CMSC 132: OBJECT-ORIENTED PROGRAMMING II

Single Source Shortest Path Algorithm

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Single Source Shortest Path

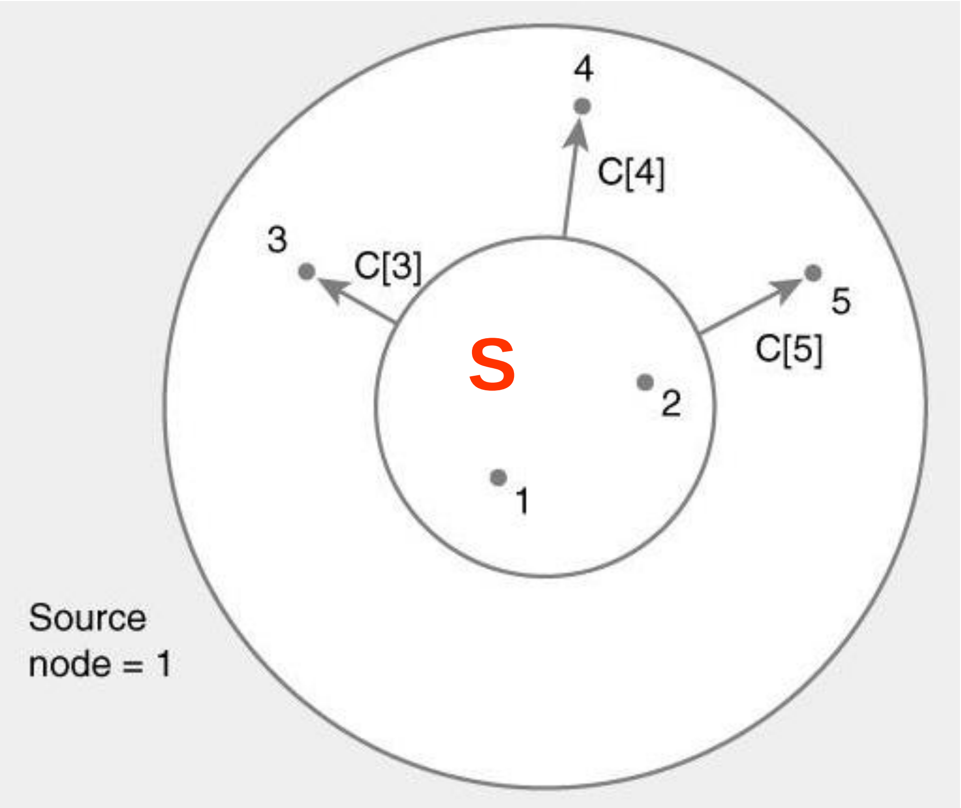
- Common graph problems
  - Problem1 → Find path from X to Y with lowest edge weight
  - Problem2 → Find path from X to any Y with lowest edge weight
- This is not the same as the Traveling Salesman Problem
- Single Source Shortest Path - Useful for many applications
  - Shortest route in map (Similar to GPS)
  - Lowest cost trip
  - Most efficient internet route
- Dijkstra's algorithm
  - Finds path from X to any Y with lowest edge weight
    - Computes shortest path from X to any other node, but not the shortest path from any node to any other node

Shortest Path – Dijkstra's Algorithm

- Maintain
  - Nodes with known shortest path from start → S
  - Cost of shortest path to node K from start → C[K]
    - Only for paths through nodes in S
  - Predecessor to K on shortest path ⇒ P[K]
    - Updated whenever new (lower) C[K] discovered
    - Remembers actual path with lowest cost

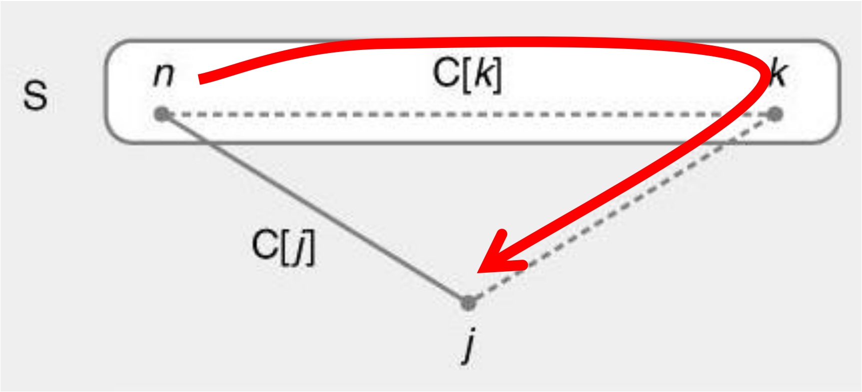
Shortest Path – Intuition for Dijkstra's

- At each step in the algorithm
  - Shortest paths are known for nodes in S
  - Store in C[K] length of shortest path to node K (for all paths through nodes in { S })
  - Add to { S } next lowest cost node



Shortest Path – Intuition for Djikstra's

- Update distance to J after adding node K
  - Previous shortest path to K already in C[K]
  - Possibly shorter path to J by going through node K
  - Compare C[J] with C[K] + weight of (K,J), update C[J] if needed



Shortest Path – Dijkstra's Algorithm

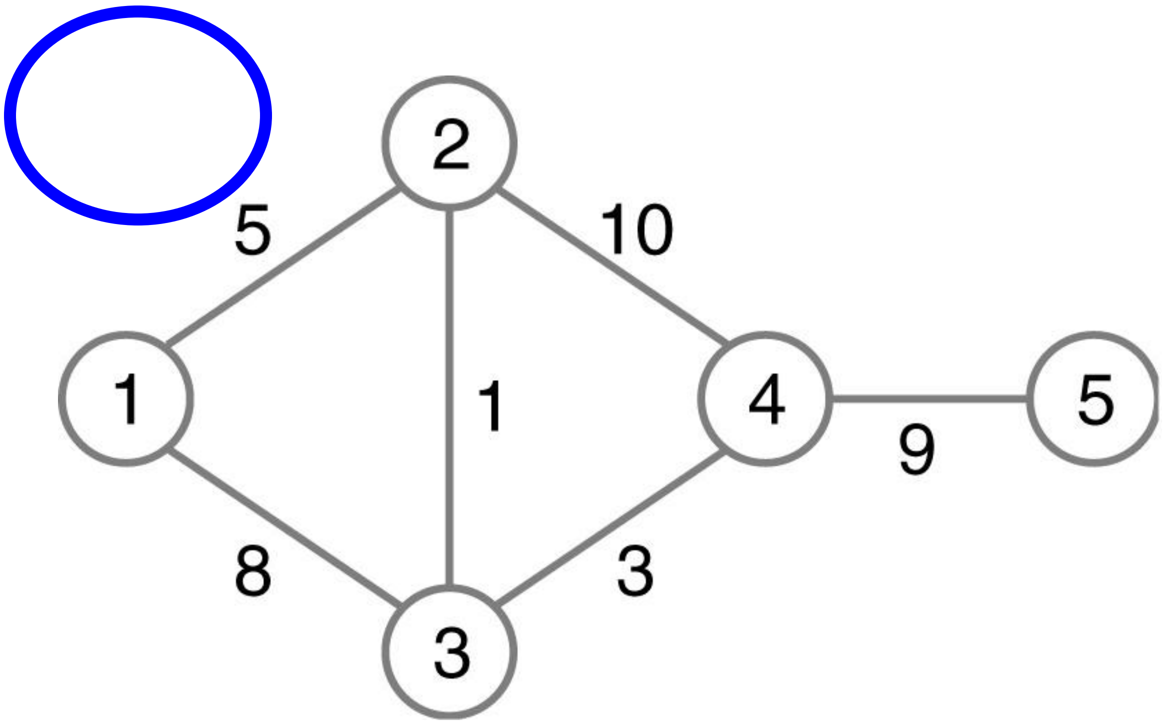
```
S = ∅
P[ ] = none for all nodes
C[start] = 0, C[ ] = ∞ for all other nodes
while ( nodes can be added to S )
  find node K not in S with smallest C[K]
  add K to S
  for each node J not in S  adjacent to K
      if ( C[K] + cost of (K,J) < C[J] )
         C[J] = C[K] + cost of (K,J)
         P[J] = K
```

Optimal solution computed with greedy algorithm

Dijkstra's Shortest Path Example

- Initial state
- S = ∅

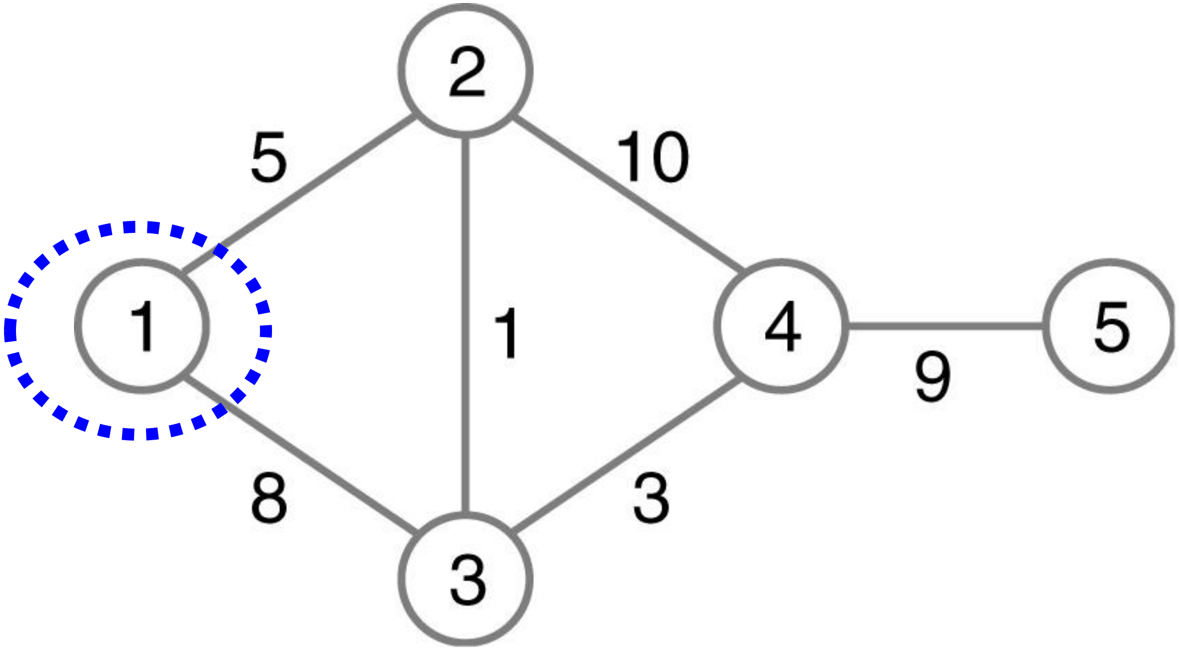
	C	P
1	0	none
2	∞	none
3	∞	none
4	∞	none
5	∞	none



Dijkstra's Shortest Path Example

- Find shortest paths starting from node 1
- S = 1

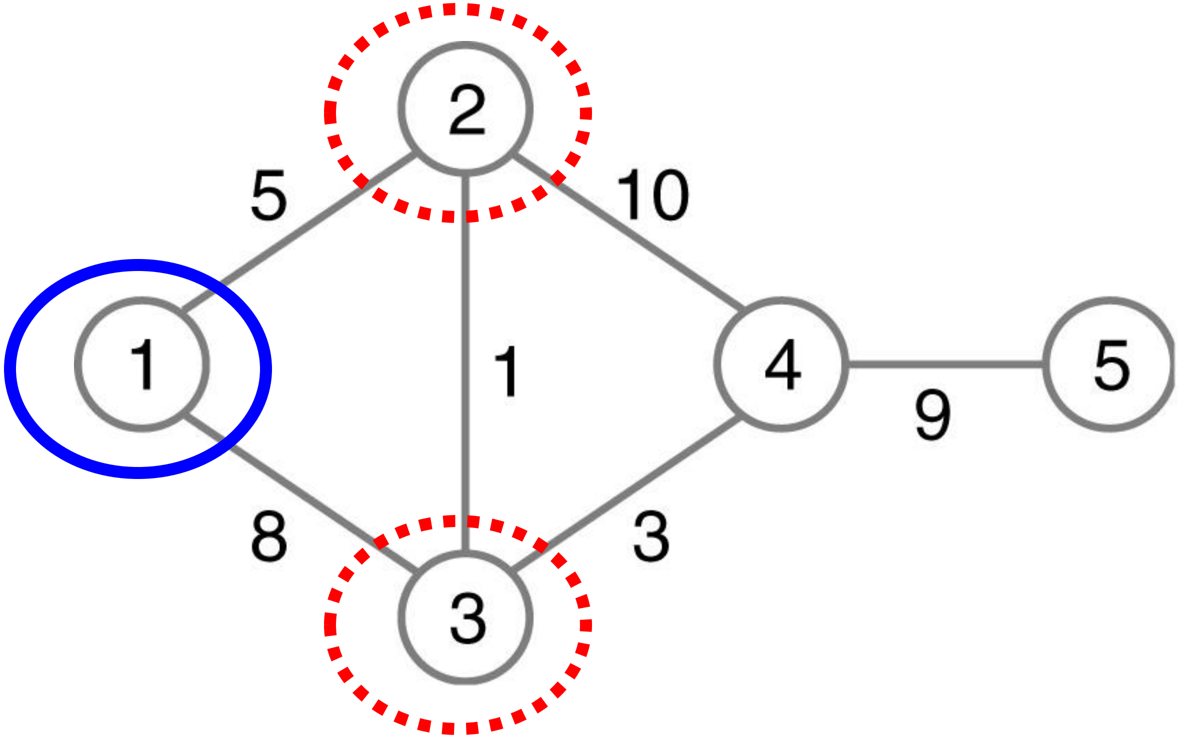
	C	P
1	0	none
2	∞	none
3	∞	none
4	∞	none
5	∞	none



Djikstra's Shortest Path Example

- Update C[K] for all neighbors of 1 not in { S }
- S = { 1 }

	C	P
1	0	none
2	5	1
3	8	1
4	∞	none
5	∞	none

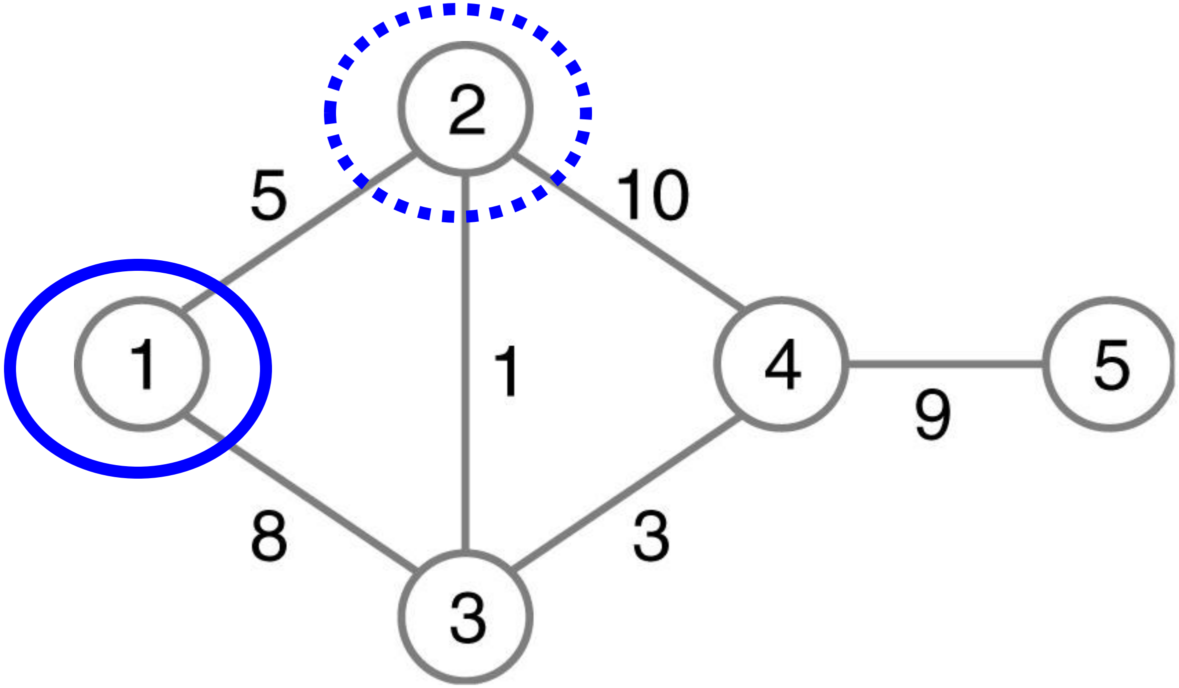


C[2] = min (∞ , C[1] + (1,2)) = min (∞ , 0 + 5) = 5
C[3] = min (∞ , C[1] + (1,3)) = min (∞ , 0 + 8) = 8

Djikstra's Shortest Path Example

- Find node K with smallest C[K] and add to S
- S = { 1, 2 }

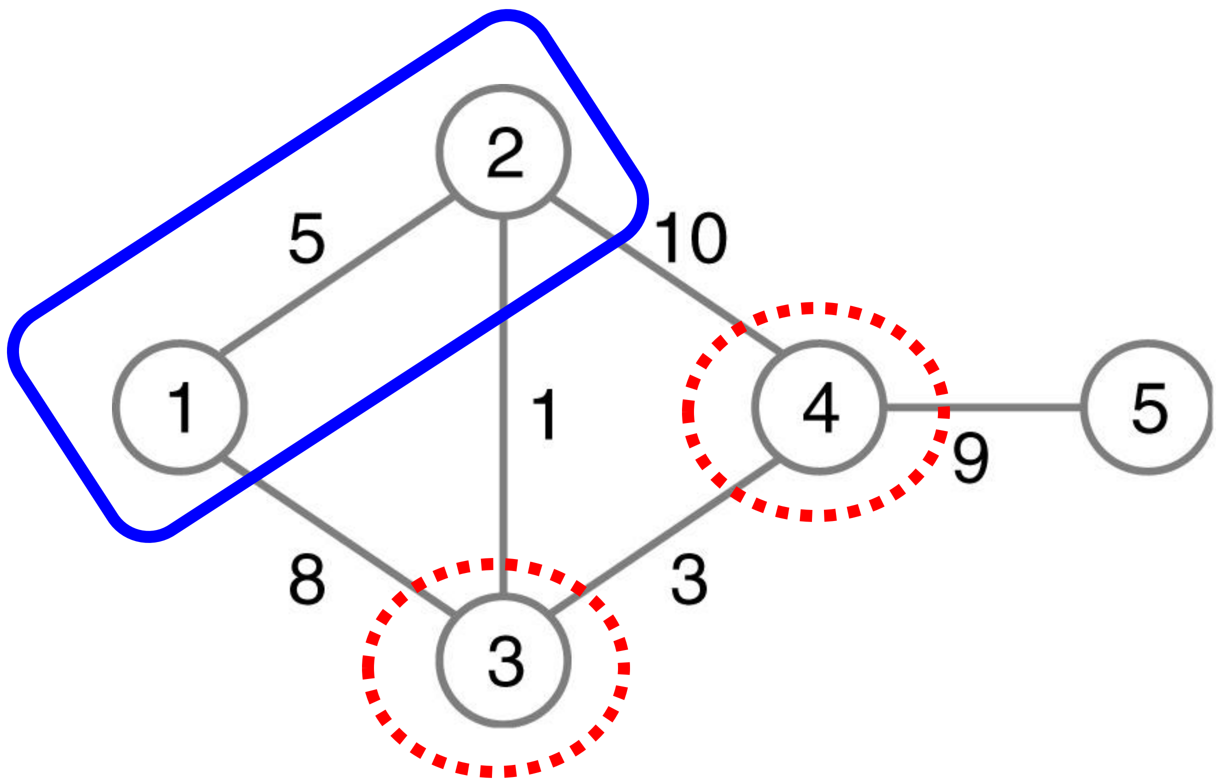
	C	P
1	0	none
2	5	1
3	8	1
4	∞	none
5	∞	none



Dijkstra's Shortest Path Example

- Update C[K] for all neighbors of 2 not in S
- S = { 1, 2 }

	C	P
1	0	none
2	5	1
3	6	2
4	15	2
5	∞	none

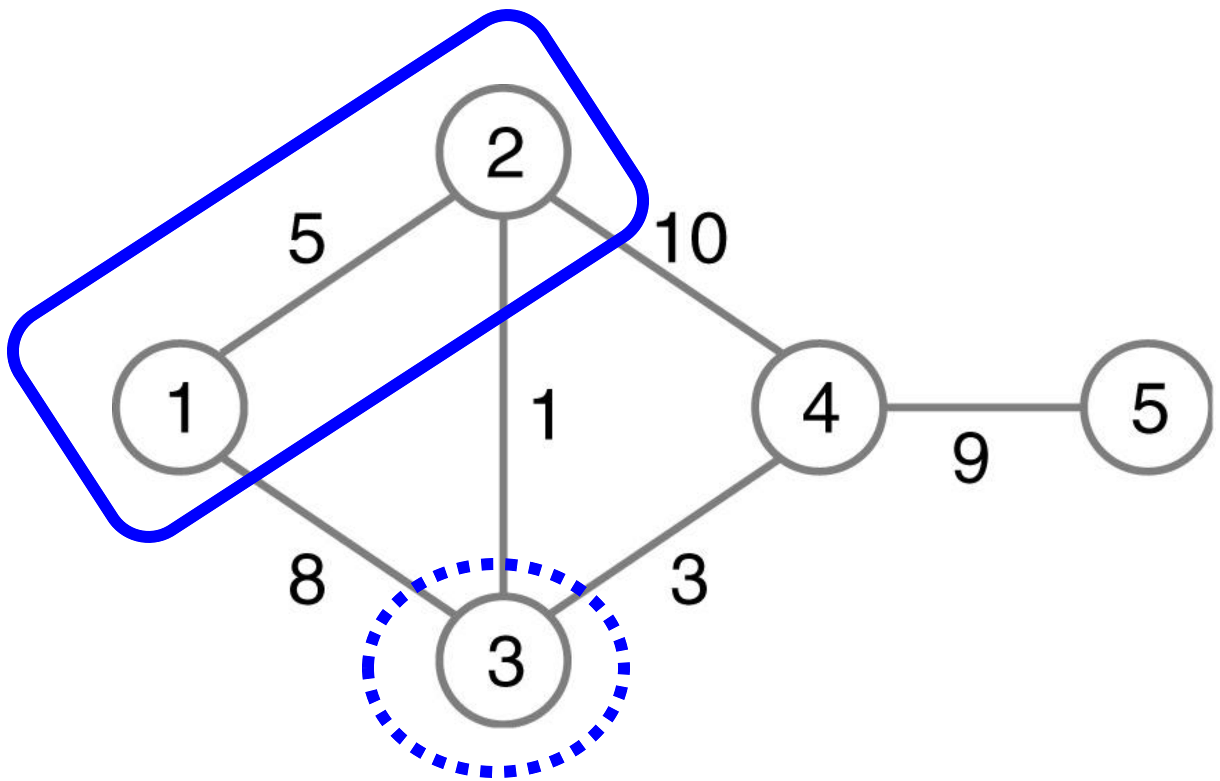


C[3] = min (8 , C[2] + (2,3)) = min (8 , 5 + 1) = 6
C[4] = min (∞ , C[2] + (2,4)) = min (∞ , 5 + 10) = 15

Dijkstra’s Shortest Path Example

- Find node K with smallest C[K] and add to S
- S = { 1, 2, 3 }

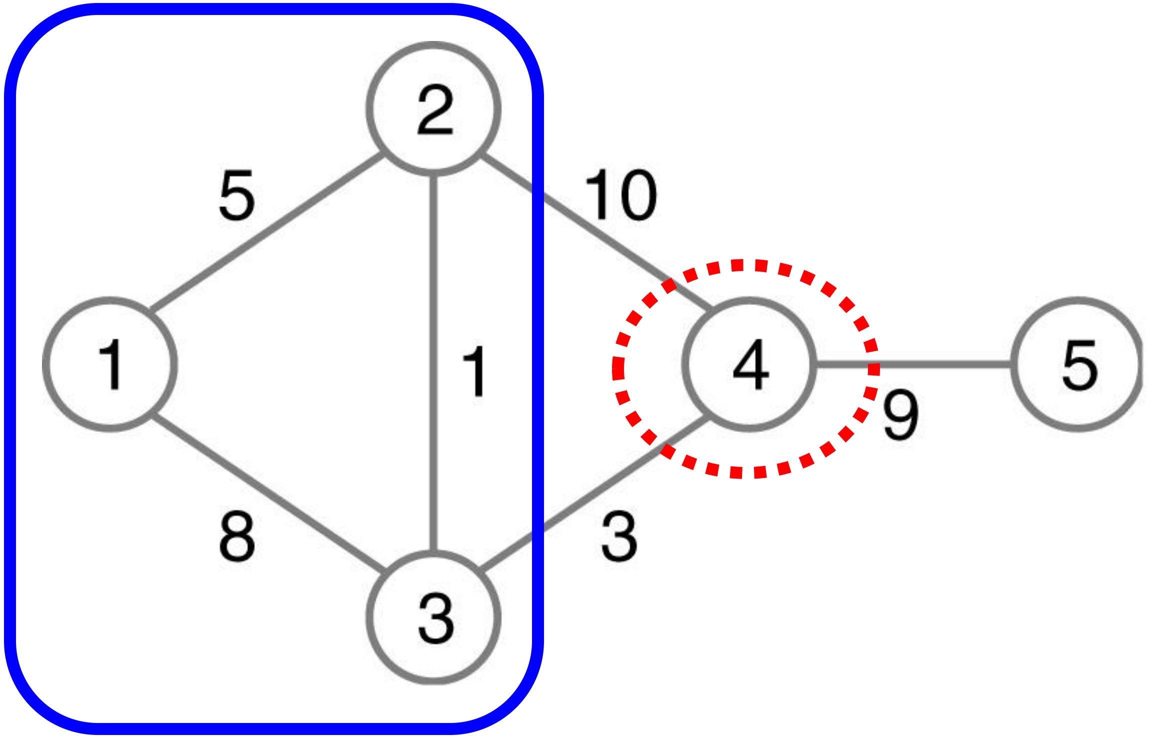
	C	P
1	0	none
2	5	1
3	6	2
4	15	2
5	∞	none



Dijkstra's Shortest Path Example

- Update C[K] for all neighbors of 3 not in S
- { S } = 1, 2, 3

	C	P
1	0	none
2	5	1
3	6	2
4	9	3
5	∞	none

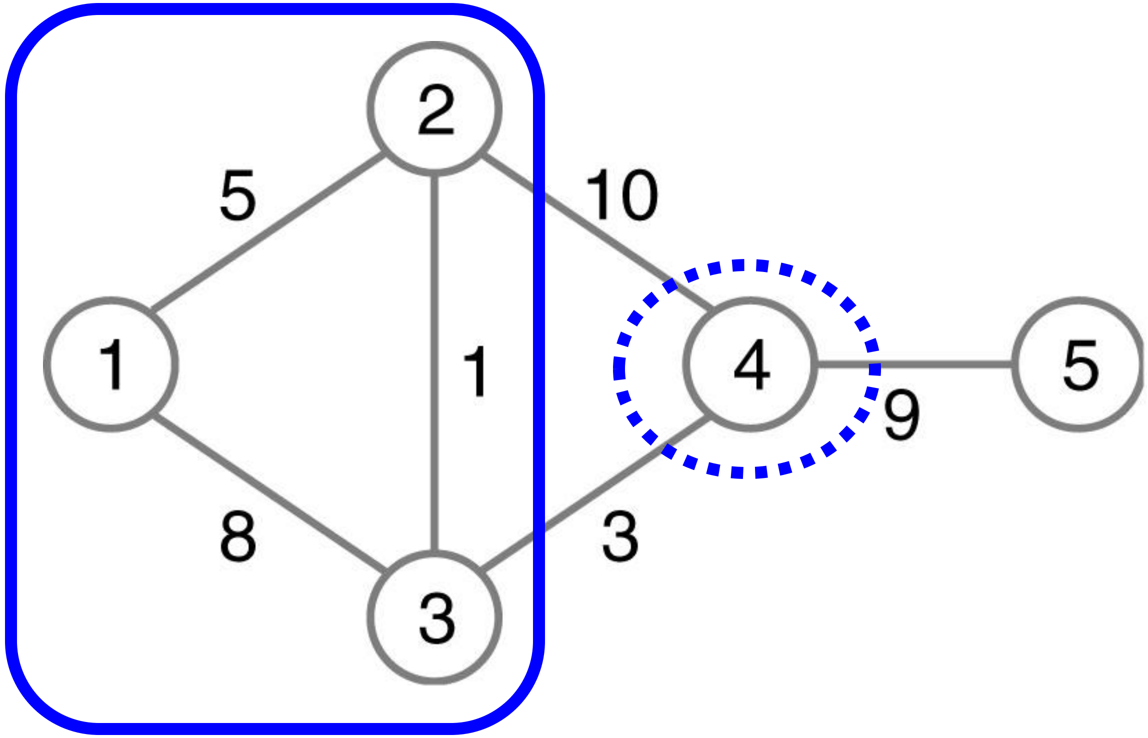


C[4] = min (15 , C[3] + (3,4)) = min (15 , 6 + 3) = 9

Dijkstra's Shortest Path Example

- Find node K with smallest C[K] and add to S
- { S } = 1, 2, 3, 4

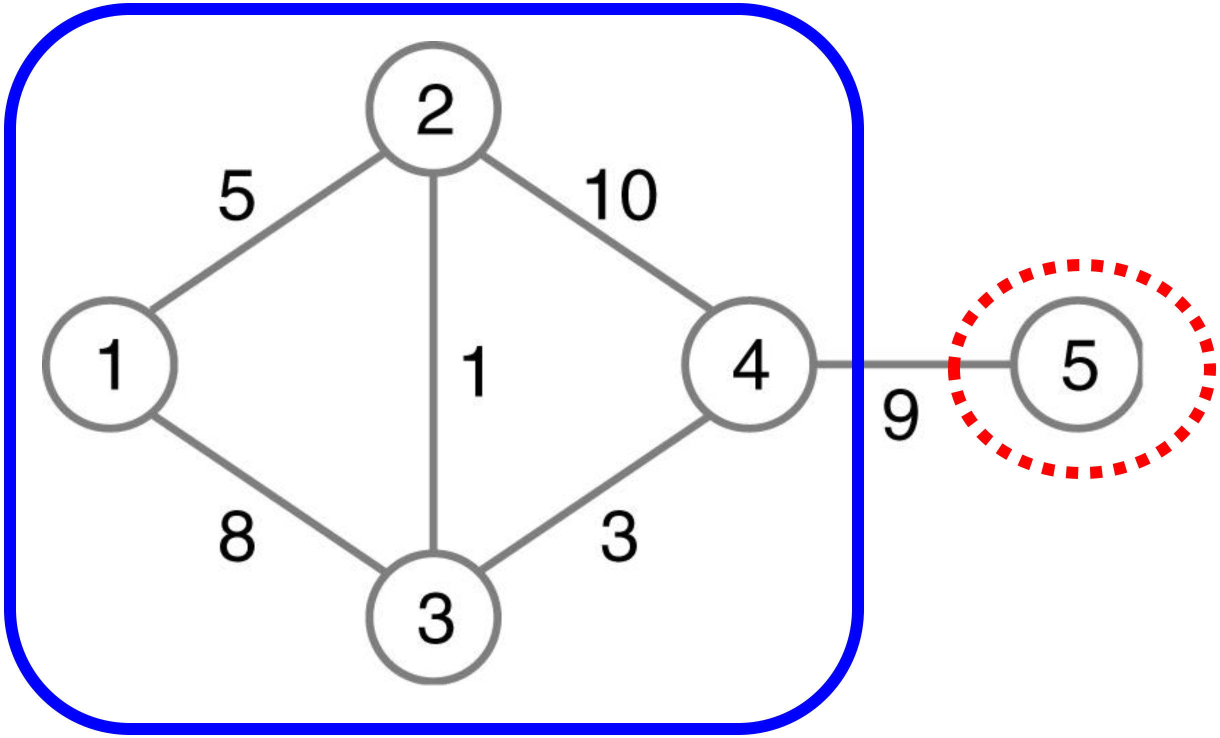
	C	P
1	0	none
2	5	1
3	6	2
4	9	3
5	∞	none



Dijkstra’s Shortest Path Example

- Update C[K] for all neighbors of 4 not in S
- S = { 1, 2, 3, 4 }

	C	P
1	0	none
2	5	1
3	6	2
4	9	3
5	18	4

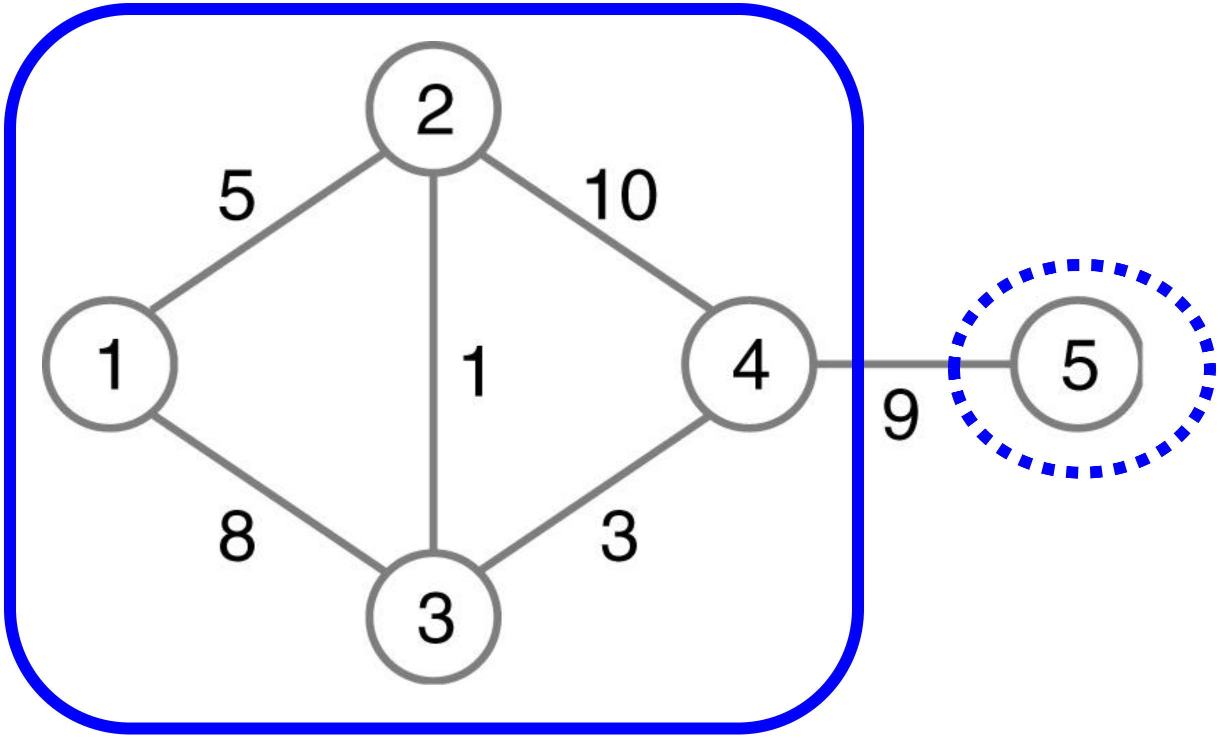


C[5] = min (∞ , C[4] + (4,5)) = min (∞ , 9 + 9) = 18

Dijkstra's Shortest Path Example

- Find node K with smallest C[K] and add to S
- S = { 1, 2, 3, 4, 5 }

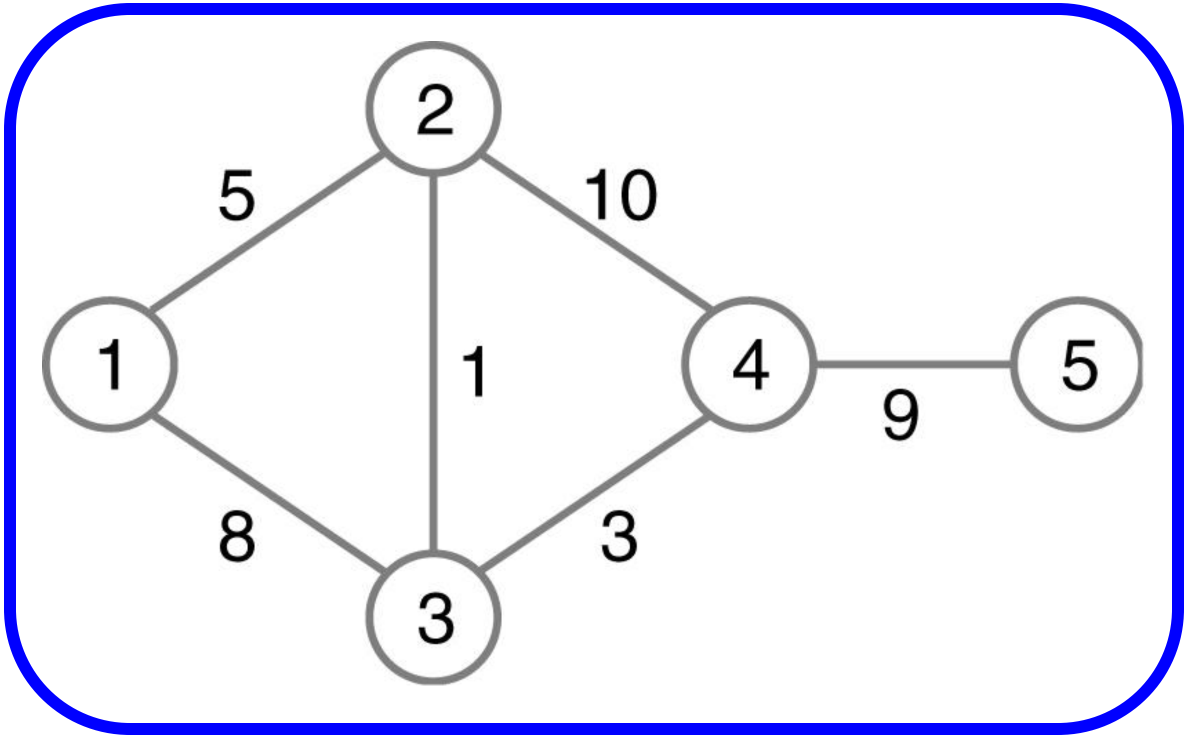
	C	P
1	0	none
2	5	1
3	6	2
4	9	3
5	18	4



Dijkstra's Shortest Path Example

- All nodes in S, algorithm is finished
- S = { 1, 2, 3, 4, 5 }

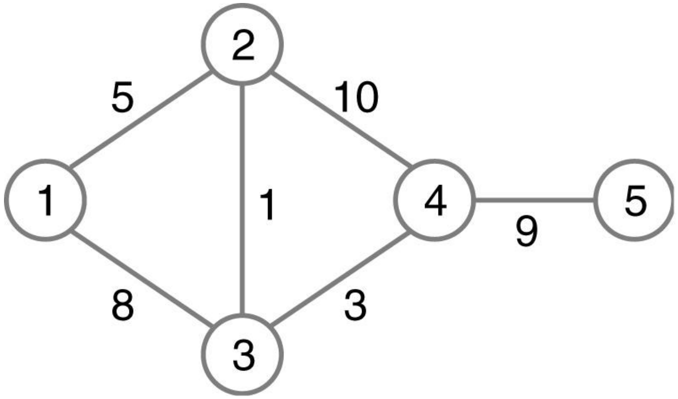
	C	P
1	0	none
2	5	1
3	6	2
4	9	3
5	18	4



Dijkstra's Shortest Path Example

- Find shortest path from start to K
  - Start at K
  - Trace back predecessors in P[ ]
- Example paths (in reverse)
  - 2 → 1
  - 3 → 2 → 1
  - 4 → 3 → 2 → 1
  - 5 → 4 → 3 → 2 → 1

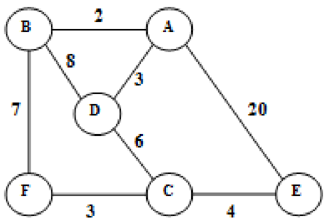
	C	P
1	0	none
2	5	1
3	6	2
4	9	3
5	18	4



About Dijkstra's Algorithm

- You always select the next node with the lowest cost
  - **Not necessarily adjacent to the last one processed**
- What if while processing a node, one of the adjacent nodes belongs to the set S?
- What if you have a value not reachable from the start vertex? What is the cost?
- What if there is a node with an edge pointing to itself?
- What if there are two nodes with the same cost? Which one is selected next?
- What happens if the edge costs are negative?
  - Use Bellman-Ford algorithm
- You can stop Dijkstra's once you have computed the path/cost to the node of interest
- **Running the algorithm using one vertex (start) does not generate the shortest paths from any vertex to any other vertex.**
- Big O using min-priority queue → **O(|E| + |V| log |V|)**

Typical Problem for Exam/Quiz



Apply Dijkstra's algorithm using B as the starting (source) node. Indicate the cost and predecessor for each node in the graph after processing 1, 2 and 3 nodes (B and 2 other nodes) have been added to the set of processed nodes (Remember to update the appropriate table entries after processing the 3rd node added). An empty table entry implies an infinite cost or no predecessor. *Note: points will be deducted if you simply fill in the entire table instead showing the table at the first three steps.*

Answer:

After processing 1 node:

Node	A	B	C	D	E	F
Cost	2	0		8		7
Predecessor	B			B		B

After processing 2 nodes:

Node	A	B	C	D	E	F
Cost	2	0		5	22	7
Predecessor	B			A	A	B

After processing 3 nodes:

Node	A	B	C	D	E	F
Cost	2	0	11	5	22	7
Predecessor	B		D	A	A	B

Java Priority Queue

- Java Priority Queue
  - https://docs.oracle.com/en/java/javase/11/docs/api/java.base/java/util/PriorityQueue.html
- **Example:** PriorityQueueCode.zip