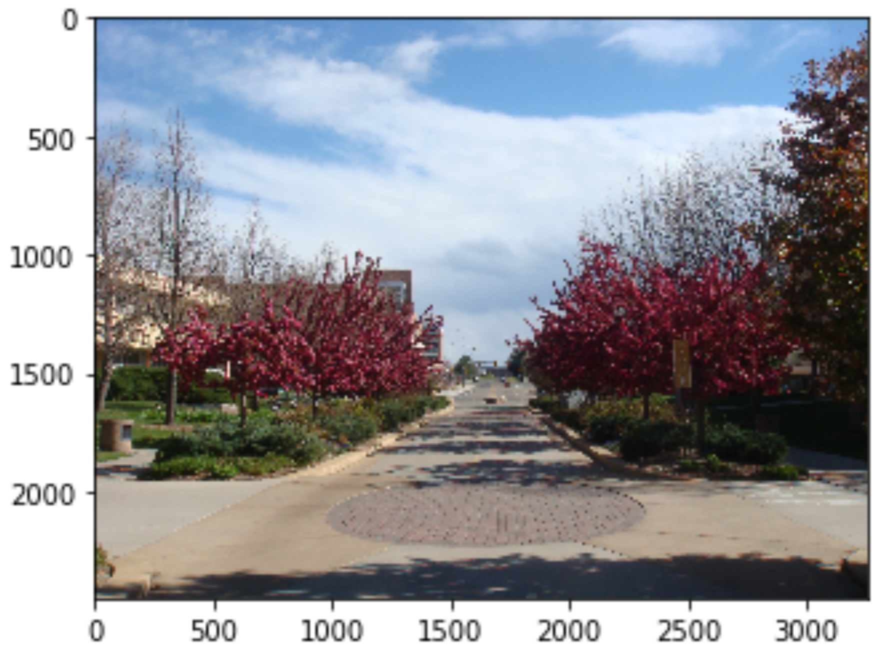


0
500
1000
1500
2000
0 500 1000 1500 2000 2500 3000




```
array([[ 127.086,  128.086,  129.086,  129.086,  130.086,  130.086,
         131.086,  131.086,  129.972],
       [ 128.086,  129.086,  130.086,  130.086,  130.086,  130.972,
         130.972,  130.972,  130.972],
       [ 128.972,  130.972,  130.972,  130.972,  129.972,  130.972,
         130.972,  131.972,  131.972],
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         130.972,  131.972,  131.972],
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         131.972,  131.972,  132.271],
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         132.972,  133.271,  132.271],
       [ 130.972,  130.972,  130.972,  131.972,  132.972,  132.972,
         134.271,  134.271,  134.271],
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         129.972,  132.972,  132.972],
       [ 131.798,  131.798,  131.798,  131.798,  131.972,  130.972,
         130.972,  132.972,  132.972]])
```

Human perception of Color

- Human eye is visually sensitive to electromagnetic spectrum between 400 - 700 nm.
- HSL - hue, saturation and luminance
- measured on a scale of 0-240
- Hue provides the color choice - arranged in a linear strip Red (240), violet (200), indigo(170), blue(128), green(85), yellow(42), orange(21)

Human perception of Color

- Hue provides the color choice - arranged in a linear strip Red (240), violet (200), indigo(170), blue(128), green(85), yellow(42), orange(21)
- Saturation indicates pureness, or intensity or the amount of gray.
- luminance indicates lightness of the color. A value of 0 indicates black, value of 240 indicates pure white.
- The relationship between grayscale reflectance of a surface and its RGB color equivalent is given by:

  $Y = 0.299 *R + 0.587*G + 0.114*B$

Image Processing

- An **image processing** operation typically defines a new image g in terms of an existing image f.
- We can transform either the domain or the range of f.

$$g(x, y) = t(f(x, y))$$

- **Range transformation:**

  What kinds of operations can this perform?

- Some operations preserve the range but **change the domain** of f :

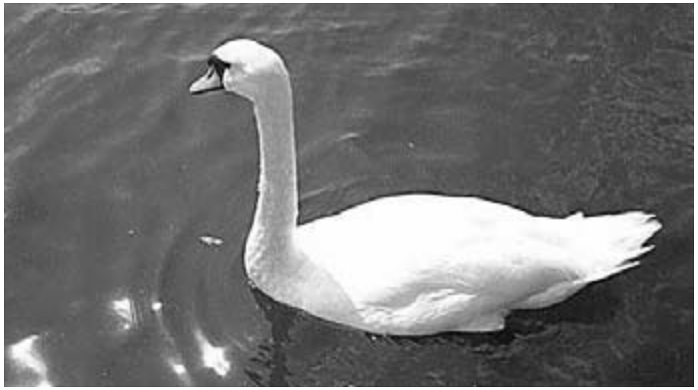
$$g(x, y) = f(t_x(x, y), t_y(x, y))$$

  What kinds of operations can this perform?

Thresholding

One of the simplest operations we can perform on an image is *thresholding*.

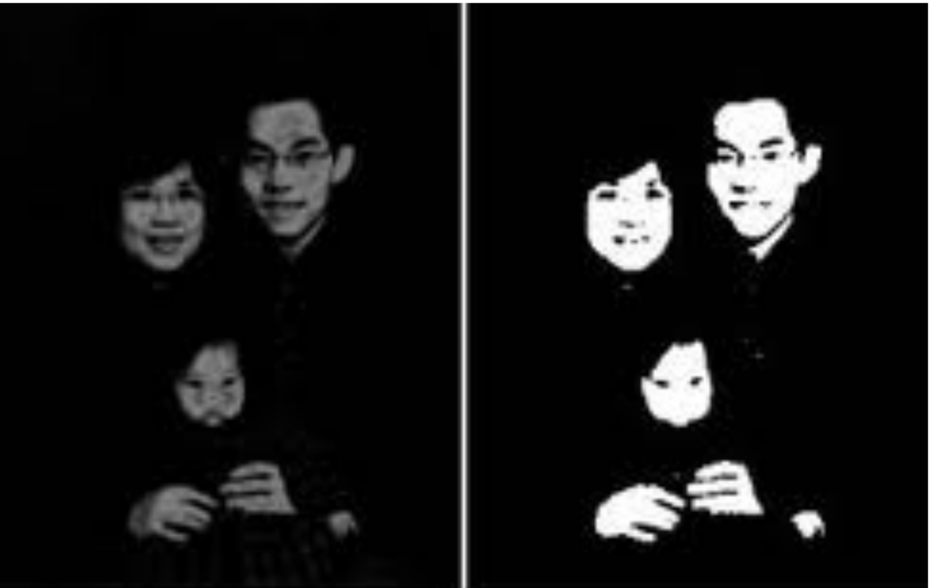
For example, if we take the swan image and threshold it with a threshold of T, we make all pixels >= T into 1, and all pixels < T into 0.

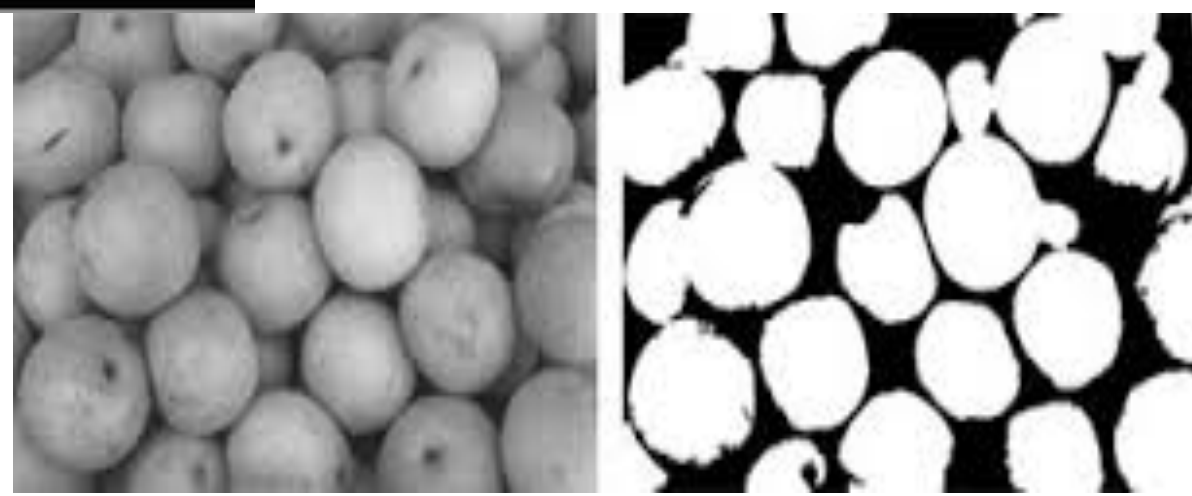


Threshold T=128

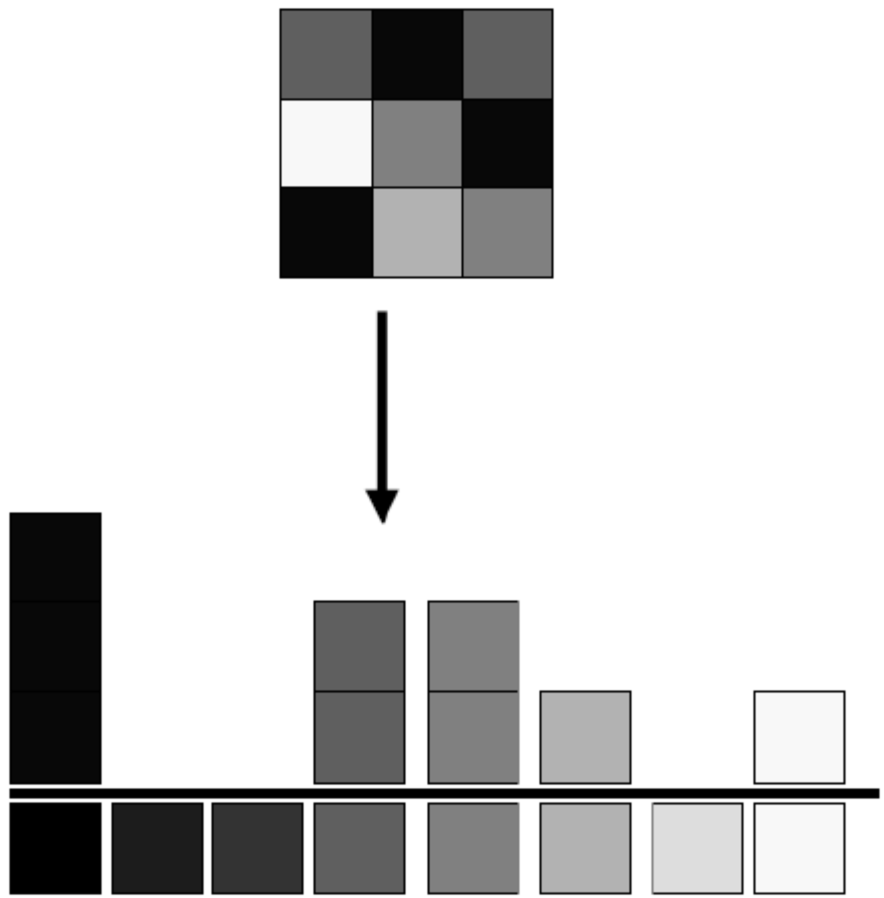
$$im[im > 128] = 255$$
$$im[im \leq 128] = 0$$

Examples





A simple image and its histogram

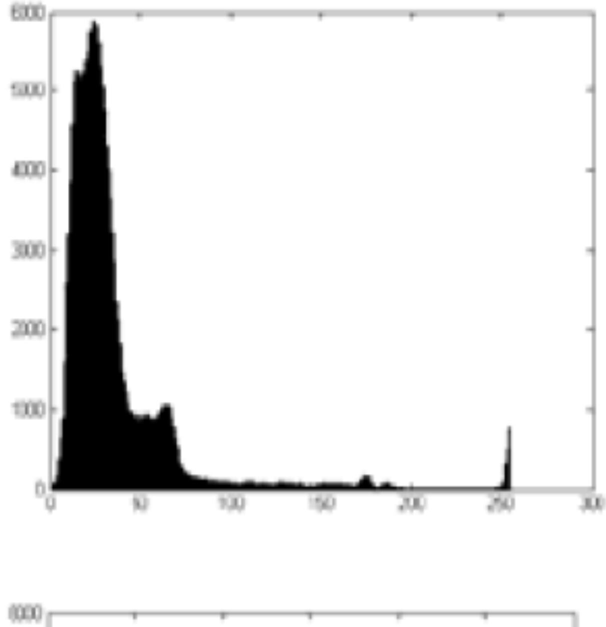


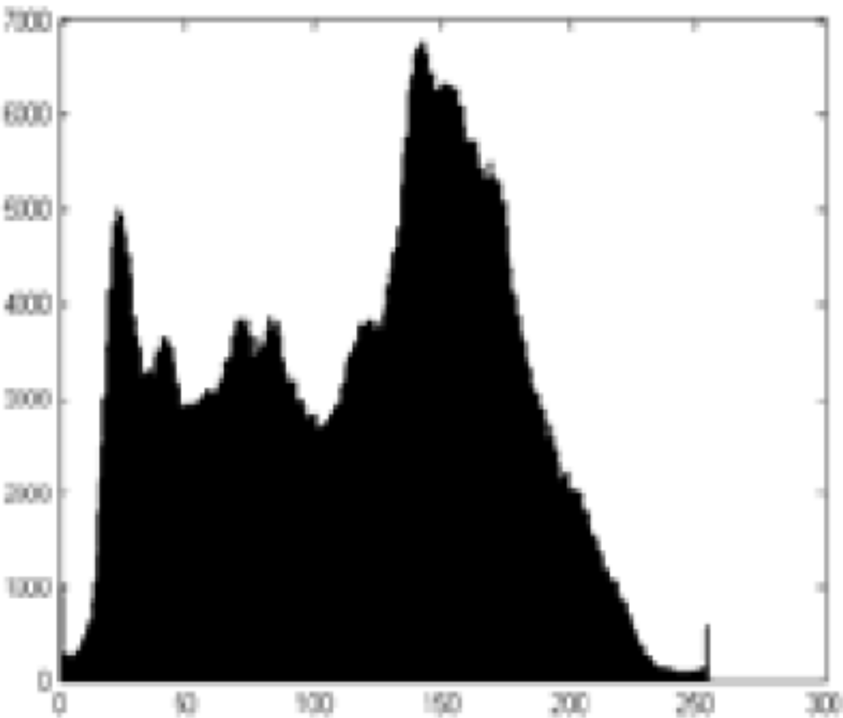
Definition of histogram

- To write this down, we might say that we have an image, I, in which the intensity at pixel with coordinates (x,y) is I(x,y). We would write the histogram h, as h(i) indicating that intensity i, appears h(i) times in the image. If we let the expression (a=b) have the value 1 when a=b, and 0 otherwise, we can write for histogram h(i):

$$h(i) = \sum_x \sum_y I(x,y) = i$$

Examples




7000
6000
5000
4000
3000
2000
1000
0
0
50
100
150
200
250
300

Histograms allow image manipulation

- One reason to compute a histogram is that it allows us to manipulate an image by changing its histogram. We do this by creating a new image, J, in which:

$$J(x,y) = f(I(x,y))$$

- The trick is to choose an f that will generate a nice or useful image. Typically, we choose f to be monotonic. This means that: if $u<v$ then $f(u) < f(v)$. Non-monotonic functions tend to make an image look truly different, while monotonic changes will be more subtle.

Histogram Equalization

- The idea is to spread out the histogram so that it makes full use of the dynamic range of the image.
- For example, if an image is very dark, most of the intensities might lie in the range 0-50. By choosing f to spread out the intensity values, we can make fuller use of the available intensities, and make darker parts of an image easier to understand.
- If we choose f to make the histogram of the new image, J, as uniform as possible, we call this **histogram equalization.**

How to do it

- **Cumulative Distribution Function (CDF).**

This encodes the fraction of pixels with an intensity that is equal to or less than a specific value.

If h is a histogram and C is a CDF, then h(i) indicates the number of pixels with intensity of i, while ,

$$C(i) = \sum_{j<=i} h(j)/N$$

indicates the fraction of pixels with intensity less than or equal to i, assuming the image has N pixels.

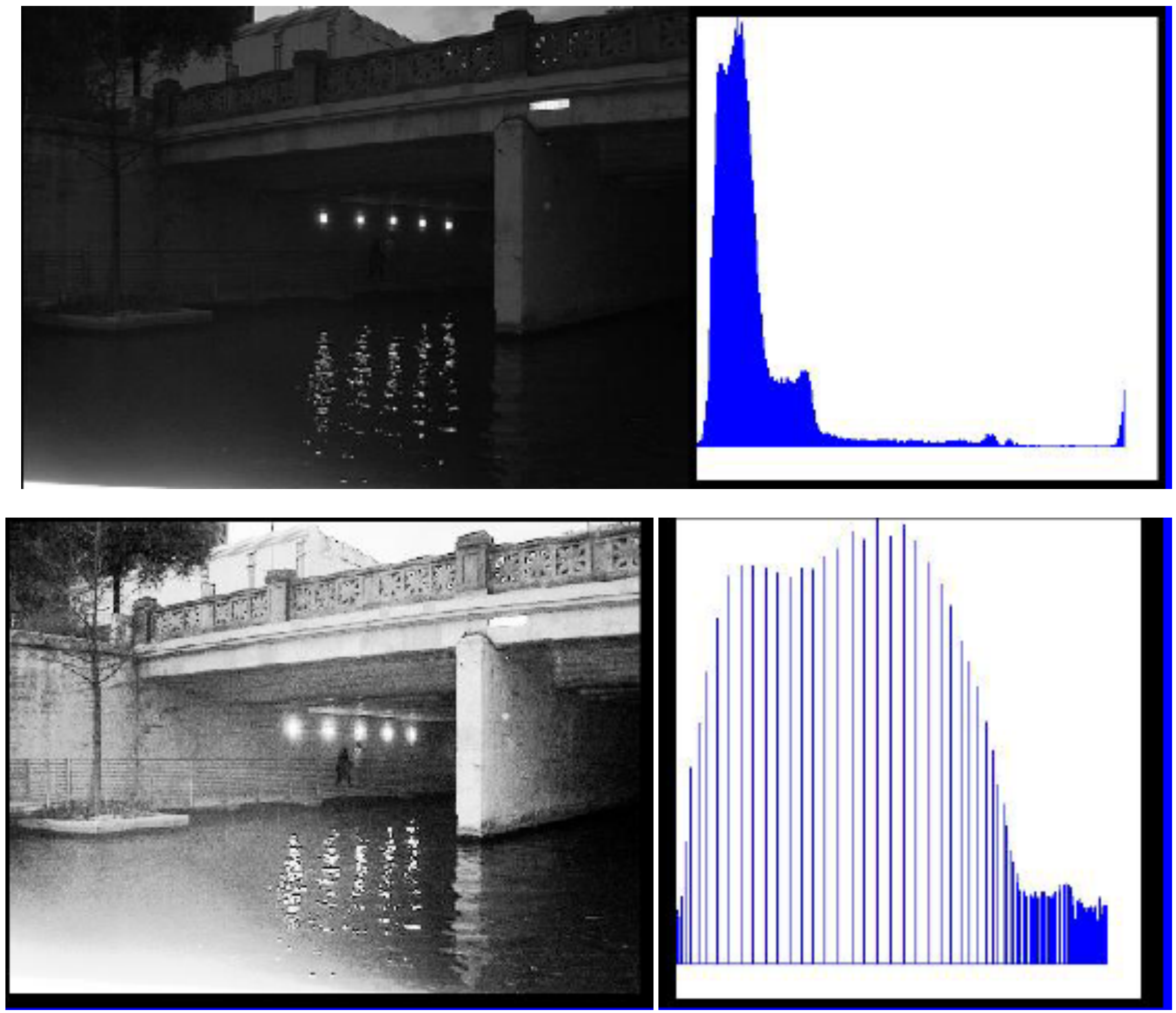
Actual construction

- g is the histogram of J, and D its CDF.

- If there are k intensity levels in an image, then we want g=(N/k, N/k, …), and D=(1/k, 2/k,3/k, …). This means that we want D(i) = i/k. Notice that C(i) = D(f(i)). That is, all the pixels in I that have an intensity less than or equal to i will have an intensity less than or equal to f(i) in J (since f is monotonic, if j<i, f(j) < f(i)).

- Putting these together, we have D(f(i))=f(i)/k=C(i), so f(i) = kC(i).

Examples of histogram equalization



8 x 8 Image

$$\begin{bmatrix} 52 & 55 & 61 & 59 & 70 & 61 & 76 & 61 \\ 62 & 59 & 55 & 104 & 94 & 85 & 59 & 71 \\ 63 & 65 & 66 & 113 & 144 & 104 & 63 & 72 \\ 64 & 70 & 70 & 126 & 154 & 109 & 71 & 69 \\ 67 & 73 & 68 & 106 & 122 & 88 & 68 & 68 \\ 68 & 79 & 60 & 79 & 77 & 66 & 58 & 75 \\ 69 & 85 & 64 & 58 & 55 & 61 & 65 & 83 \\ 70 & 87 & 69 & 68 & 65 & 73 & 78 & 90 \end{bmatrix}$$

8 x 8 Image

$$\begin{bmatrix} 52 & 55 & 61 & 59 & 70 & 61 & 76 & 61 \\ 62 & 59 & 55 & 104 & 94 & 85 & 59 & 71 \\ 63 & 65 & 66 & 113 & 144 & 104 & 63 & 72 \\ 64 & 70 & 70 & 126 & 154 & 109 & 71 & 69 \\ 67 & 73 & 68 & 106 & 122 & 88 & 68 & 68 \\ 68 & 79 & 60 & & & & & \\ 69 & 85 & 64 & & & & & \\ 70 & 87 & 69 & & & & & \end{bmatrix}$$

Value	Count	Value	Count	Value	Count	Value	Count	Value	Count
52	1	64	2	72	1	85	2	113	1
55	3	65	3	73	2	87	1	122	1
58	2	66	2	75	1	88	1	126	1
59	3	67	1	76	1	90	1	144	1
60	1	68	5	77	1	94	1	154	1
61	4	69	3	78	1	104	2		
62	1	70	4	79	2	106	1		
63	2	71	2	83	1	109	1		

8 x 8 Image

Value	Count	Value	Count	Value	Count
52	1	64	2	72	1
55	3	65	3	73	2
58	2	66	2	75	1
59	3	67	1	76	1
60	1	68	5	77	1
61	4	69	3	78	1
62	1	70	4	79	2
63	2	71	2	83	1

v, Pixel Intensity	cdf(v)	h(v), Equalized v
52	1	0
55	4	12
58	6	20
59	9	32
60	10	36
61	14	53
62	15	57
63	17	65
64	19	73
65	22	85
66	24	93

$$h(v) = round\left(\frac{cdf(v) - cdf_{min}}{(M \times N) - cdf_{min}} \times (L-1)\right)$$

M – width M – height L – Number of gray levels

Comparing histograms

- SSD

Let h and g be two histograms.

$$\|\mathrm{h} - \mathrm{g}\| = \sum_{i=1}^{N} \left(h(i) - g(i)\right)^2$$

- Cosine

$$\cos(h, g) = \frac{\langle h, g \rangle}{\|h\| \|g\|}$$

Treating histograms as probability distributions

- Chi-Squared

$$\chi^2(h_i, h_j) = \frac{1}{2}\sum_{m=1}^{K}\frac{[h_i(m) - h_j(m)]^2}{h_i(m) + h_j(m)}$$

- Smoothing probability distributions
  - Use bigger buckets.
  - Add a constant value (eg., 1) to every bucket.
  - Gaussian smoothing

Mean filtering (average over a neighborhood)

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Correlation & Convolution

Mohammad Nayeem Teli

Mean filtering (average over a neighborhood)

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Correlation & Convolution

- Basic operation to extract information from an image.

- These operations have two key features:

  - shift invariant
  - linear

- Applicable to 1-D and multi dimensional images.

Correlation Example - 1D

Image I $\quad \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 2 & 3 & 6 & 5 & 5 & 1 & 8 & 9 & 7 \\ \hline \end{array}$ $\quad G = f(I)$

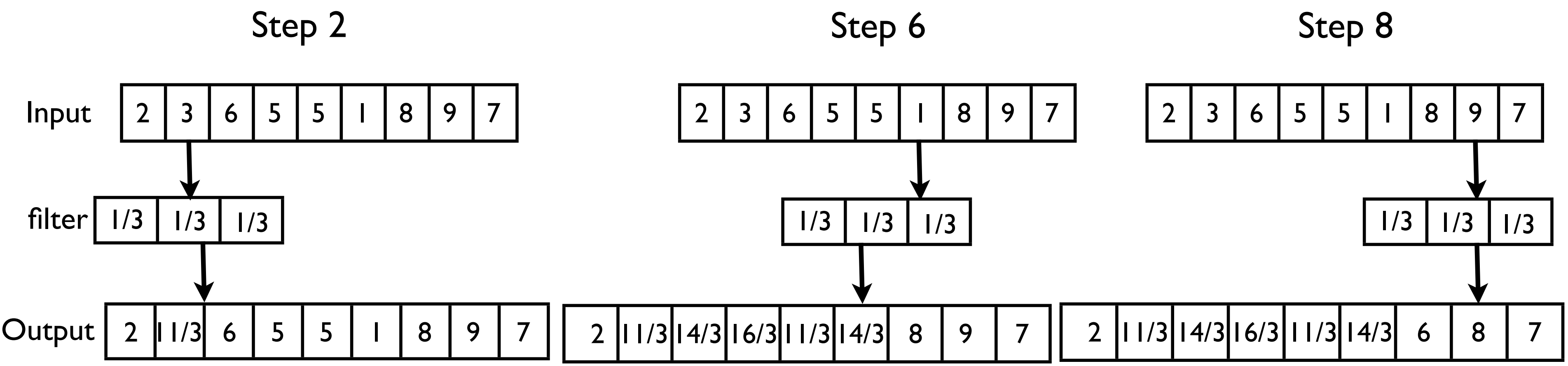
$I[2] = 3 \quad G[2] = \frac{2+3+6}{3} = \frac{11}{3} \quad \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 2 & \frac{11}{3} & 6 & 5 & 5 & 1 & 8 & 9 & 7 \\ \hline \end{array}$

$I[3] = 6 \quad G[3] = \frac{3+6+5}{3} = \frac{14}{3} \quad \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 2 & \frac{11}{3} & \frac{14}{3} & 5 & 5 & 1 & 8 & 9 & 7 \\ \hline \end{array}$

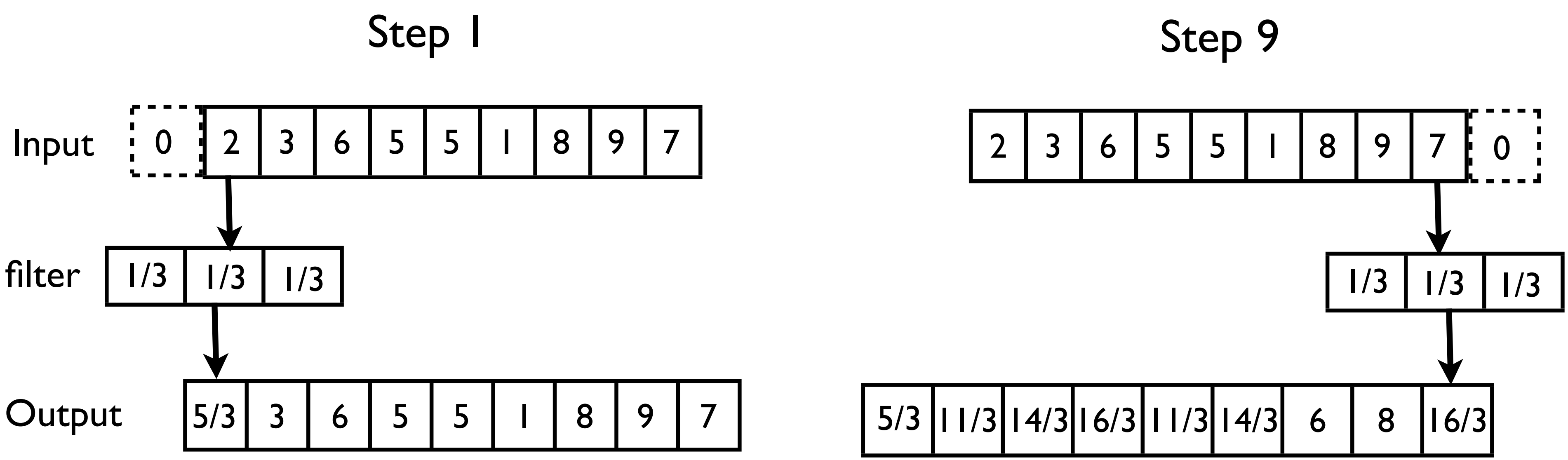
$\vdots \qquad \vdots \qquad \vdots$

$I[8] = 9 \quad G[8] = \frac{8+9+7}{3} = 8 \quad \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 2 & \frac{11}{3} & \frac{14}{3} & \frac{16}{3} & \frac{11}{3} & \frac{14}{3} & 6 & 8 & 7 \\ \hline \end{array}$

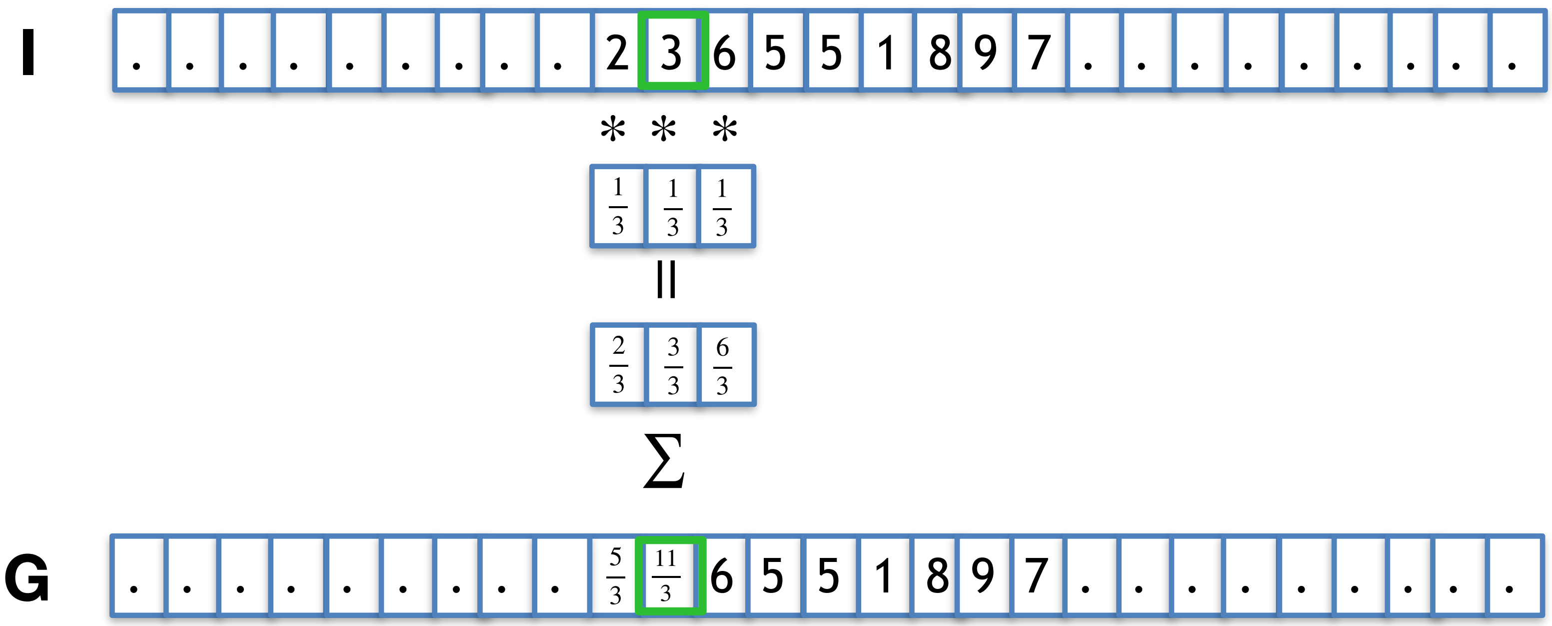
Correlation Example - 1D



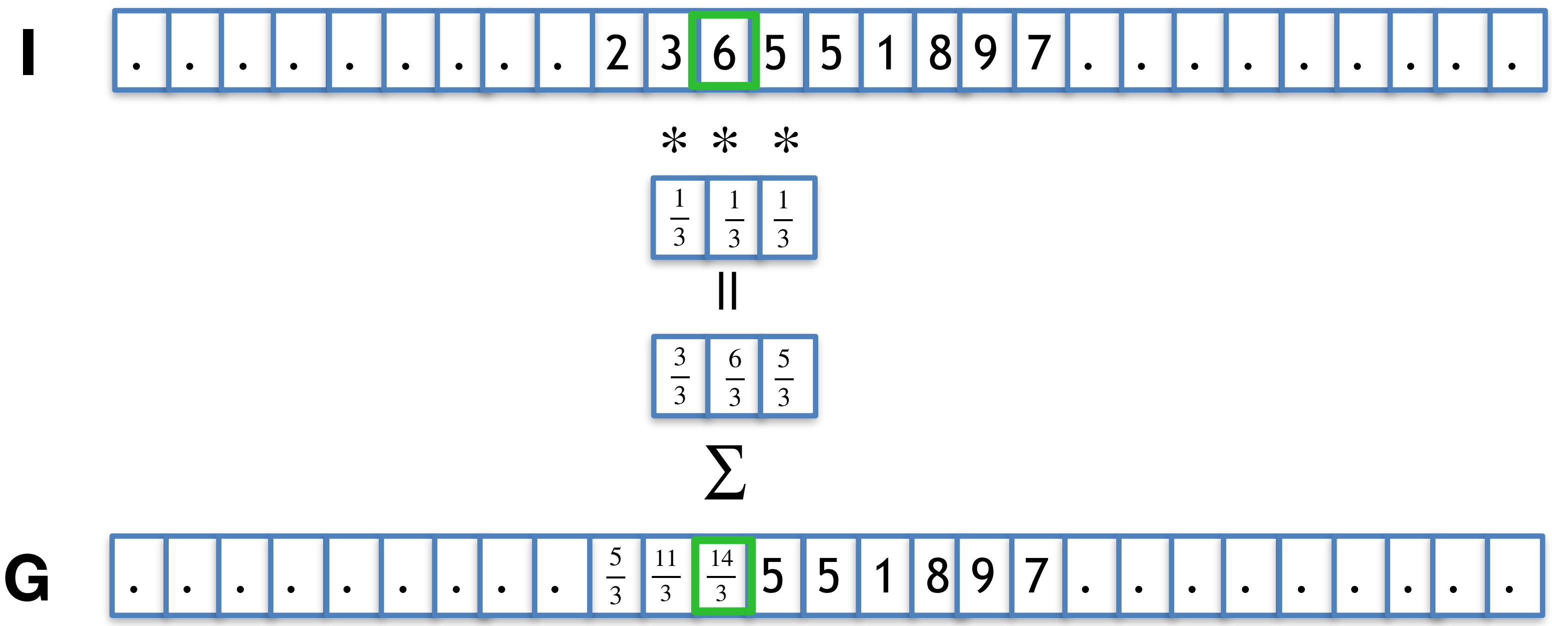
Correlation Example - 1D



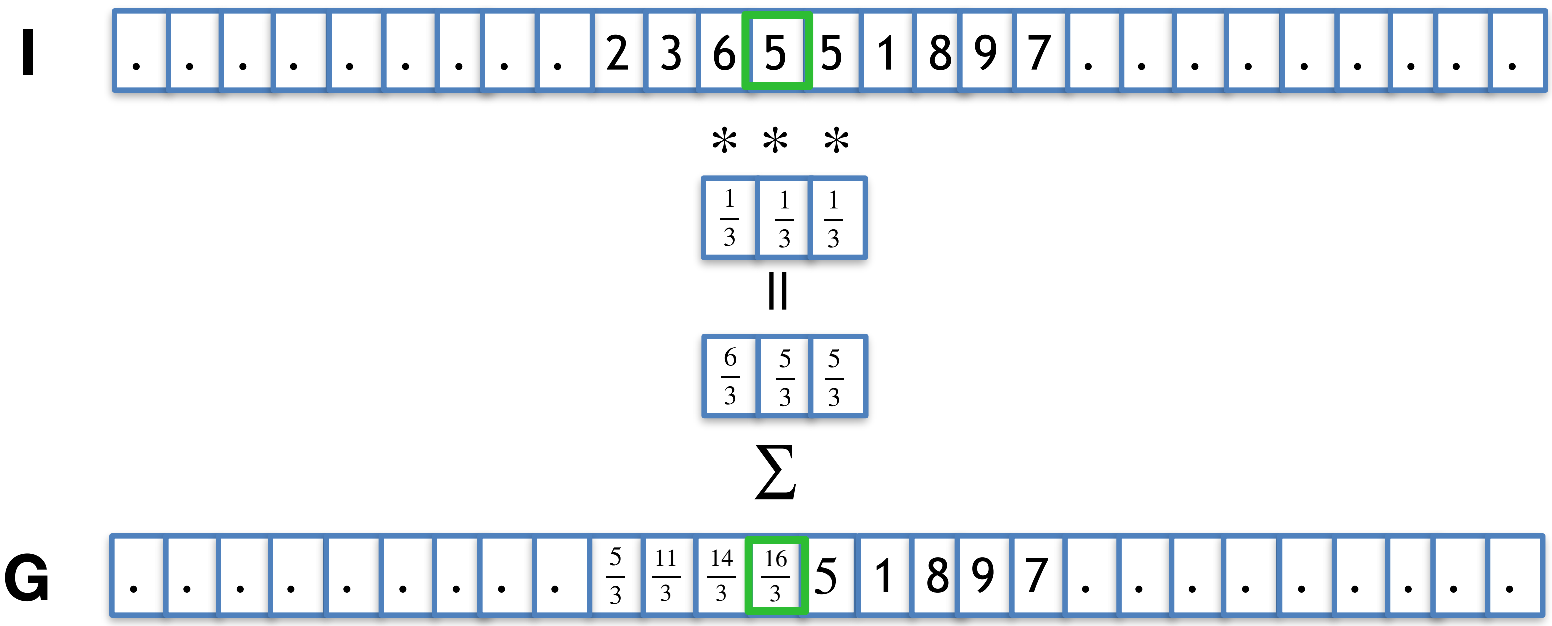
Correlation Example - 1D



Correlation Example - 1D



Correlation Example - 1D



Correlation Example - 1D

I

.	.	.	.	.	.	.	.	.	2	3	6	5	$\boxed{5}$	1	8	9	7	.	.	.	.	.	.	.	.	.

$* \quad * \quad *$

$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$||$

$\frac{5}{3}$	$\frac{5}{3}$	$\frac{1}{3}$

Σ

G

.	.	.	.	.	.	.	.	.	$\frac{5}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	$\boxed{\frac{11}{3}}$	1	8	9	7	.	.	.	.	.	.	.	.	.

Correlation Example - 1D

I

.	.	.	.	.	.	.	.	.	2	3	6	5	5	1	8	9	7	.	.	.	.	.	.	.	.	.

$$* \quad * \quad *$$

$$\begin{array}{|c|c|c|}\hline \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline \end{array}$$

$$\|$$

$$\begin{array}{|c|c|c|}\hline \frac{5}{3} & \frac{1}{3} & \frac{8}{3} \\ \hline \end{array}$$

$$\Sigma$$

G

.	.	.	.	.	.	.	.	.	$\frac{5}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	8	9	7	.	.	.	.	.	.	.	.	.

Correlation Example - 1D

I	.	.	.	.	.	.	.	.	.	2	3	6	5	5	1	8	9	7	.	.	.	.	.	.	.	.	.

$$* \quad * \quad *$$

$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$||$$

$\frac{1}{3}$	$\frac{8}{3}$	$\frac{9}{3}$

$$\Sigma$$

G	.	.	.	.	.	.	.	.	.	$\frac{5}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	6	9	7	.	.	.	.	.	.	.	.	.

Correlation Example - 1D

I	.	.	.	.	.	.	.	.	.	2	3	6	5	5	1	8	$\boxed{9}$	7	.	.	.	.	.	.	.	.	.

$$* \quad * \quad *$$

$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$\|$$

$\frac{8}{3}$	$\frac{9}{3}$	$\frac{7}{3}$

$$\Sigma$$

G	.	.	.	.	.	.	.	.	.	$\frac{5}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	6	$\boxed{8}$	7	.	.	.	.	.	.	.	.	.

Correlation Example - 1D

I	.	.	.	.	.	.	.	.	.	2	3	6	5	5	1	8	9	7	.	.	.	.	.	.	.	.	.

$$* \quad * \quad *$$

$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$\|$$

$\frac{9}{3}$	$\frac{7}{3}$	$\frac{0}{3}$

$$\sum$$

G	.	.	.	.	.	.	.	.	.	$\frac{5}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	6	8	$\frac{16}{3}$	.	.	.	.	.	.	.	.	.

Cross-Correlation and Convolution

$$
\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}
$$

Cross-Correlation and Convolution

$$2 \times 15 + 1 \times 1 = 31$$

$$\begin{array}{|c|c|c|c|c|}\hline 5 & 15 & 4 & 0 & -1 \\ \hline 10 & 1 & 5 & 1 & 0 \\ \hline 6 & 9 & 11 & 1 & -1 \\ \hline 0 & -1 & 5 & 15 & 4 \\ \hline 1 & 0 & 10 & 1 & 5 \\ \hline \end{array} \circ \begin{array}{|c|c|c|}\hline -1 & 0 & 1 \\ \hline -2 & 0 & 2 \\ \hline -1 & 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|}\hline 31 & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Cross-Correlation and Convolution

$$-2 \times 5 + 2 \times 4 - 1 \times 10 + 5 \times 1 = 7$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7			

Cross-Correlation and Convolution

$$-2 \times 15 - 1 \times 1 + 1 \times 1 = -30$$

$$\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -7 & -30 & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

Cross-Correlation and Convolution

$$-2 \times 4 - 1 \times 2 - 5 \times 1 = -15$$

$$\begin{array}{|c|c|c|c|c|} \hline 5 & 15 & 4 & 0 & -1 \\ \hline 10 & 1 & 5 & 1 & 0 \\ \hline 6 & 9 & 11 & 1 & -1 \\ \hline 0 & -1 & 5 & 15 & 4 \\ \hline 1 & 0 & 10 & 1 & 5 \\ \hline \end{array} \circ \begin{array}{|c|c|c|} \hline -1 & 0 & 1 \\ \hline -2 & 0 & 2 \\ \hline -1 & 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 31 & -7 & -30 & -15 & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Cross-Correlation and Convolution

$$-1 \times 1 = -1$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1

Cross-Correlation and Convolution

$$15 \times 1 + 2 \times 1 + 9 \times 1 = 26$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26				

Cross-Correlation and Convolution

$$5 \times (-1) + 10 \times (-2) + 6 \times (-1) + 4 \times 1 + 5 \times 2 + 11 \times 1 = -6$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6			

Cross-Correlation and Convolution

$$15 \times (-1) + 1 \times (-2) + 9 \times (-1) + 0 \times 1 + 1 \times 2 + 1 \times 1 = -23$$

$$\begin{array}{|c|c|c|c|c|}\hline 5 & 15 & 4 & 0 & -1 \\ \hline 10 & 1 & 5 & 1 & 0 \\ \hline 6 & 9 & 11 & 1 & -1 \\ \hline 0 & -1 & 5 & 15 & 4 \\ \hline 1 & 0 & 10 & 1 & 5 \\ \hline \end{array} \circ \begin{array}{|c|c|c|}\hline -1 & 0 & 1 \\ \hline -2 & 0 & 2 \\ \hline -1 & 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|}\hline 31 & -7 & -30 & -15 & -1 \\ \hline 26 & -6 & -23 & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Cross-Correlation and Convolution

$$4 \times (-1) + 5 \times (-2) + 11 \times (-1) + (-1) \times 1 + 0 \times 2 + (-1) \times 1 = -27$$

$$\begin{array}{|c|c|c|c|c|} \hline 5 & 15 & 4 & 0 & -1 \\ \hline 10 & 1 & 5 & 1 & 0 \\ \hline 6 & 9 & 11 & 1 & -1 \\ \hline 0 & -1 & 5 & 15 & 4 \\ \hline 1 & 0 & 10 & 1 & 5 \\ \hline \end{array} \circ \begin{array}{|c|c|c|} \hline -1 & 0 & 1 \\ \hline -2 & 0 & 2 \\ \hline -1 & 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 31 & -7 & -30 & -15 & -1 \\ \hline 26 & -6 & -23 & -27 & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Cross-Correlation and Convolution

$$0 \times (-1) + 1 \times (-2) + 1 \times (-1) = -3$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3

Cross-Correlation and Convolution

$$1 \times 1 + 9 \times 2 + -1 \times 1 = 18$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18				

Cross-Correlation and Convolution

$$10 \times (-1) + 6 \times (-2) + 5 \times 1 + 11 \times 2 + 5 \times 1 = 10$$

$$\begin{array}{|c|c|c|c|c|}\hline 5 & 15 & 4 & 0 & -1 \\ \hline 10 & 1 & 5 & 1 & 0 \\ \hline 6 & 9 & 11 & 1 & -1 \\ \hline 0 & -1 & 5 & 15 & 4 \\ \hline 1 & 0 & 10 & 1 & 5 \\ \hline \end{array} \circ \begin{array}{|c|c|c|}\hline -1 & 0 & 1 \\ \hline -2 & 0 & 2 \\ \hline -1 & 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|}\hline 31 & -7 & -30 & -15 & -1 \\ \hline 26 & -6 & -23 & -27 & -3 \\ \hline 18 & 10 & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Cross-Correlation and Convolution

$$1 \times (-1) + 9 \times (-2) + (-1) \times (-1) + 1 \times 1 + 2 \times 1 + 15 \times 1 = 0$$

$$\begin{array}{|c|c|c|c|c|} \hline 5 & 15 & 4 & 0 & -1 \\ \hline 10 & 1 & 5 & 1 & 0 \\ \hline 6 & 9 & 11 & 1 & -1 \\ \hline 0 & -1 & 5 & 15 & 4 \\ \hline 1 & 0 & 10 & 1 & 5 \\ \hline \end{array} \circ \begin{array}{|c|c|c|} \hline -1 & 0 & 1 \\ \hline -2 & 0 & 2 \\ \hline -1 & 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 31 & -7 & -30 & -15 & -1 \\ \hline 26 & -6 & -23 & -27 & -3 \\ \hline 18 & 10 & 0 & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Cross-Correlation and Convolution

$$5 \times (-1) + 11 \times (-2) + 5 \times (-1) + 0 \times 1 + (-1) \times 2 + 4 \times 1 = -30$$

$$\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -7 & -30 & -15 & -1 \\ 26 & -6 & -23 & -27 & -3 \\ 18 & 10 & 0 & -30 & \\ & & & & \\ & & & & \end{bmatrix}$$

Cross-Correlation and Convolution

$$1 \times (-1) + 1 \times (-2) + 15 \times (-1) + 0 \times 1 + 0 \times 2 + 0 \times 1 = -18$$

$$\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -7 & -30 & -15 & -1 \\ 26 & -6 & -23 & -27 & -3 \\ 18 & 10 & 0 & -30 & -18 \\ & & & & \\ & & & & \end{bmatrix}$$

Cross-Correlation and Convolution

$$0 \times (-1) + 0 \times (-2) + 0 \times (-1) + 9 \times 1 + (-1) \times 2 + 0 \times 1 = 7$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7				

Cross-Correlation and Convolution

$$6 \times (-1) + 0 \times (-2) + 1 \times (-1) + 11 \times 1 + 5 \times 2 + 10 \times 1 = 24$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24			

Cross-Correlation and Convolution

$$9 \times (-1) + -1 \times (-2) + 0 \times (-1) + 1 \times 1 + 15 \times 2 + 1 \times 1 = 25$$

$$\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -7 & -30 & -15 & -1 \\ 26 & -6 & -23 & -27 & -3 \\ 18 & 10 & 0 & -30 & -18 \\ 7 & 24 & 25 & & \\ & & & & \end{bmatrix}$$

Cross-Correlation and Convolution

$$11 \times (-1) + 5 \times (-2) + 10 \times (-1) + (-1) \times 1 + 4 \times 2 + 5 \times 1 = -19$$

$$\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -7 & -30 & -15 & -1 \\ 26 & -6 & -23 & -27 & -3 \\ 18 & 10 & 0 & -30 & -18 \\ 7 & 24 & 25 & -19 & \\ & & & & \end{bmatrix}$$

Cross-Correlation and Convolution

$$1 \times (-1) + 15 \times (-2) + 1 \times (-1) + 0 \times 1 + 0 \times 2 + 0 \times 1 = -32$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32

Cross-Correlation and Convolution

$$0 \times (-1) + 0 \times (-2) + 0 \times (-1) + (-1) \times 1 + 0 \times 2 + 0 \times 1 = -1$$

$$\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -7 & -30 & -15 & -1 \\ 26 & -6 & -23 & -27 & -3 \\ 18 & 10 & 0 & -30 & -18 \\ 7 & 24 & 25 & -19 & -32 \\ -1 & & & & \end{bmatrix}$$

Cross-Correlation and Convolution

$$0 \times (-1) + 1 \times (-2) + 0 \times (-1) + 5 \times 1 + 10 \times 2 + 0 \times 1 = 23$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32
-1	23			

Cross-Correlation and Convolution

$$-1\times(-1)+0\times(-2)+0\times(-1)+15\times 1+01\times 2+0\times 1=16$$

$$\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -7 & -30 & -15 & -1 \\ 26 & -6 & -23 & -27 & -3 \\ 18 & 10 & 0 & -30 & -18 \\ 7 & 24 & 25 & -19 & -32 \\ -1 & 23 & 16 & & \end{bmatrix}$$

Cross-Correlation and Convolution

$$5 \times (-1) + 10 \times (-2) + 0 \times (-1) + 4 \times 1 + 5 \times 2 + 0 \times 1 = -11$$

$$\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -7 & -30 & -15 & -1 \\ 26 & -6 & -23 & -27 & -3 \\ 18 & 10 & 0 & -30 & -18 \\ 7 & 24 & 25 & -19 & -32 \\ -1 & 23 & 16 & -11 & \end{bmatrix}$$

Cross-Correlation and Convolution

$$15 \times (-1) + 1 \times (-2) + 0 \times (-1) + 0 \times 1 + 0 \times 2 + 0 \times 1 = -17$$

$$\begin{bmatrix} 5 & 15 & 4 & 0 & -1 \\ 10 & 1 & 5 & 1 & 0 \\ 6 & 9 & 11 & 1 & -1 \\ 0 & -1 & 5 & 15 & 4 \\ 1 & 0 & 10 & 1 & 5 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -7 & -30 & -15 & -1 \\ 26 & -6 & -23 & -27 & -3 \\ 18 & 10 & 0 & -30 & -18 \\ 7 & 24 & 25 & -19 & -32 \\ -1 & 23 & 16 & -11 & -17 \end{bmatrix}$$

Cross-Correlation and Convolution

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32
-1	23	16	-11	-17

Image, I — Filter/template — Output image

Cross-Correlation - Mathematically

$1D$

$$G = F \circ I[i] = \sum_{u=-k}^{k} F[u]I[i+u]$$ F has $2k+1$ elements

Box filter $F[u] = \frac{1}{3}$ for $u = -1,0,1$ and 0 otherwise

Cross-correlation filtering - 2D

Let's write this down as an equation. Assume the averaging window is (2k+1)x(2k+1):

$$G[i,j] = \frac{1}{(2k+1)^2} \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[i+u, j+v]$$

We can generalize this idea by allowing different weights for different neighboring pixels:

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[u,v] I[i+u, j+v]$$

This is called a **cross-correlation** operation and written:

$$G = F \circ I$$

F is called the "filter," "kernel," or "mask."

Convolution

Filter is flipped before correlating

$1D$ $\quad$ F has $2k+1$ elements

$$G = F * I[i] = \sum_{u=-k}^{k} F[u]I[i-u]$$

Box filter $F[u] = \dfrac{1}{3}$ for $u = -1,0,1$ and 0 otherwise

for example, convolution of 1D image with the filter [3,5,2]

is exactly the same as correlation with the filter [2,5,3]

Convolution filtering - 2D

For 2D the filter is flipped and rotated

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[u,v] I[i-u, j-v]$$

Correlation and convolution are identical for symmetrical filters

Convolution with the filter

1	2	1
0	0	0
-1	-2	-1

is the same as Correlation with the filter

-1	0	1
-2	0	2
-1	0	1