Data structures are FUNDAMENTAL!

- All fields of CS involve storing, retrieving and processing data

- Information retrieval
- Geographic Inf. Systems
- Machine Learning
- Text/String processing
- Computer Graphics
- ....

Basic Elements in Study of data structures

- Modeling: How real world objects are encoded
- Operations: Allowed functions to access + modify structure
- Representation: Mapping to memory
- Algorithms: How are operations performed?

Course Overview:

- Fundamental data structures + algorithms
- Mathematical techniques for analyzing them
- Implementation

Introduction to Data Structures

- Elements of data structures
- Our approach
- Short review of asymptotics

Our approach:

- Theoretical: Algorithms + Asymptotic Analysis
- Practical: Implementation + practical efficiency

Asymptotic Analysis:

- Run time as function of n: no. of items
- Worst-case, average case, randomized,...
- Amortized - average over series of ops.

Asymptotic: "Big-O"

- Ignore constants
- Focus on large n

$T(n) = 34n^2 + 15n\log n + 143$

$T(n) = O(n^2)$

Common:

$O(1)$: constant time ☺ [Hashmap]

$O(\log n)$: log-time (good) [Binary search]

$O(n^p)$: p = constant: poly time

$O(\sqrt{n})$

Basic Data Structures I

- ADTs
- Lists, Stacks, Queues
- Sequential Allocation

Abstract Data Type (ADT)

- Abstracts the functional elements of a data structure (math) from its implementation (algorithm / programming)

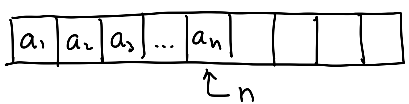
Linear List ADT:

Stores a sequence of elements $\langle a_1, a_2, \ldots, a_n \rangle$. Operations:

- init() - create an empty list
- get(i) - returns a_i
- set(i,x) - sets i^{th} element to x
- insert(i,x) - inserts x prior to i^{th} (moving others back)
- delete(i) - deletes i^{th} item (moving others up)
- length() - returns num. of items

Implementations:

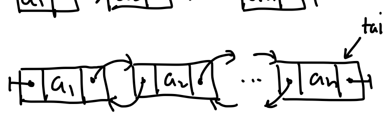
Sequential: Store items in an array



Linked allocation: linked list

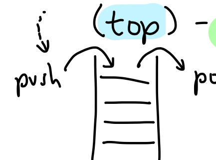
Singly: head →

Doubly: head →

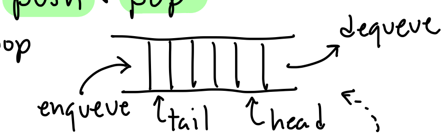


Performance varies with implementation

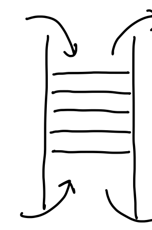
Stack: All access from one side (top) - push + pop



Queue: FIFO list: enqueue inserts at tail and dequeue deletes from head



Deque ("deck"): Can insert or delete from either end



Dynamic Lists + Sequential Allocation: What to do when your array runs out of space?

Doubling Reallocation:

When array of size n overflows

- allocate new array size 2n
- copy old to new
- remove old array

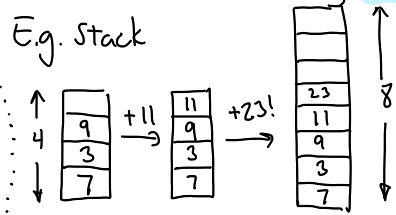
Basic Data Structures II

- Amortized analysis of dynamic stack

Dynamic (Sequential) Allocation

- When we overflow, double

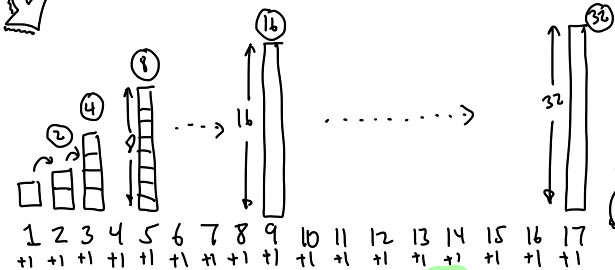
E.g. Stack



Cost model (Actual cost)

Cheap: No reallocation → 1 unit

Expensive: Array of size n is reallocated to size 2n } → 2n+1



Total = 17+(2+4+8+16+32) = 79

Amortized Cost:

Starting from an empty structure, suppose that any sequence of m ops takes time T(m). The amortized cost is T(m)/m.

Thm: Starting from an empty stack, the amortized cost of our stack operations is at most 5.

[ie. any seq. of m ops has cost ≤ 5·m]

Charging Argument:

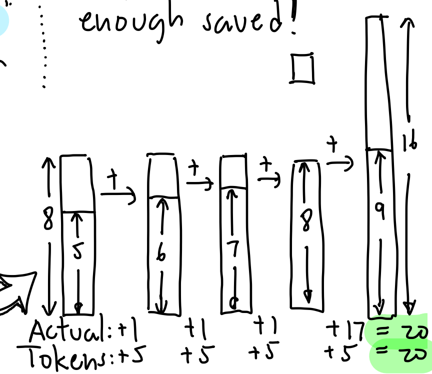
- Each request of push/pop we charge user 5 "work tokens"
- We use 1 token to pay for the operation + put other 4 in bank account.
- Will show there is enough in bank account to pay actual costs.

Proof:

- Break the full sequence after each reallocation → run

1 2 | 3 | 4 5 | 6 7 8 9 | 10 11 ... 16 17 |

- At start of a run there are n+1 items in stack and array size is 2n
- There are at least n ops before the end of run
- During this time we collect at least 5n tokens
  - → 1 for each op
  - → 4 for deposit
- Next reallocation costs 4n, but we have enough saved! □



Basic Data Structures III

- Dynamic Stack - Wrap-up
- Multilists + Sparse Matrices

Dynamic Stack:

- Showed doubling ⇒ Amortized $O(1)$
- Other strategies?

Fixed Increment: Increase by a fixed constant

$n \longrightarrow n + 100$

Fixed factor: Increase by a fixed constant factor (not nec. 2)

$n \rightarrow 5 \cdot n$

Squaring: Square the size (or some other power)

$n \rightarrow n^2$ or $n \rightarrow \lceil n^{1.5} \rceil$

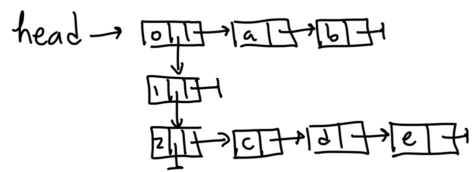
Which of these provide $O(1)$ amortized cost per operation?

Leave as exercise ☹

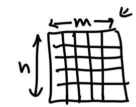
(Spoiler alert!)

- Fixed increment → no
- Fixed factor → yes
- Squaring → yes

Multilists: Lists of lists



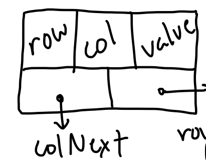
Sparse Matrices: An $n \times m$ matrix has $n \cdot m$ entries and takes (naively) $O(n \cdot m)$ space



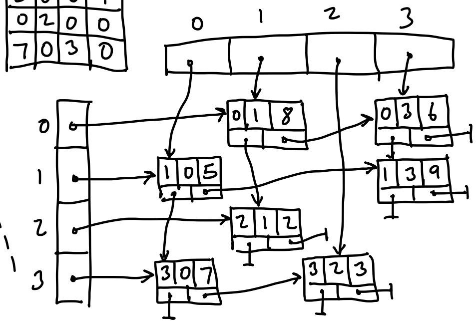
Sparse matrix: Most entries are zero

Idea: Store only nonzero entries linked by row and column

Node:



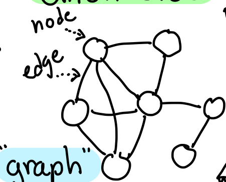
0	8	0	6
5	0	0	9
0	2	0	0
7	0	3	0



Trees: Basic Concepts and Definitions

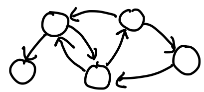
Graph: $G = (V, E)$
V = finite set of **vertices** (**nodes**)
E = set of **edges** (pairs of vertices)

Undirected



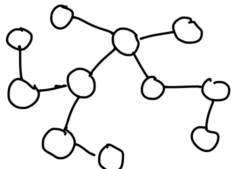
"**graph**"

Directed

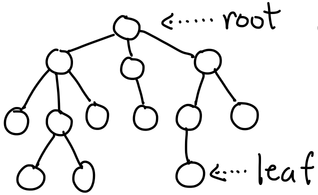


"**digraph**"

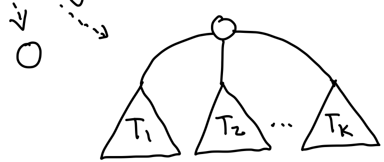
Tree (or "**Free Tree**")
- undirected
- connected
- acyclic graph



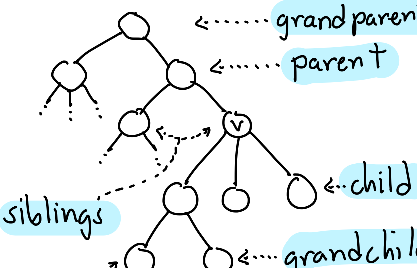
Rooted tree: A free tree with **root** node



Formal definition:
Rooted tree: is either
- single node (root)
- set of one or more rooted trees ("subtrees") joined to a common root

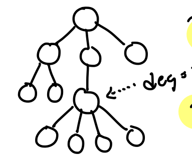


"**Family**" **Relations**

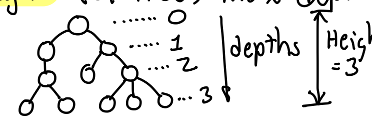


leaf: no children

Degree (of node): number of children
Degree (of tree): max. degree of any node



Depth: path length from root
Height: (of tree) max depth

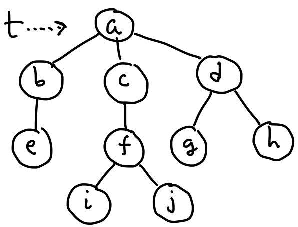


Trees Representation + Binary Trees (1)

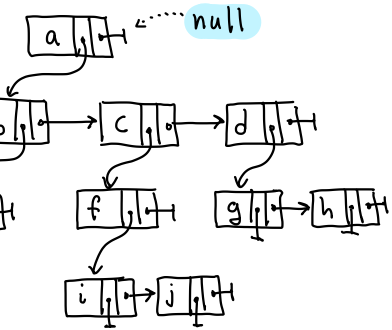
Representing rooted trees:

Each node stores a (linked) list of its children

E.g. root ⋯→

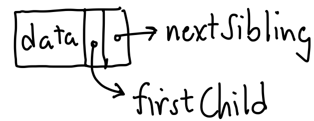


root:



called the Binary representation

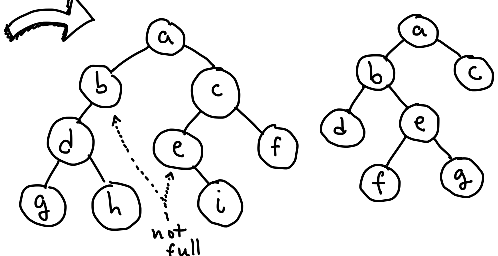
Node structure:



Binary tree:

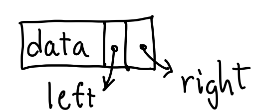
A rooted tree of degree 2, where each node has two children (possibly null) left + right

(Not full) Full:

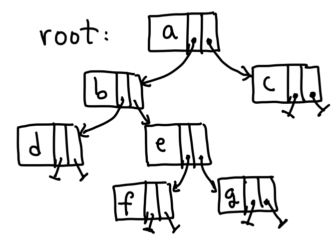


Full: Every non-leaf node has 2 children

Node structure:



root:



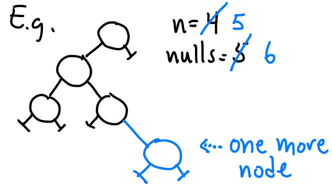
In Java:

```
class BTNode<E> {      ⋯ generic data
    E data;
    BTNode<E> left;
    BTNode<E> right;
    ....
}
```

Wasted space?

Theorem: A binary tree with n nodes has n+1 null links

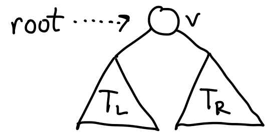
E.g.



```
traverse(BTNode v) {
  if (v == null) return;
  visit/process v        <- Preorder
  traverse(v.left)
  visit/process v        <- Inorder
  traverse(v.right)
  visit/process v        <- Postorder
}
```

Traversals: How to (systematically) visit the nodes of a rooted tree?

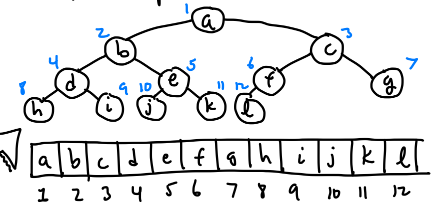
Binary Tree Traversals (can be generalized)



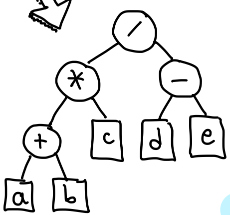
- process/visit v
- traverse T_L
- traverse T_R

recursive

Complete Binary Tree: All levels full (except last)



parent(i) = ⌊i/2⌋
left(i) = 2·i
right(i) = 2·i+1



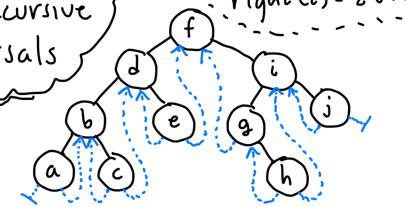
Preorder:
/ * + a b c − d e

Postorder:
a b + c * d e − /

Inorder:
a + b * c / d − e

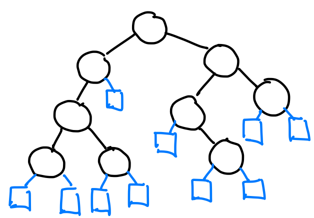
Binary Trees: Traversals, Extension, and More

Challenge: Nonrecursive traversals



Those wasteful null links....

Extended binary tree: Replace each null link with a special leaf node: **external node**



Thm: An extended binary tree with n internal nodes (black) has n+1 external nodes (blue)

Observation: Every extended binary tree is **full**

Another way to save space...

Threaded binary tree: Store (useful) links in the null links. (Use a mark bit to distinguish link types.)

E.g. **Inorder Threads:**

Null left → inorder predecessor

Null right → " successor

Binary Search Trees I
- Basic definitions
- Finding keys

Dictionary:

insert(Key x, Value v)
- insert (x, v) in dict. (No duplicates)

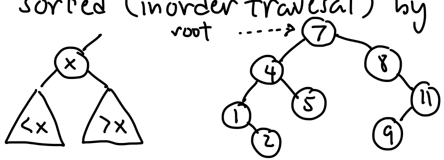
delete(Key x)
- delete x from dict. (Error if x not there)

find(Key x)
- returns a reference to associated value v, or **null** if not there.

Sequential Allocation?
- Store in array sorted by key
  - → **Find**: O(log n) by binary search
  - → **Insert/Delete**: O(n) time

Can we achieve O(log n) time for all ops? ☆ **Binary Search Trees** ☆

Idea: Store entries in binary tree sorted (inorder traversal) by key



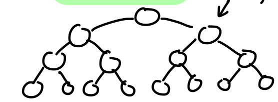
Search: Given a set of n **entries** each associated with **key** x_i and **value** v_i
- Store for quick access + updates
- **Ordered**: Assume that keys are totally ordered: <, >, ==

Find: How to find a key in the tree?
- Start at root p ← root
- if (x < p.key) search left
- if (x > p.key) search right
- if (x == p.key) found it!
- if (p == null) not there!

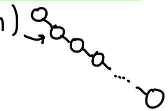
```
Value find(Key x, BSTNode p){
    if (p == null) return null
    else if (x < p.key)
        return find(x, p.left)
    else if (x > p.key)
        return find(x, p.right)
    else return p.value
}
```

Efficiency: Depends on tree's height

Balanced: O(log n)



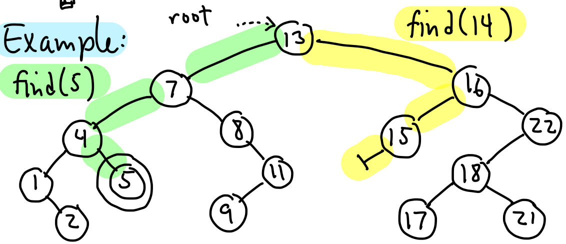
Unbalanced: O(n)



Example:

find(5)

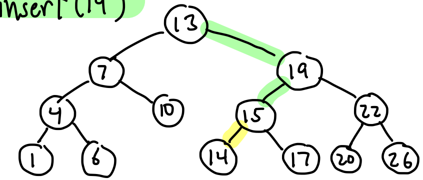
find(14)



Binary Search Trees II
- insertion
- deletion

Insert (Key x, Value v)
- find x in tree
- if found ⇒ error! duplicate key
- else: create new node where we "fell out"

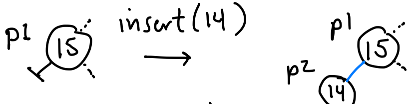
insert(14)




```
BSTNode insert(Key x, Value v, BSTNode p){
    if (p==null)
        p = new BSTNode(x,v)
    else if (x < p.key)
        p.left = insert(x,v, p.left)
    else if (x > p.key)
        p.right = insert(x,v, p.right)
    else throw exception → Duplicate!
    return p
}
```

Why did we do:
p.left = insert(x,v, p.left)?

Be sure you understand this!



p1.left = insert(14, v, p1.left)

p2 = new BSTNode
return p2

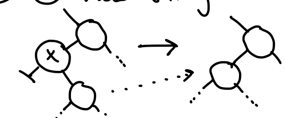
Delete (Key x)
- find x
- if not found → error
- else: remove this node + restore BST structure

How?

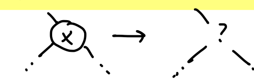
3 cases:

① ⓧ is a leaf

② ⓧ has single child

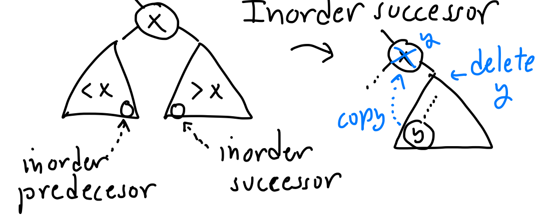


3. ⓧ has two children



Find replacement node ⓨ, copy to ⓧ, and then delete ⓨ

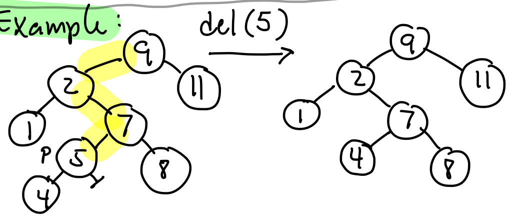
Replacement Node?



Binary Search Trees III
- deletion
- analysis
- Java

```
BSTNode delete(Key x, BSTNode p){
    if (p == null) ERROR! Key not found
    else
        if (x < p.key)
            p.left = delete(x, p.left)
        else if (x > p.key)
            p.right = delete(x, p.right)
        else if (either p.left or p.right null)
            if (p.left == null)
                return p.right
            if (p.right == null)
                return p.left
        else
            r = findReplacement(p)
            copy r's contents to p
            p.right = delete(r.key, p.right)
    return p
}
```

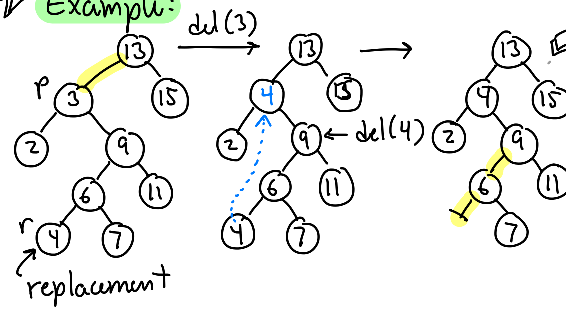
Example:



Find Replacement Node

```
BSTNode findReplacement(BSTNode p){
    BSTNode r = p.right
    while (r.left ≠ null)
        r = r.left
    return r
}
```

Example:



Java Implementation:
- Parameterize Key + Value types: extends Comparable
  class BinSearchTree<K,V>{...
- BSTNode - inner class
- Private data: BSTNode root
- insert, delete, find: local
- provide public fns insert, delete, find

But height can vary from $O(\log n)$ to $O(n)$...

Expected case is good

Thm: If n keys are inserted in random order, expected height is $O(\log n)$.

Analysis:

All operations (find, insert, delete) run in $O(h)$ time, where h = tree's height

Java implementation (see notes for details)

```
public class BSTree<Key extends Comparable, Value> {
    class Node {
        Key key
        Value value
        Node left, right

        .... constructor, toString...
    }

    Value find(Key x, Node p) {...}
    Node insert(Key x, Value v, Node p) {...}
    Node delete (Key x, Node p) {...}

    private Node root;

    public Value find(Key x) {...}
    public void insert(Key x, Value v) {...}
    public void delete (Key x) {...}
}
```

Inner class for node (protected)

Local helpers (private or protected)

Data (private)

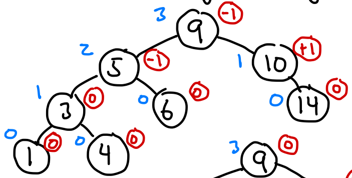
Public members (invoke helpers)

AVL Trees I
- Basic defs
- Height props
- Rotations

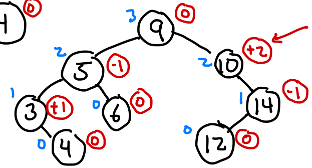
AVL Height Balance
- for each node v, the heights of its subtrees differ by ≤ 1.

AVL tree: A binary search tree that satisfies this condition

Balance factor:
bal(v) = hgt(v.right) − hgt(v.left)



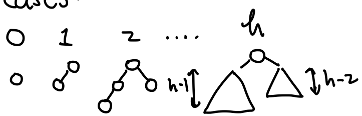
← This is an AVL tree



Not an AVL tree

Does this imply O(log n) height?
Worst cases:

height: $h =$ 0 1 2 .... h



nodes: $n =$ 1 2 4 7 12 20 ...

$n+1 =$ 2 3 5 8 13 21 ...

Recall: $F_0 = 0$, $F_1 = 1$, $F_h = F_{h-1} + F_{h-2}$

Conjecture: Min no. of nodes in AVL tree of height h is $F_{h+3} - 1$

Theorem: An AVL tree of height h has at least $F_{h+3} - 1$ nodes.

Proof: (Induct. on h)

$h = 0$: $n(h) = 1 = F_3 - 1$

$h = 1$: $n(h) = 2 = F_4 - 1$

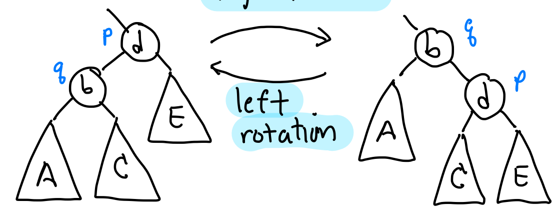
$h \geq 2$: $n(h) = 1 + n(h-1) + n(h-2)$

$= 1 + (F_{h+2} - 1) + (F_{h+1} - 1)$

$= (F_{h+2} + F_{h+1}) - 1 = F_{h+3} - 1$ □


```
BSTNode rotateRight(BSTNode p) {
    BSTNode q = p.left
    p.left = q.right
    q.right = p
    return q
}
```

How to maintain the AVL property?



Corollary: An AVL tree with n nodes has height O(log n)

Proof: Fact: $F_h \approx \varphi^h / \sqrt{5}$ where $\varphi = (1+\sqrt{5})/2$ "Golden ratio"

$n \geq \varphi^{h+3} = c \cdot \varphi^h \Rightarrow h \leq \log_\varphi n + c'$

$\Rightarrow h \leq \log_2 n / \log_2 \varphi$

$= O(\log n)$ □

AVL Trees II
- double rotations
- insertion

AVL Tree:

AVL Node: Same as BSTNode (from Lect 4) but add: int height

simpler than bal factor

Utilities:

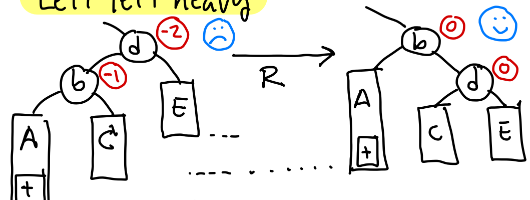
```
int height(AVLNode p)
  return { p == null  → -1
         { o.w.       → p.height

void updateheight(AVLNode p)
  p.height = 1 + max(height(p.left),
                     height(p.right))

int balanceFactor(AVLNode p)
  return height(p.right) -
         height(p.left)
```

Find: Same as B.S.T.

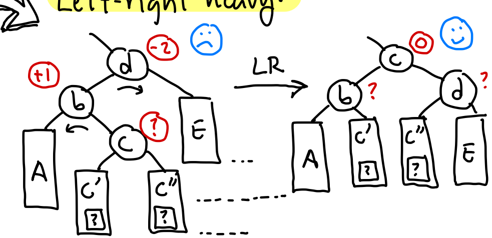
Insert: Same as BST but as we "back out" rebalance

How to rebalance? Bal = -2

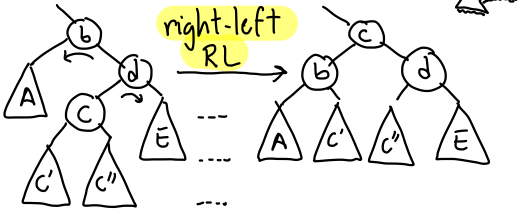
Left-left heavy

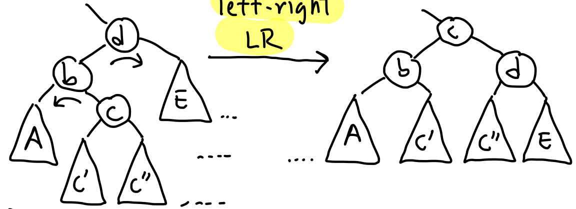


Left-right heavy:



Double rotations:






```
BSTNode rotateLeftRight(BSTNode p)
  p.left = rotateLeft(p.left)
  return rotateRight(p)
```

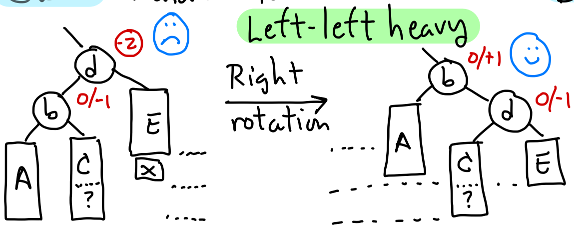
```
AVLNode rebalance(AVLNode p)
  if (p==null) return p
  if (balanceFactor(p) < -1)
    if (ht(p.left.left) ≥ ht(p.left.right))
      p = rotateRight(p)
    else p = rotateLeftRight(p)
  else if (balanceFact(p) > +1)
    ...(symmetrical)
  updateHeight(p); return p
```

```
AVLNode insert(Key x, Value v, AVLNode p) {
  if (p == null) p = new AVLNode(x, v)
  else if (x < p.key)
    p.left = insert(x, v, p.left)
  else if (x > p.key)
    p.right = insert(x, v, p.right)
  else throw -Error- Duplicate!
  return rebalance(p)
```

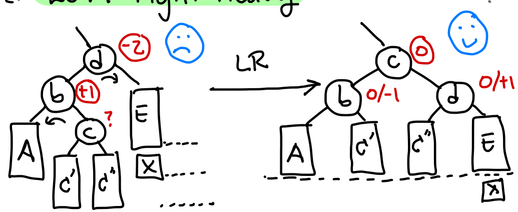
AVL Trees III
- Deletion
- Examples

Cases: Balance factor -2

Left-left heavy

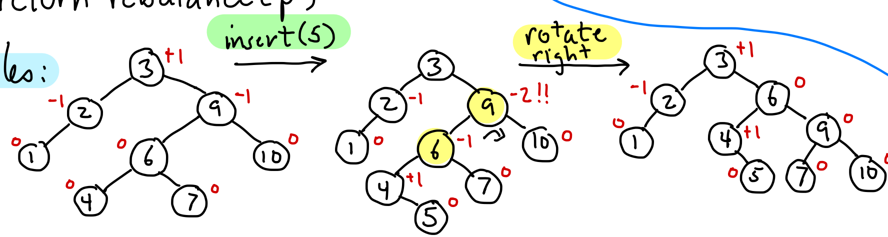


Left-right heavy



AVLNode delete (Key x, AVLNode p)
  same as BST delete
  return rebalance(p)

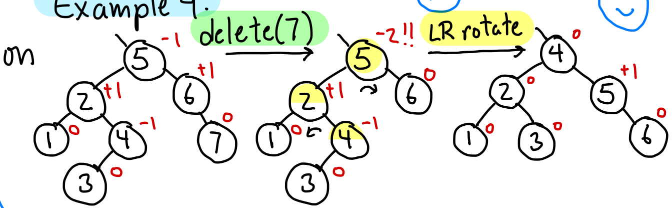
Examples:



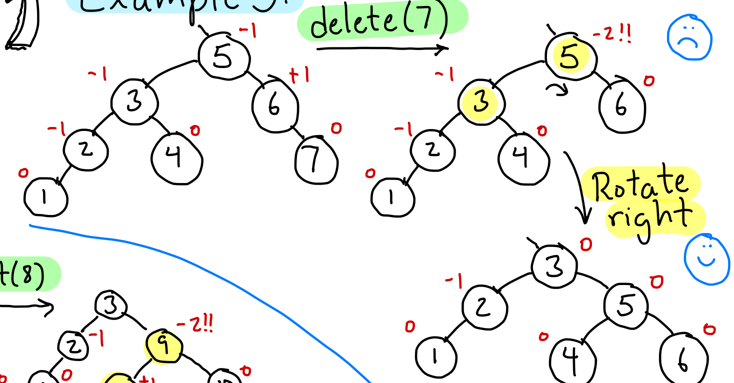
Deletion: Basic plan

- Apply standard BST deletion
  - find key to delete
  - find replacement node
  - copy contents
  - delete replacement
- rebalance

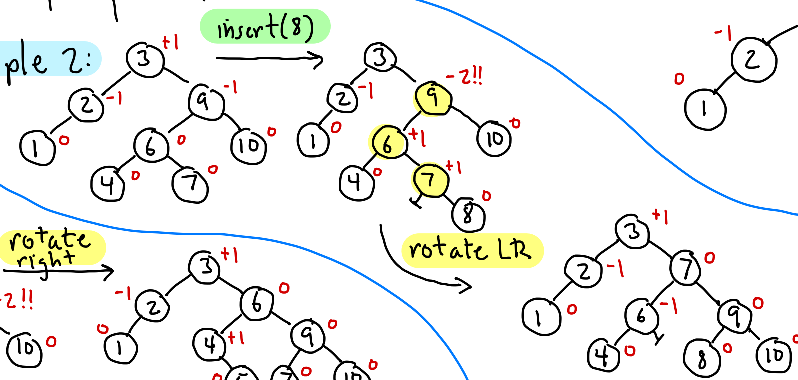
Example 4:



Example 3:



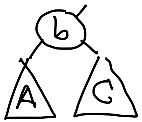
Example 2:



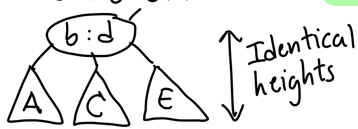
2-3 Trees I

Node types:

2-Node
1 key
2 children



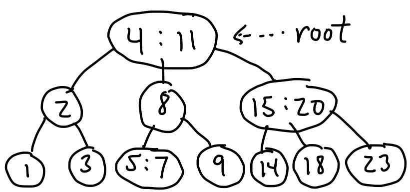
3-Node
2 keys
3 children



Def: A **2-3 tree** of height h is either:

- Empty ($h = -1$)
- A 2-Node root and two subtrees, each 2-3 tree of height $h-1$
- A 3-Node root and three subtrees... height $h-1$.

Example: 2-3 tree of height 2



Recap:

AVL: Height balanced Binary

2-3 tree: Height exact Variable width

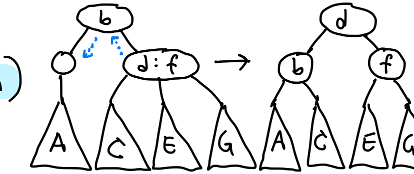
Thm: A 2-3 tree of n nodes has height $O(\log n)$

Roughly: $\log_3 n \leq h \leq \log_2 n$

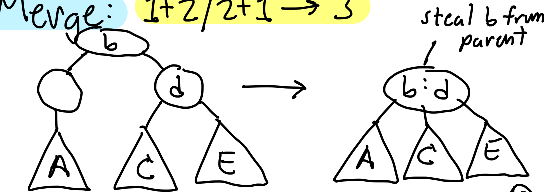
How to maintain balance?

- Split
- Merge
- Adoption (Key rotation)

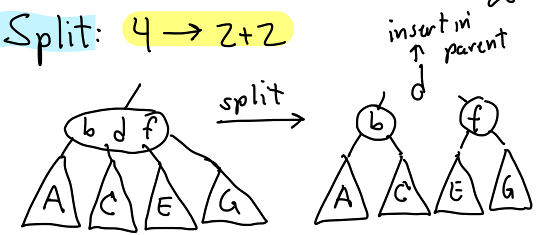
Adoption (Key-Rotation) $1+3 = 2+2$



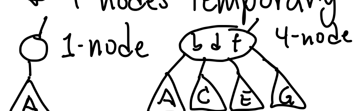
Merge: $1+2 / 2+1 \to 3$



Split: $4 \to 2+2$

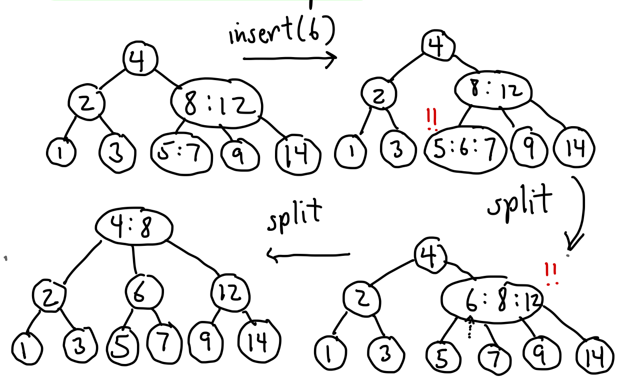


Conceptual tool: We'll allow 1-nodes + 4-nodes temporary



2-3 Trees II

Insertion example:



Dictionary operations:

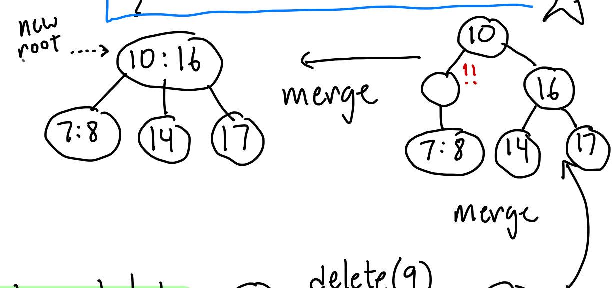
Find - straightforward

Insert - find leaf node where key "belongs" + add it (may split)

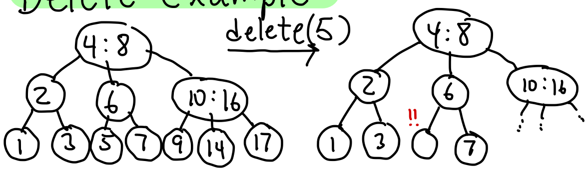
Delete - find/replacement/ merge or adopt

Implementation?

```
class TwoThreeNode {
    int nChildren
    TwoThreeNode children[3]
    Key key[2]
}
```



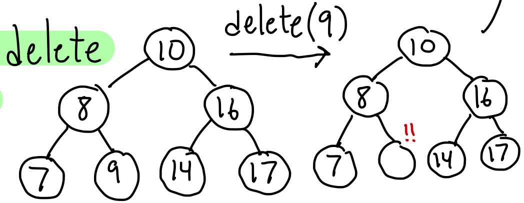
Delete Example:



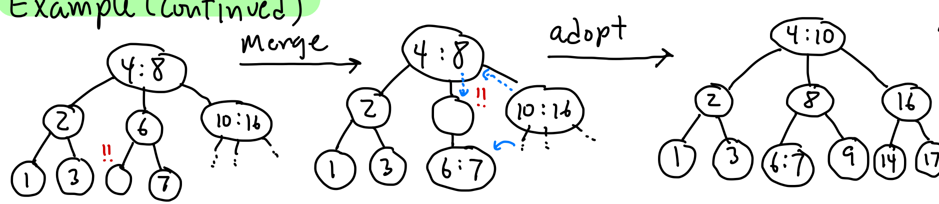
Deletion remedy:

- Have a 3-node neighboring sibling → adopt
- O.w.: Merge with either sibling + steal key from parent

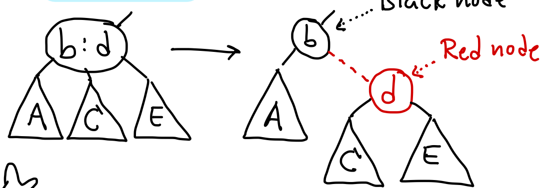
Another delete example:



Example (continued)



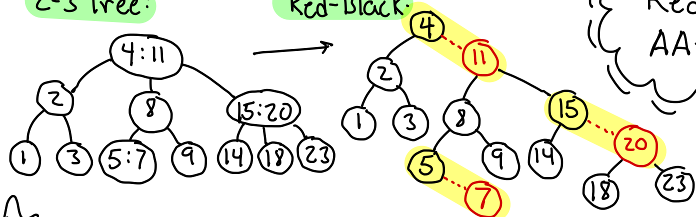
Encoding 3-node as binary tree node



Example:

2-3 Tree:

Red-Black:



Rules:

1. Every node labeled red/black
2. Root is black
3. Nulls treated as if black
4. If node is red, both children are black
5. Every path from root to null has same no. of black

Some history:

2-3 Trees: Bayer 1972

Red-black Trees: Guibas & Sedgewick 1978 (a binary variant of 2-3)

Rumor - Guibas had two pens - red & black to draw with

Red-Black and AA-Trees I

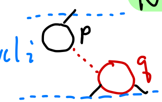
Lemma: A red-black tree with n keys has height $O(\log n)$

Proof: It's at most twice that of a 2-3 tree.

Q: Is every Red-Black Tree the encoding of some 2-3 tree?

AA-Trees: Simpler to code

- **No null pointers**: Create a sentinel node, nil, and all nulls point to it → nil
- **No colors**: Each node stores level number. Red child is at same level as parent.



q is red ⟺ q.level == p.level

What we need are stricter rules!

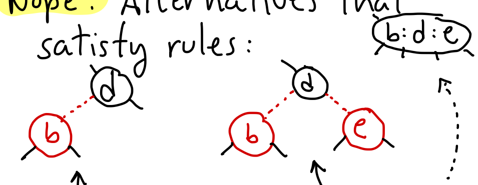
AA-tree:

Arne Anderson 1993

New rule:

6. Each red node can arise only as right child (of a black node)

Nope! Alternatives that satisfy rules:



A "left-skewed" encoding

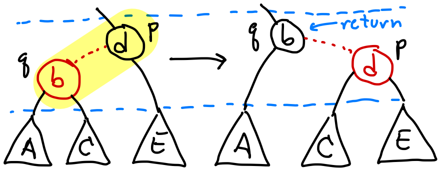
Corresponds to 2-3-4 trees

Red-Black + AA Trees II

Restructuring Ops:

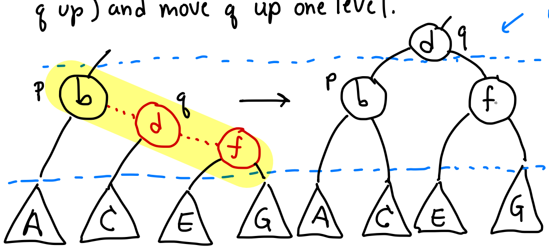
Skew: Restore right skew

→ If black node has red left child, rotate



How to test? p.left.level == p.level

Split: If a black node has a right-right red chain, do a left rotation at p (bringing its right child q up) and move q up one level.



what 2-3 op does this remind you of?

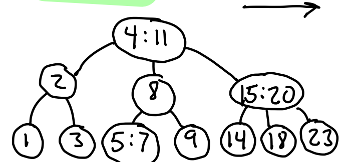
How to test?

p.level == p.right.level == p.right.right.level

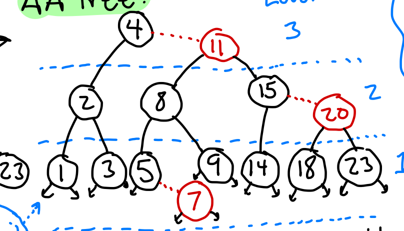
not needed (levels are monotone)

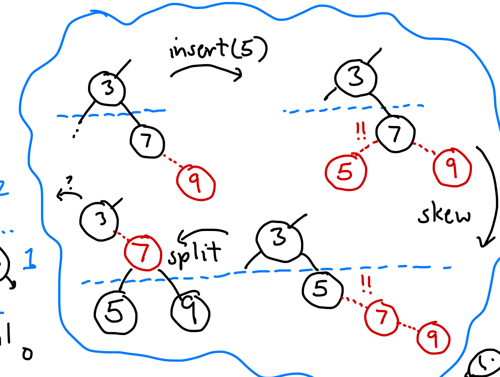
Example:

2-3 Tree:



→ AA tree:





AA Insertion:

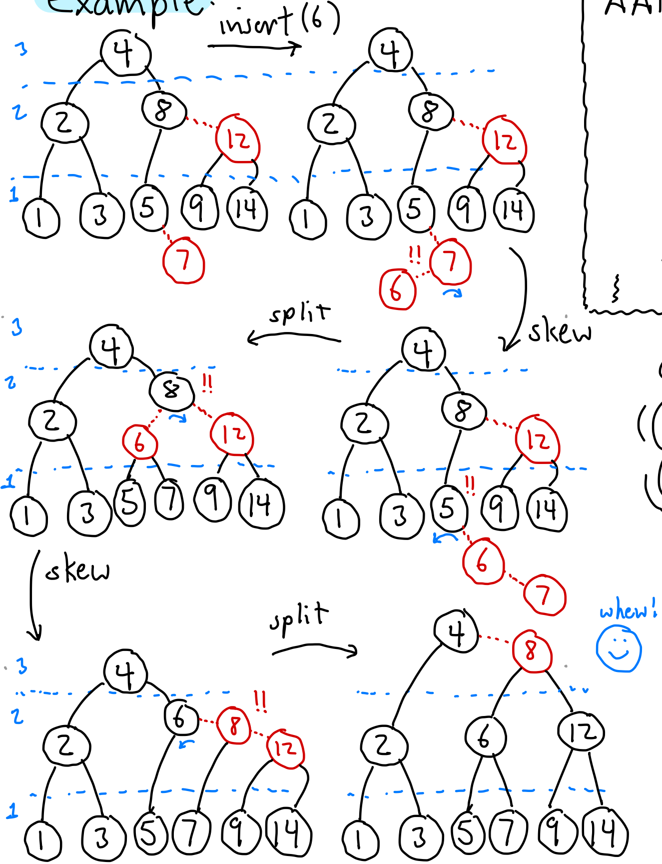
- Find the leaf (as usual)
- Create new red node
- Back out applying skew + split

```
AANode skew(AANode p) {
    if (p == nil) return p
    if (p.left.level == p.level) {          ← right rotate p
        AANode q = p.left
        p.left = q.right; q.right = p
        return q        ← new subtree root
    } else return p     ← everything's fine
}
```

```
AANode split(AANode p) {
    if (p == nil) return p
    if (p.right.right.level == p.level) {
        AANode q = p.right
        p.right = q.left                  ← left rotation at p
        q.left = p
        q.level += 1     ← move q up a level
        return q
    } else return p      ← all okay
}
```

Red-Black and AA Trees III

Example:




```
AANode insert(Key x, Value v, AANode p) {
    if (p == nil)
        p = new AANode(x, v, 1, nil, nil)
    else if (x < p.key) ... insert on left
    else if (x > p.key) ... insert on right
    else Duplicate Key!
    return split(skew(p))
}
```

leaf level

left, right

Deletion:

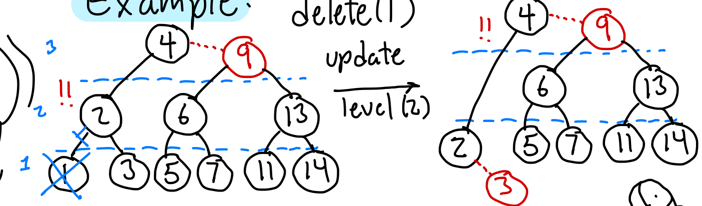
Two more helpers:

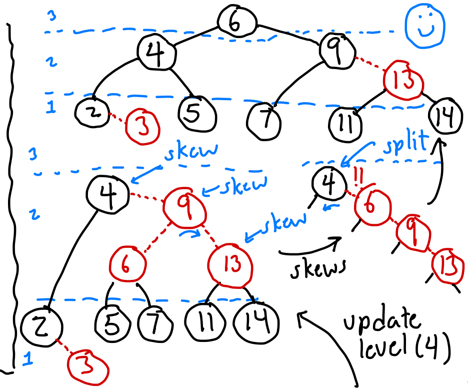
updateLevel: If p's level exceeds $\ell = 1 + \min$ (p.left.level, p.right.level) then set p's level to ℓ & also p's right child

fixAfterDelete(p):
- update p's level
- skew(p), skew(p.right), skew(p.right.right)
- split(p), split(p.right)

deletion: Same as AVL deletion, but end with: **return fixAfterDelete(p)**

Example: delete(1)





Randomized Data Structures

- Use a random number generator
- Running in expectation over all random choices
- Often simpler than deterministic

Treaps I

History:

1989: Seidel + Aragon
[Explosion of randomized algorithms]

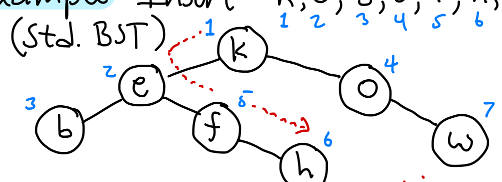
Later discovered this was already known: Priority Search Trees from different context (geometry)
McCreight 1980

Intuition:

- Random insertion into BSTs $\Rightarrow O(\log n)$ expected height
- Worst case can be very bad $O(n)$ height
- Treap: A tree that behaves as if keys are inserted in random order

Example: Insert: k, e, b, o, f, h, w
1 2 3 4 5 6 7

(Std. BST)



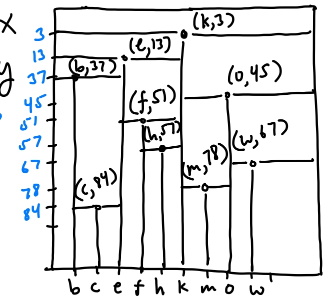
Along any path - Insertion times increase

Obs: In a standard BST, keys are by inorder + insert times are in heap order (parent < child)

Geometric Interpretation:

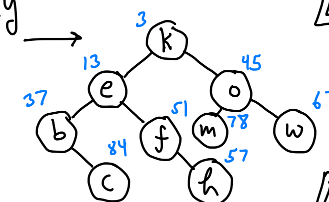
key → x
priority → y

McCreight's Priority Search Tree



Example:

Key	Priority
b	37
c	84
e	13
f	51
h	57
k	3
m	78
o	45
w	67



Treap: Each node stores a key + a random priority.

Keys are in inorder.
Priorities are in heap order

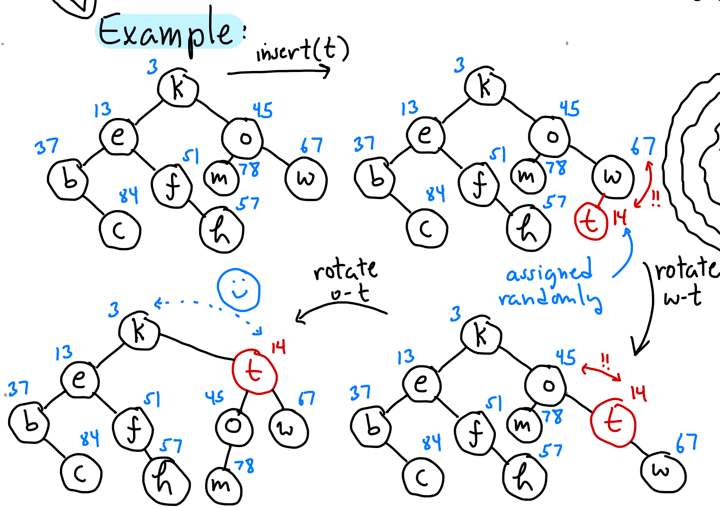
? Is it always possible to do both?

Yes: Just consider the corresponding BST

Treaps II

Insertion: As usual, find the leaf + create a new leaf node.
- Assign random priority
- On backing out - check heap order + rotate to fix.

Example:

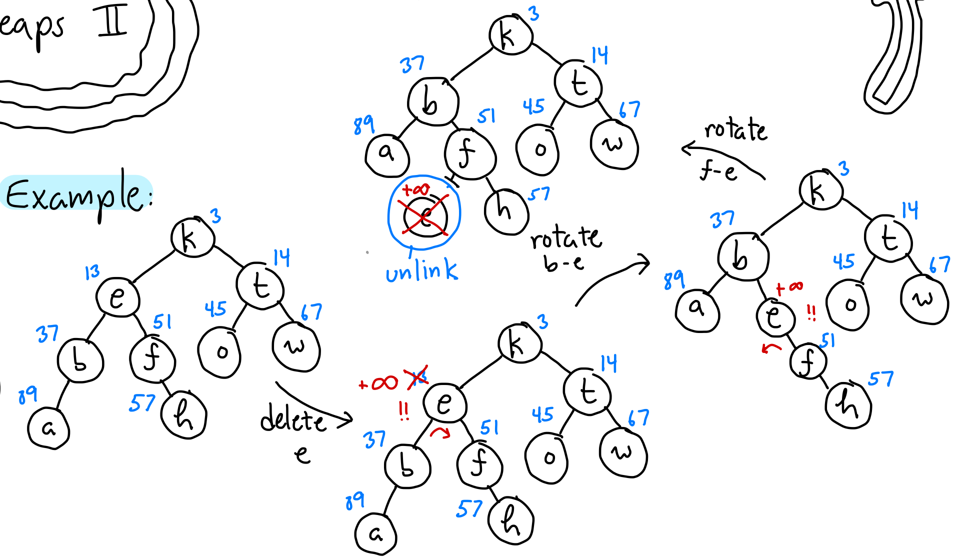


Theorem: A treap containing n entries has height $O(\log n)$ in expectation (averaged over all assignments of random priorities)

Proof: Follows directly from BST analysis

Deletion: (Cute solution) Find node to delete. Set its priority to $+\infty$. Rotate it down to leaf level + unlink.

Example:

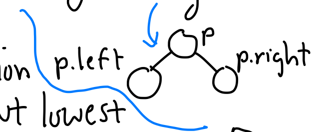


Implementation: (See pdf notes)

Node: Stores priority + usual...

Helpers:

lowest priority(p) returns node of lowest priority among:



restructure: performs rotation (if needed) to put lowest priority node at p.

Skip Lists I

Ideal Skip List:

- Organize list in levels
- Level 0: Everything
  - 1: Every other
  - 2: Every fourth
  - ⋮
  - i: Every 2^i

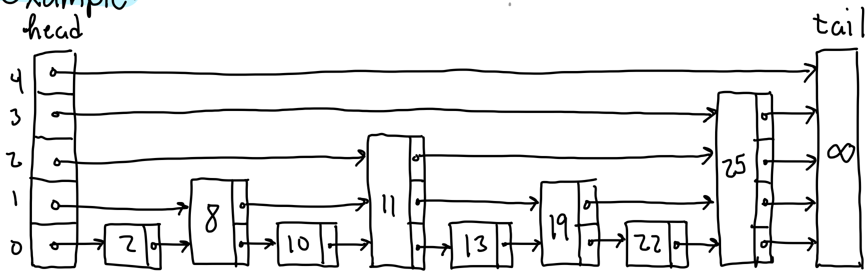
Sorted linked lists:

- Easy to code
- Easy to insert/delete
- Slow to search… $O(n)$

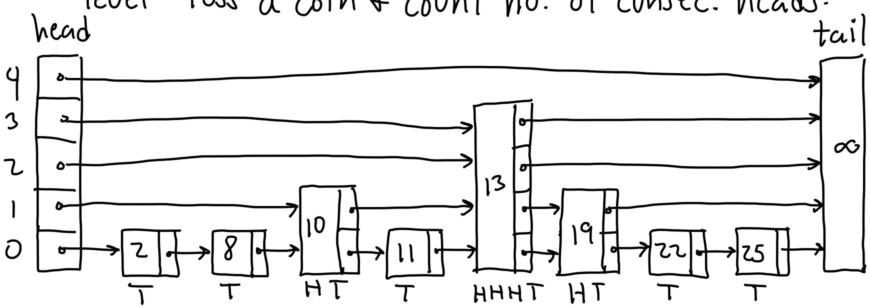
Idea: Add extra links to skip

How to generalize?

Example:



Too rigid → **Randomize!** To determine level – toss a coin + count no. of consec. heads:



Node Structure: (Variable sized)

```
class SkipNode {
    Key key
    Value value
    SkipNode[] next
}
```

In constructor, set level and size

```
Value find(Key x) {
    i = topmost Level
    SkipNode p = head
    while (i ≥ 0) {
        if (p.next[i].key ≤ x) p = p.next[i]
        else  i--
    }
    if (p.key == x) return p.value
        else return null
}
```

current node

until we hit base level

advance horizontal

drop down a level

we are at base level

Skip Lists II

Thm: A skip list with n nodes has $O(\log n)$ levels in expectation

Proof: Will show that probability of exceeding $c \cdot \lg n$ is $\leq 1/n^{c-1}$.

→ Prob that any given node's level exceeds ℓ is $1/2^{\ell}$ [ℓ consecutive heads]

→ Prob that any of n node's level exceeds ℓ is $\leq n/2^{\ell}$ [n trials with prob $1/2^{\ell}$]

→ Let $\ell = c \cdot \lg n$ $(\lg \equiv \log_2)$

Prob that max level exceeds $c \cdot \lg n$ is:

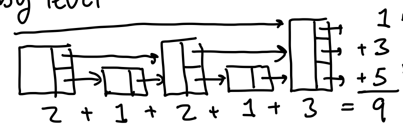
$$\leq n/2^{\ell} = n/2^{(c \cdot \lg n)} = n/(2^{\lg n})^c = n/n^c = 1/n^{c-1}$$

$\square$

Obs: Prob. level exceeds $3 \cdot \lg n$ is $\leq 1/n^2$.
(If $n \geq 1{,}000$, chances are less than 1 in million!)

Thm: Total space for n-node skip list is $O(n)$ expected.

Proof: Rather than count node by node, we count level by level:



- Let n_i = no. of nodes that contrib. to level i.
- Prob that node at level $\geq i$ is $1/2^i$
- Expected no. of nodes that contrib. to level i = $n/2^i$

$\Rightarrow E(n_i) = n/2^i$

Total space (expected) is:

$$E\left(\sum_{i=0}^{\infty} n_i\right) = \sum_{i=0}^{\infty} E(n_i) = \sum_{i=0}^{\infty} n/2^i = n \sum_{i=0}^{\infty} 1/2^i = 2n$$

$\square$

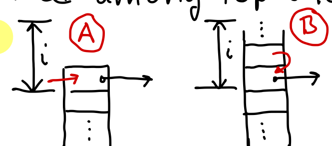
Thm: Expected search time is $O(\log n)$

Proof:
- We have seen no. levels is $O(\log n)$
- Will show that we visit 2 nodes per level on average

Obs – Whenever search arrives first time to a node, it's at top level. (Can you see why?)

Def: $E(i)$ = Expect. num. nodes visited among top i levels.

Cases:



$$E(i) = 1 + (\text{Prob}(A))\,E(i) + (\text{Prob}(B))\,E(i-1)$$

current node — same level — from prior level

$$= 1 + \tfrac{1}{2}E(i) + \tfrac{1}{2}E(i-1)$$

$$\Rightarrow E(i)(1 - \tfrac{1}{2}) = 1 + \tfrac{1}{2}E(i-1)$$

$$\Rightarrow E(i) = [1 + \tfrac{1}{2}E(i-1)]2 = 2 + E(i-1)$$

Basis: $E(0) = 0 \Rightarrow E(i) = 2 \cdot i$

Let ℓ = max level. **Total visited** $= E(\ell) = 2 \cdot \ell$

$\Rightarrow$ We visit 2 nodes per level on average.

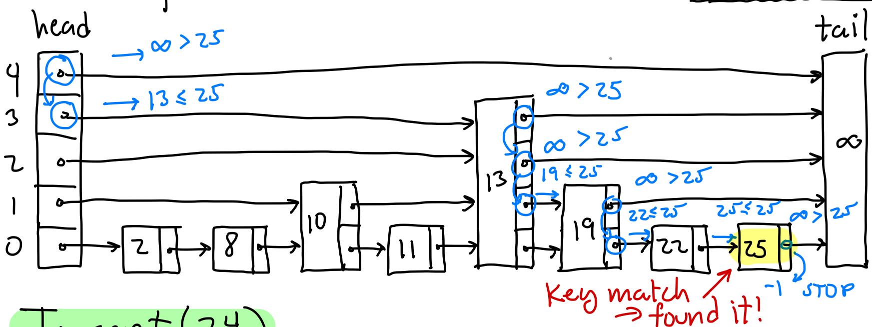
$\square$

Skip Lists III

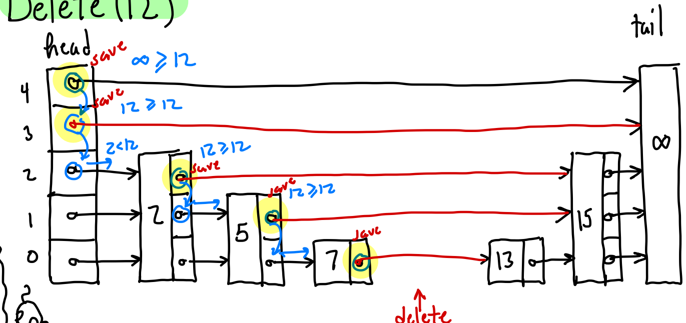
Delete:
- Start at top
- Search each level saving last node < key
- On reaching node at level 0, remove it and unlink from saved pointers

Insert: (Similar to linked lists)
- Start at top level
- At each level:
  - Advance to last node ≤ key
  - Save node + drop level
- At level 0:
  - Create new node (flip coins to determine height)
  - Link into each saved node

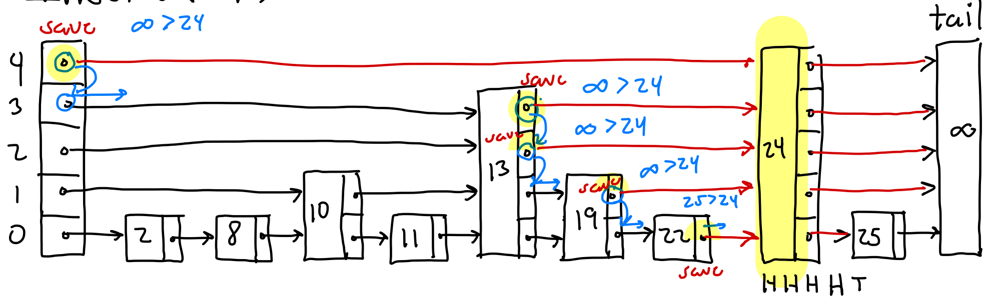
Example: find(25)



Delete(12)



Insert(24)



Analysis: All operations run in time ~ find ⇒ $O(\log n)$ expected

Note: Variation in running times due to randomness only – not sequence
⇒ User cannot force poor performance.

SPLAY TREES I

Recap: Lots of search trees
- Unbalanced BSTs
- AVL Trees
- 2-3, Red-black, AA Trees
- Treaps + Skip lists

→ **Focus:** Worst-case or randomized expected case

Other/Better Criteria?

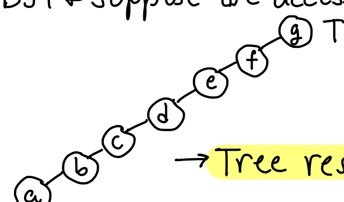
Expected case: Some keys more popular than others

Self-adjusting: Tree adapts as popularity changes

How to design/analyze?

Splay Tree: A self-adjusting binary search tree
- **No rules!** (yay anarchy!)
  - No balance factors
  - No limits on tree height
  - No colors/levels/priorities
- **Amortized efficiency:**
  - Any single op - slow
  - Long series - efficient on avg.

Intuition: Let T be an unbalanced BST + suppose we access its deepest key

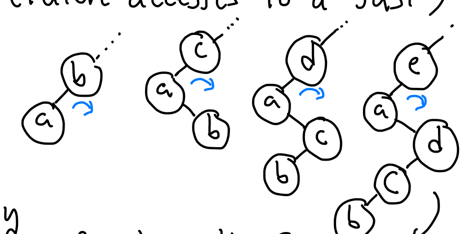


find("a")

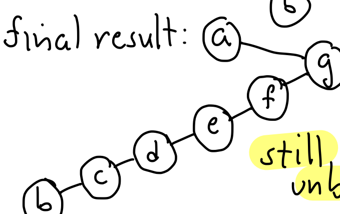
ugh!

→ **Tree restructures itself**

Idea I: Rotate "a" to top (Future accesses to "a" fast)

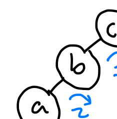


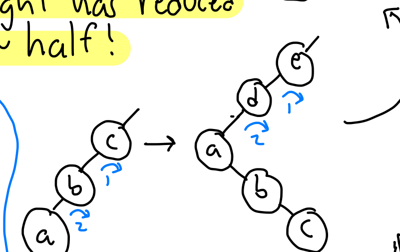
.... final result:



still unbalanced!

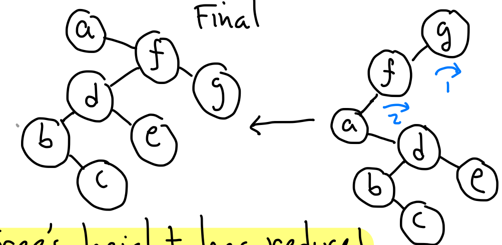
Idea II: Rotate 2 at a time - upper + lower





Tree's height has reduced by ~ half!

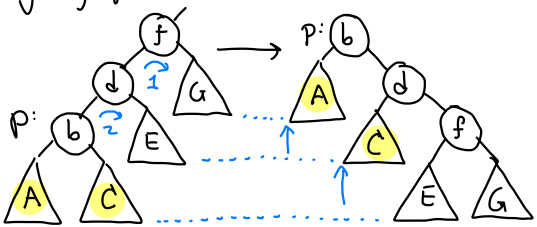
Final



Lesson: Different combinations of rotations can:
- bring given node to root
- significantly change (improve?) tree structure.

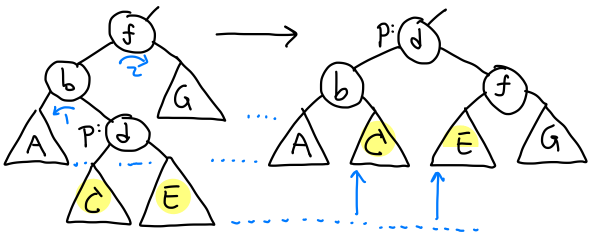
Splay Trees II

ZigZig(p): [LL case]



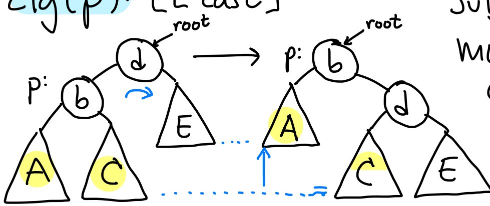
Subtrees A,C move up↑

ZigZag(p): [LR case]



Subtrees C, E of p move up↑

Zig(p): [L case]



Subtree A moves up↑
C unchanged

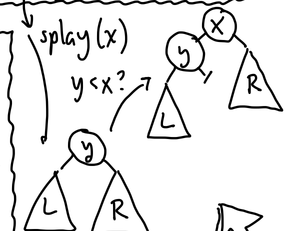
Splay(Key x):

```
Node p ← find x by standard BST search
while (p ≠ root) {
   if (p == child of root) zig(p)
   else /* p has grand parent */
      if (p is LL or RR grand child) zigzig(p)
      else /* p is LR or RL gr. child */ zigzag(p)
```

insert(x):

root.key ≠ x or error!

```
{ splay(x)
  q = new Node(x)
  if (root.key < x)
     x.left = root
     x.right = root.right
     root.right = null
  else ... symmetrical...
```

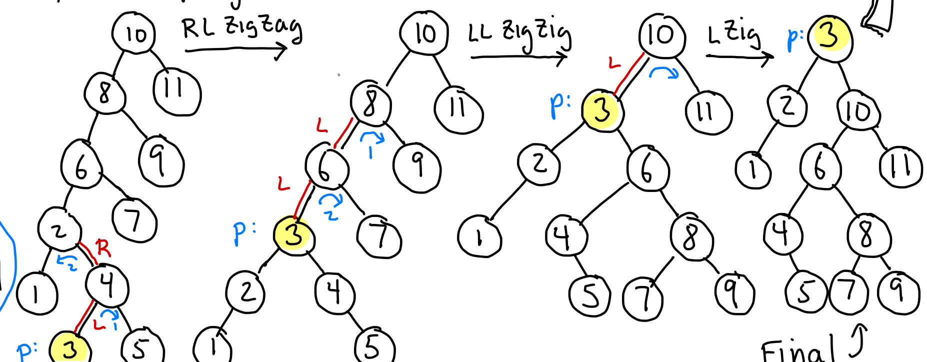


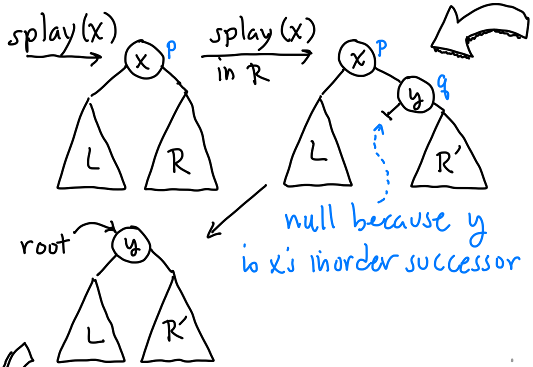
find(x):

```
splay(x)
if (root.key == x)
   found!
else not found
```

Example:

splay(3)





SPLAY TREES III

delete(x):

splay(x) [x now at root]
p = root
if (p.key ≠ x) error!
splay(x) in p's right subtree
q = p.right [q's key is x's successor]
q.left = p.left [q.left == null]
root = q

Analysis:

- Amortized analysis
- Any one op might take $O(n)$
- Over a long sequence, average time is $O(\log n)$ each
- Amortized analysis is based on a sophisticated **potential argument**
- Potential: A function of the tree's structure
  - Balanced ⇒ Low potential
  - Unbalanced ⇒ High potential
- Every operation tends to reduce the potential

Splay Trees are Amazingly Adaptive!

Balance Theorem: Starting with an empty dictionary, any sequence of m accesses takes total time

$$O(m \log n + n \log n)$$

where n = max. entries at any time.

Dynamic Finger Theorem:

Keys: $x_1 < \dots < x_n$. We perform accesses $x_{i_1}, x_{i_2}, \dots, x_{i_m}$

Let $\Delta_j = i_j - i_{j-1}$: distance between consecutive items

Thm: Total access time is

$$O\left(m + n \log n + \sum_{j=1}^{m} (1 + \lg \Delta_j)\right)$$

Static Optimality:

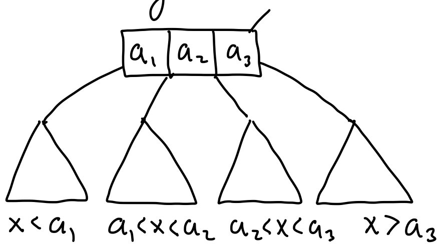
- Suppose key x_i is accessed with prob p_i ($\sum_{i=1}^{n} p_i = 1$)
- **Information Theory:** Best possible binary search tree answers queries in expected time $O(H)$ where $H = \sum_i p_i \lg 1/p_i$ ← Entropy

Static Optimality Theorem: Given a seq. of m ops. on splay tree with keys $x_1 \dots x_n$, where x_i is accessed q_i times. Let $p_i = q_i/m$. Then total time is

$$O\left(m \sum_i p_i \lg 1/p_i\right)$$

B-Trees I

Multiway Search Trees:



Secondary Memory:

- Most large data structures reside on disk storage
- Organized in blocks - pages
- Latency: High start-up time
- Want to minimize no. of blocks accessed

Node Structure:

constant int M=...

```
class BTreeNode {
    int       nChild       // no. of children
    BTreeNode child[M]     // children
    Key       key[M-1]     // keys
    Value     value[M-1]   // values
}
```

B-Tree:

- Perhaps the most widely used search tree
- 1970 - Bayer + McCreight
- Databases
- Numerous variants

Theorem:

A B-tree of order m with n keys has height at most $(\lg n)/\gamma$, where $\gamma = \lg(m/2)$

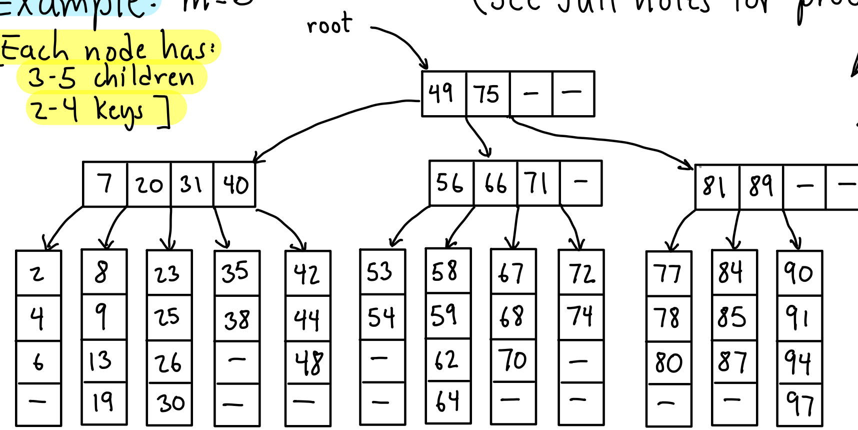
(See full notes for proof)

B-Tree: of order m (≥ 3)

- Root is leaf or has ≥ 2 children
- Non-root nodes have $\lceil m/2 \rceil$ to m children [null for leaves]
- k children $\Rightarrow$ $k-1$ key-values
- All leaves at same level

Example: $m=5$

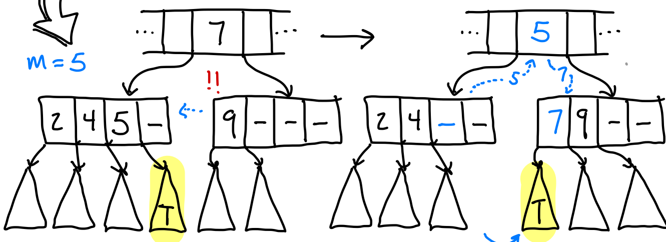
[Each node has: 3-5 children, 2-4 keys]



B-Trees II

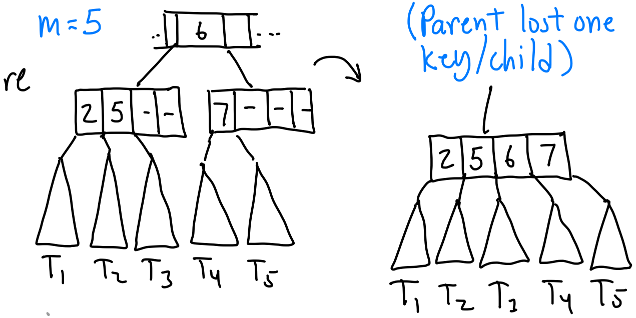
Key Rotation (Adoption)

- A node has too few children $\lceil m/2 \rceil - 1$
- Does either immediate sibling have extra? $\geq \lceil m/2 \rceil + 1$
- Adopt child from sibling & rotate keys
- When applicable - preferred



B-Tree restructuring:

- Generalizes 2-3 restructure
- Key rotation (Adoption)
- Splitting (insertion)
- Merging (deletion)



Lemma: For all $m \geq 2$,

$$\lceil m/2 \rceil \leq 2\lceil m/2 \rceil - 1 \leq m$$

$\Rightarrow$ Resulting node is valid

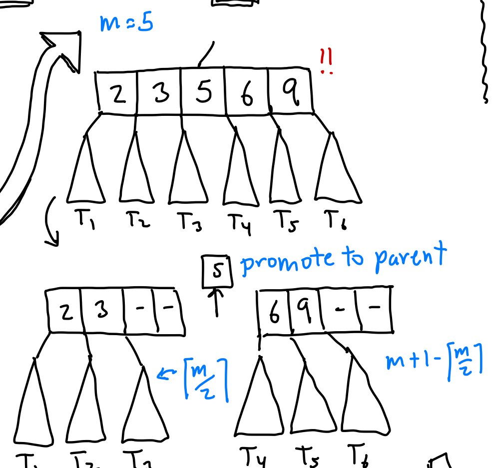
Node Splitting:

- After insertion, a node has too many children... $m+1$
- We split into two nodes of sizes $m' = \lceil m/2 \rceil$ and $m'' = m+1-\lceil m/2 \rceil$

Lemma: For all $m \geq 2$,

$$\lceil m/2 \rceil \leq m+1-\lceil m/2 \rceil \leq m$$

$\Rightarrow$ $m' + m''$ are valid node sizes



Node Merging:

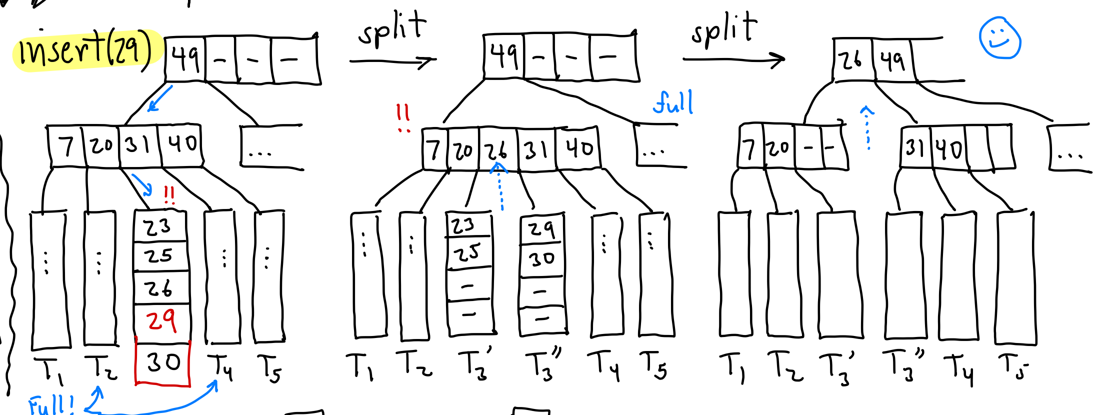
- A node has too few children $\lceil m/2 \rceil - 1$
- Neither sibling has extra (both $\lceil m/2 \rceil$)
- Merge with either sibling to produce node with $(\lceil m/2 \rceil - 1) + \lceil m/2 \rceil$ child

B-Trees III

Insertion:

- Find insertion point (leaf level)
- Add key/value here
- If node overfull (m keys, m+1 children)
  - → Can either sibling take a child (< m)?
    - ⇒ Key rotation [done]
  - → Else, split
    - → Promotes key
    - → If root splits add new root

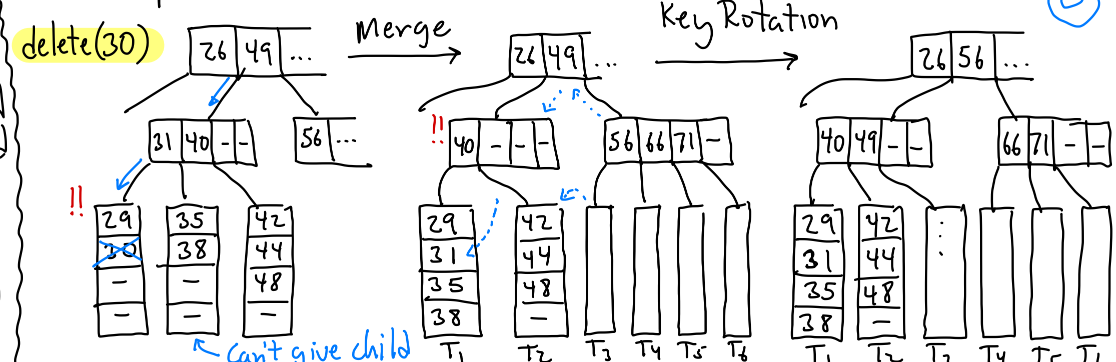
Example: m=5



Deletion:

- Find key to delete
- Find replacement / copy
- If underfull ($\lceil m/2 \rceil - 1$) child
  - → If sibling can give child
    - → Key rotation
  - → Else (sibling has $\lceil m/2 \rceil$)
    - → Merge with sibling
- → Propagates → If root has 1 child → collapse root

Example: m=5



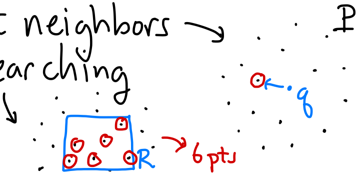
Quadtrees & kd-Trees I

So far: 1-dimensional keys

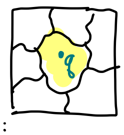
- Multi-dimensional data
- Applications:
  - Spatial databases + maps
  - Robotics + Auton. Systems
  - Vision / Graphics / Games
  - Machine Learning
- ...

Geometric Search:

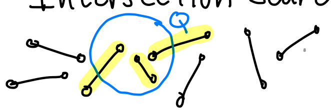
- Nearest neighbors
- Range searching



- Point Location



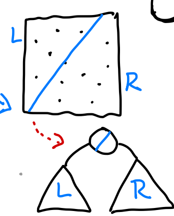
- Intersection Search



Multi-Dim vs. 1-dim Search?

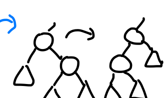
Similarities:

- Tree structure
- Balance $O(\log n)$
- Internal nodes - split
- External nodes - data



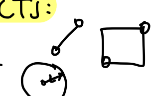
Differences:

- No (natural) total order
- Need other ways to discriminate + separate
- Tree rotation may not be meaningful



Representations:

- **Scalars:** Real numbers for coordinates, etc. float
- **Points:** $p = (p_1, \dots, p_d)$ in real d-dim space $\mathbb{R}^d$
- **Other geom objects:** Built from these

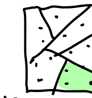


Point: A d-vector in $\mathbb{R}^d$

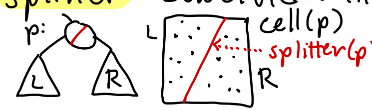
$p = (p_1, \dots, p_d) \quad p_i \in \mathbb{R}$

```
class Point {
  float[] coord          // coords
  Point(int d)
      ....> coord = new float[d]
  int getDim() --> coord.length
  float get(int i) --> coord[i]
  .... others: equality, distance
       toString ...
}
```

Partition Trees:



- Tree structure based on hierarchical space partition
- Each node is associated w. a region — **cell**
- Each internal node stores a **splitter** — subdivides the cell



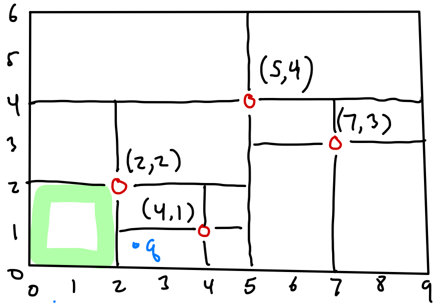
- External nodes store pts.

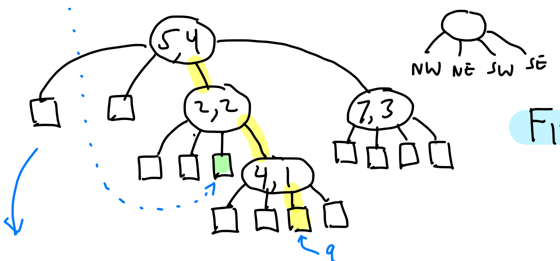
Quadtrees & kd-Trees II

Point Quadtree:

- Each internal node stores a point
- Cell is split by horiz. + vertic. lines through point

(5,4)
(2,2)
(7,3)
(4,1)

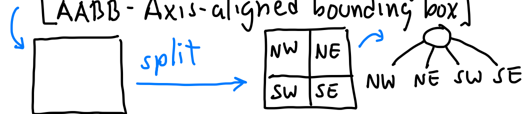




Each external node corresps. to cell of final subdivision

Quadtrees: (abstractly)

- Partition trees
- **Cell**: Axis-parallel rectangle [AABB - "Axis-aligned bounding box]



- **Splitter**: Subdivides cell into four (genlly 2^d) subcells

Find/Pt Location:

Given a query point q, is it in tree, and if not which leaf cell contains it?

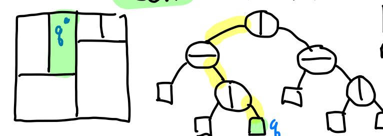
→ Follow path from root down (generalizing BST find)

History: Bentley 1975

- called it 2-d tree ($\mathbb{R}^2$)
  3-d tree ($\mathbb{R}^3$)
  ⋮
- In short **kd-tree** (any dim)
- Where/which direction to split? → next

kd-Tree: Binary variant of quadtree

- **splitter**: Horiz. or vertic. line in 2-d (orthogonal plane o.w.)
- **cell**: Still AABB



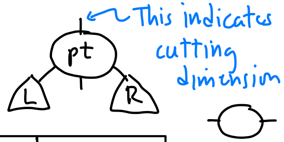
left: left/below
right: right/above

Quadtrees - Analysis

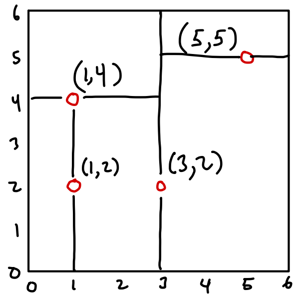
- Numerous variants! PR, PMR, QR, QX, ... see Samet's book
- Popular in 2-d apps (in 3-d, octtrees)
- Don't scale to high dim
  - out degree = 2^d
- What to do for higher dims?

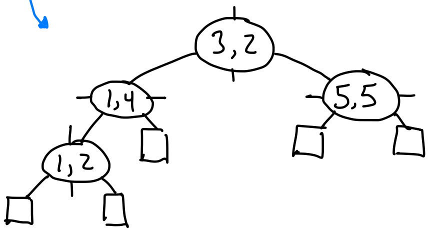
Quadtrees & kd-Trees III

Example:



(3,2)
(1,4)
(5,5)
(1,2)



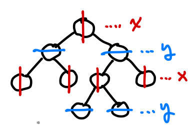


Kd-Tree Node:

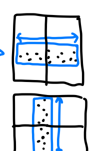
```
class KDNode {
    Point pt       // splitting point
    int cutDim     // cutting coordinate
    KDNode left    // low side
    KDNode right   // high side
}
```

How do we choose cutting dim?

- **Standard kd-tree**: cycle through them (eg. d=3: 1,2,3,1,2,3...) based on tree depth



- **Optimized kd-tree**: (Bentley)
  - Based on widest dimension of pts in cell.



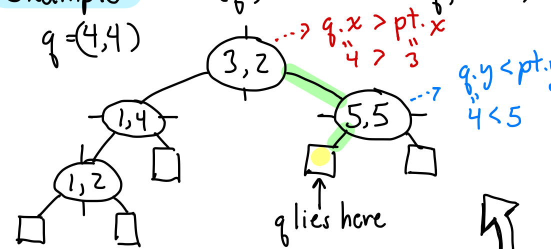
Find:

- Descend the tree
- Compare query pt with node pt along cutDim

```
class KDNode {
    :
    boolean onLeft(Point q)
    :{return q[cutDim] < pt[cutDim]}
}
```

Example: find(q) $\xrightarrow{\text{calls}}$ find(q, root)

q = (4,4)



Analysis:

Find runs in time O(h), where h is height of tree.

Theorem: If pts are inserted in random order, expected height is O(log n)

```
Value find(Point q, KDNode p) {
    if (p == null) return null;
    else if (q == p.pt)          ···> all coords match?
        return p.value
    else if (p.onLeft(q))
        return find(q, p.left)
    else
        return find(q, p.right)
}
```

Quadtrees & kd-Trees IV

Kd-Tree Insertion:

(Similar to std. BSTs)

- Descend tree until
  - → find pt → Error - duplicate
  - → falling out ← (Although we draw extended trees, lets assume standard trees)
    - → create new node
    - → set cutting dim

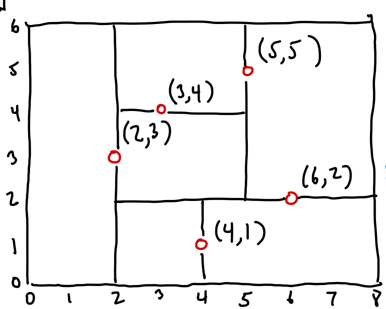
```
KDNode insert(Point x,
    Value v, KDNode p, int cd) {
  if (p == null)  // fell out?
    p = new KDNode(x, v, cd)
        // new leaf node
  else if (p.pt == x)
    Error! Duplicate key
  else if (p.onLeft(x))
    p.left = insert(x, v, p.left, (cd+1) % dim)
  else
    p.right = insert(x, v, p.right,
              (cd+1) % dim)
  return p
}
```

cutting dimension to use

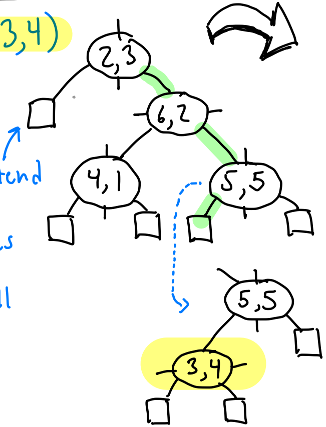
dimension of point

Example:

insert(3,4)



pretend ext. nodes are null



Deletion:

- Descend path to leaf
- If found:
  - leaf node → just remove
  - internal node
    - → find replacement
    - → copy here
    - → recur. delete replacement

This is the hardest part. See Latex notes.

Rebalance by Rebuilding:

- Rebuild subtrees as with scapegoat trees
- O(log n) amortized
- Find: O(log n) guaranteed.

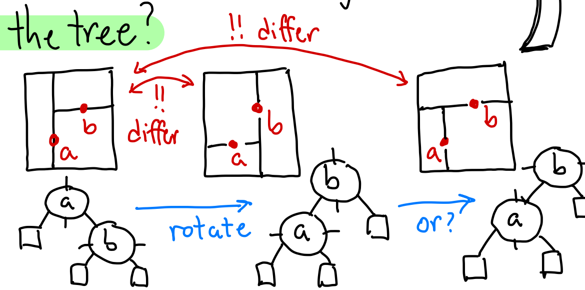
Analysis:

Run time: O(h)

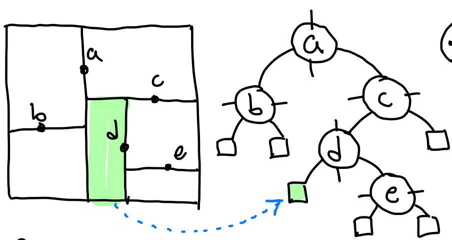
Tree height

Can we balance the tree?

- Rotation does not make sense

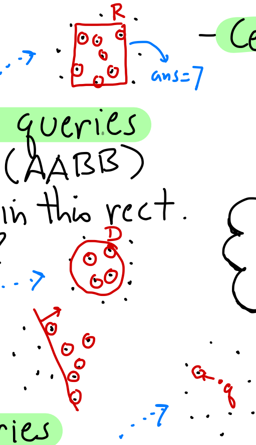


Kd-Tree Queries I



Queries?

- Orthogonal range queries
  - Given query rect. (AABB) count/report pts in this rect.
- Other range queries?
  - Circular disks
  - Halfplane
  - ⋮



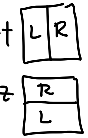
- Nearest neighbor queries
  - Given query pt, return closest pt in the set
  - Find k^{th} closest point
  - Find farthest point from q

This Lecture: $O(\sqrt{n})$ time alg for orthog. range counting queries in $\mathbb{R}^2$

→ General $\mathbb{R}^d$: $O(n^{1-1/d})$

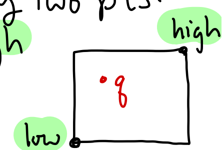
Kd-Trees:

- Partition trees
- Orthogonal split → vert / → horz
- Alternate cutting dimension $x, y, x, y, \ldots$
- Cells are axis-aligned rectangles (AABB)



Axis-Aligned Rect in $\mathbb{R}^d$

- Defined by two pts: low, high

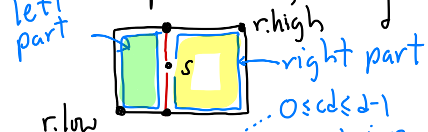


- Contains pt $q \in \mathbb{R}^d$ iff

$$low_i \leq q_i \leq high_i \qquad 1 \leq i \leq d$$

Rectangle methods for kd-cells:

- Split a cell r by a split pt $s \in r$, along cutdim cd



$0 \leq cd \leq d-1$

pt in r

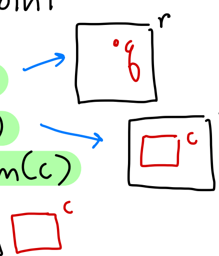
r.leftPart(cd, s)
→ returns rect with low = r.low + high = r.high but high[cd] ← s[cd]

r.rightPart(cd, s)
→ high = r.high + low = r.low but low[cd] ← s[cd]

Useful methods:

Let r, c - Rectangle
q - Point

- r.contains(q)
- r.contains(c)
- r.isDisjointFrom(c)

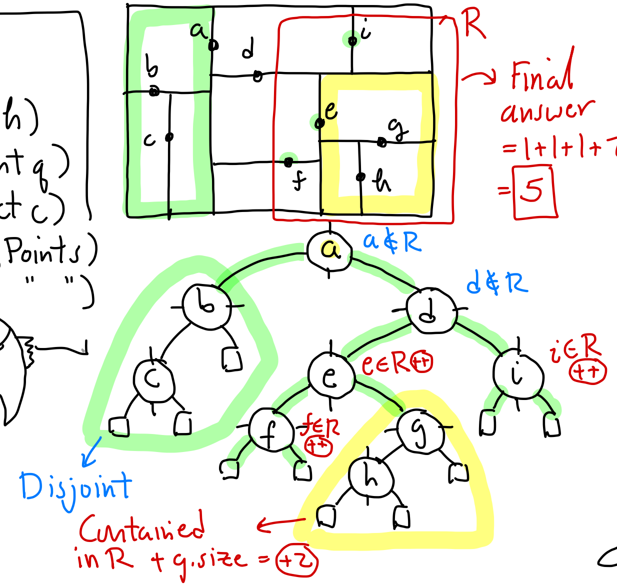


Kd-Tree Queries II

Orthog. Range Query

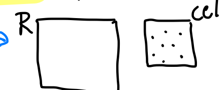
- Assume: Each node p stores:
  - p.pt : splitting point
  - p.cutDim : cutting dim
  - p.size : no. of pts in p's subtree
- Tree stores ptr. to root and bounding box for all pts.
- Recursive helper stores current node p + p's cell.

```
class Rectangle {
    private Point low, high
    public Rect(Point l, Point h)
      "     boolean contains(Point q)
      "     boolean contains(Rect c)
      "     Rect leftPart(int cd, Point s)
      "     Rect rightPart(" "  " ")
}
```

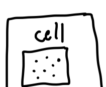


Cases:

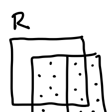
- p == null → fell out of tree → 0
- Query rect is disjoint from p's cell
  → return 0
  → no point of p contributes to answer



- Query rect contains p's cell
  → return p.size
  → every point of p's subtree contributes to answer.



- Otherwise: Rect. + cell overlap – Recurse on both children



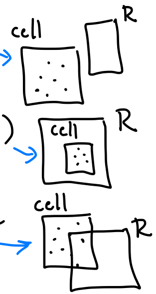

```
int rangeCount(Rect R, KDNode p, Rect cell)
    if (p == null) return 0                        // fell out of tree
    else if (R.isDijointFrom(cell)) return 0       // no overlap
    else if (R.contains(cell)) return p.size       // take all
    else { int ct = 0
        if (R.contains(p.pt)) ct++                 // p's pt in range
        ct += rangeCount(R, p.left,
                         cell.leftPart(p.cutDim, p.pt))
        ct += rangeCount(R, p.right, cell.rightPart...
    }
```

Kd-Tree Queries III

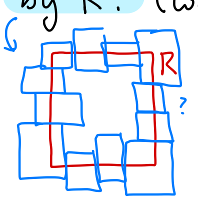
Theorem: Given a balanced kd-tree storing n pts in $\mathbb{R}^2$ (using alternating cut dim), orthog. range queries can be answered in $O(\sqrt{n})$ time.

→ Slower than $\log n$. Faster than n

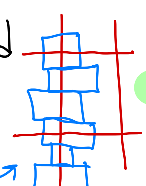
Stabbing: 3 cases
- cell is disjoint (easy)
- cell is contained (easy)
- cell partially overlaps or is stabbed by the query range (hard!)



How many cells are stabbed by R? (worst case)



Simpler: Extend R's sides to 4 lines + analyze each one.



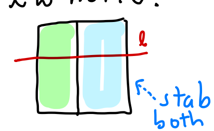
Analysis: How efficient is our algorithm?

→ Tricky to analyze
- → At some nodes we recurse on both children ⇒ $O(n)$ time?
- → At some we don't recurse at all!

Lemma: Given a kd-tree (as in Thm above) and horiz. or vert. line ℓ, at most $O(\sqrt{n})$ cells can be stabbed by ℓ

Proof: w.l.o.g. ℓ is horiz.

Cases: p splits vertically



Solving the Recurrence:
- **Macho:** Expand it
- **Wimpy:** Master Thm (CLRS)

Master Thm:

$$T(n) = aT(n/b) + n^d \quad \& \quad d < \log_b a$$
$$\Rightarrow T(n) = n^{\log_b a}$$

For us: $a = 2$, $b = 4$, $d = 0$ $\Rightarrow T(n) = n^{\log_4 2} = n^{1/2} = \sqrt{n}$

Since tree is balanced a child has half the pts + grandchild has quarter.

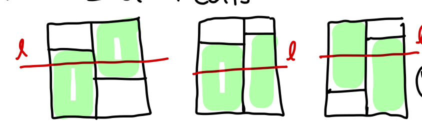
Recurrence: $T(n) = 2 + 2T(n/4)$

2 cells stabbed

Recurse on 2 grandchildren

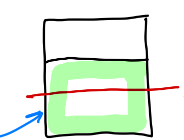
Each has n/4 pts

If we consider 2 consecutive levels of kd-tree, ℓ stabs at most 2 of 4 cells:



p splits horizontally

ℓ stabs only one



Hashing I

Recap:
So far, ordered dicts.

- insert, delete, find
- Comparison-based: <, ==, >
- getMin, getMax, getK, findUp...
- Query/Update time: $O(\log n)$
  - → Worst-case, amortized, random.

→ Can we do better? $O(1)$?

Hashing: (Unordered)
dictionary

- stores key-value pairs in array table[0..m-1]
- supports basic dict. ops. (insert, delete, find) in $O(1)$ expected time
- does not support ordered ops (getMin, findUp, ...)
- simple, practical, widely used

Overview:

- To store n keys, our table should (ideally) be a bit larger (e.g., $m \geq c \cdot n$, $c = 1.25$)
- Load factor:

  $$\lambda = n/m$$

- Running times increase as $\lambda \to 1$
- Hash function:

  $$h : \text{Keys} \longrightarrow [0..m-1]$$

  → Should scatter keys random.

  → Need to handle collisions → $x \neq y$ but $h(x) = h(y)$

Good Hash Function:

- Efficient to compute
- Produce few collisions
  - Use every bit in key
  - Break up natural clusters

E.g. Java variable names:

temp1, temp2, temp3

table: 

Common Examples:

Assume keys can be interpreted as ints

- Division hash:

  $$h(x) = x \bmod m$$

- Multiplicative hash:

  $$h(x) = (ax \bmod p) \bmod m$$

  a, p - large prime numbers

- Linear hash:

  $$h(x) = ((ax + b) \bmod p) \bmod m$$

  a, b, p - large primes

Why "mod p mod m"?

- modding by a large prime scatters keys
- m may not be prime (e.g. power of 2)

Universal Hashing:
Even better → randomize!

- Let H be a family of hash fns
- Select $h \in H$ randomly
- If $x \neq y$ then $\text{Prob}(h(x) = h(y)) = 1/m$

E.g. Let p - large prime, $a \in [1..p-1]$, $b \in [0..p-1]$ all random

- $h_{a,b}(x) = ((ax + b) \bmod p) \bmod m$

Hashing II

Overview:
- Separate Chaining
- Open Addressing:
  - Linear probing
  - Quadratic probing
  - Double hashing

simple / slow ↓ complex / fast

Collision Resolution:
If there were no collisions hashing would be trivial!

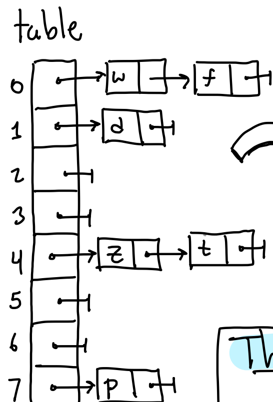
insert(x,v) → table[h(x)] = v
find(x) → return table[h(x)]
delete(x) → table[h(x)] = null

Separate Chaining:
table[i] is head of linked list of keys that hash to i.

Example:

Keys (x)	h(x)
d	1
z	4
p	7
w	0
t	4
f	0

m = 8



Analysis:
Recall load factor
$\lambda = n/m$ n = # of keys
m = table size

S_{sc} = Expected search time if x found (successful)

U_{sc} = Expect. search time if x not found (unsuccessful)

Thm: $S_{sc} = 1 + \lambda/2$ $U_{sc} = 1 + \lambda$

Proof: On avg. each list has $n/m = \lambda$
success: 1 for head + half the list
unsuccess: 1 " " + all the list

How to control λ?
- Rehashing: If table is too dense / too sparse, realloc. to new table of ideal size

Designer: $\lambda_{min}, \lambda_{max}$ - allowed λ values

$\lambda_0 = \frac{\lambda_{min} + \lambda_{max}}{2}$ "ideal"

If $\lambda < \lambda_{min}$ or $\lambda > \lambda_{max}$ ...

If $\lambda < \lambda_{min}$ or $\lambda > \lambda_{max}$? Rehash!
- Alloc. new table size = n/λ_0
- Compute new hash fn h
- Copy each x, v from old to new using h
- Delete old table

Thm: Amortized time for rehashing is $1 + (2\lambda_{max}/(\lambda_{max} - \lambda_{min}))$

Token-based — See latex notes!

Hashing III

Collision Resolution: (cont.)

- Separate chaining is efficient, but uses extra space (nodes, pointers, ...)
- Can we just use the table itself?

→ **Open Addressing**

Open Addressing:

- Special entry ("empty") means this slot is unoccupied
- Assume $\lambda \leq 1$
- To insert key: check: $h(x)$ if not empty try
  - $h(x) + i_1$
  - $h(x) + i_2$
  - ⋮

$\langle i_1, i_2, i_3, \ldots \rangle$ – **Probe sequence**

- What's the best probe sequence?

Linear Probing:

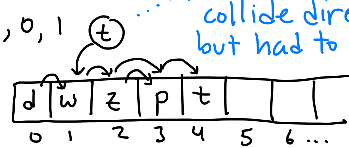
$h(x),\ h(x)+1,\ h(x)+2, \ldots$

until finding first available

Simple, but is it good?

x: d, z, p, w, t

$h(x)$: 0, 2, 2, 0, 1



t did not collide directly but had to probe 3 times!

Analysis:

Let S_{LP} = expected time for successful search

U_{LP} = " " unsuccessful "

Thm:
$$S_{LP} = \frac{1}{2}\left(1 + \frac{1}{1-\lambda}\right)$$
$$U_{LP} = \frac{1}{2}\left(1 + \frac{1}{1-\lambda}\right)^2$$

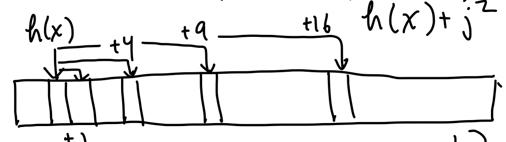
Obs: As $\lambda \to 1$ times increase rapidly

Clustering

- Clusters form when keys are hashed to nearby locations
- Spread them out!

Quadratic Probing:

$h(x),\ h(x)+1,\ h(x)+4,\ h(x)+9, \ldots\ h(x)+j^2$



Analysis:

Improves secondary clustering

- May fail to find empty entry (Try $m=4$. $j^2 \bmod 4 = 0$ or 1 but not 2 or 3)
- How bad is it? It will succeed if $\lambda < 1/2$.

Thm: If quad. probing used & m is prime, the the first $\lfloor m/2 \rfloor$ probe locations are distinct.

Pf: See latex notes.

Hashing IV

Recap:

Separate Chaining: Fastest but uses extra space (linked list)

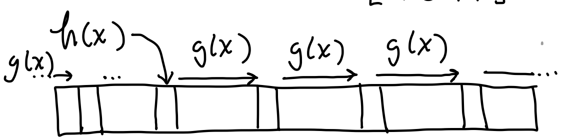
Open Addressing:
- Linear probing:
- Quadratic probing:

} clustering

Double Hashing:
(Best of the open-addressing methods)

- Probe sequence det'd by second hash fn. – $g(x)$

$$h(x) + \{0, g(x), 2\cdot g(x), 3\cdot g(x) \ldots\} \quad [\mathrm{mod}\ m]$$



(until finding an empty slot)

Why does bust up clusters?
Even if $h(x) = h(y)$ [collision] it is very unlikely that $g(x) = g(y)$

⇒ Probe sequences are entirely different!

Analysis: Defs:
S_{DH} = Expected search time of doub. hash. if successful

U_{DH} = Exp. if unsuccessful

Recall: Load factor $\lambda = n/m$

Thm:
$$\boxed{S_{DH} = \tfrac{1}{\lambda} \ln\left(\tfrac{1}{1-\lambda}\right) \qquad U_{DH} = 1/(1-\lambda)}$$

→ Proof is nontrivial (skip)

λ:	0.5	.075	0.95	0.99
U_{DH}:	2	4	20	100
S_{DH}:	1.39	1.89	3.15	4.65

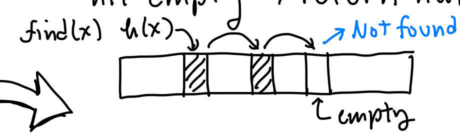
Very efficient!

Dictionary Operations:

Insert(x,v): Apply probe sequence until finding first empty slot.
- Insert (x,v) here.

(If x found along the way ⇒ duplicate key error!)

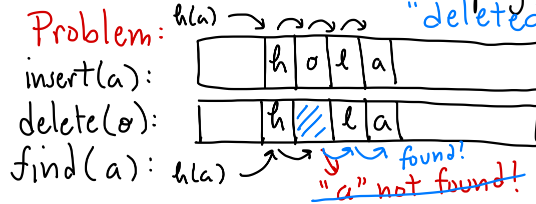
Find(x): Visit entries on probe sequence until:
- found x ⇒ return v
- hit empty ⇒ return null



Delete(x): Apply find(x)
- → Not found ⇒ error
- → Found ⇒ set to ~~"empty"~~ "deleted"

~~Is this right??~~

Problem:



Scapegoat Trees I

Scapegoat Trees:
- Arne Anderson (1989)
- Galperin + Rivest (1993) rediscovered/extended
- **Amortized analysis**
  - $O(\log n)$ for dictionary ops amortized (guaranteed for find)
  - Just let things happen
  - If subtree unbalanced – rebuild it

Recap:
- Seen many search trees
- Restructure via **rotation**
- Today: Restructure via **rebuilding**
- Sometimes rotation not possible
- Better mem. usage

Overview:
Insert:
- same as standard BST
- if depth too high
  - trace search path back
  - find unbalanced node – **scapegoat**
  - rebuild this subtree

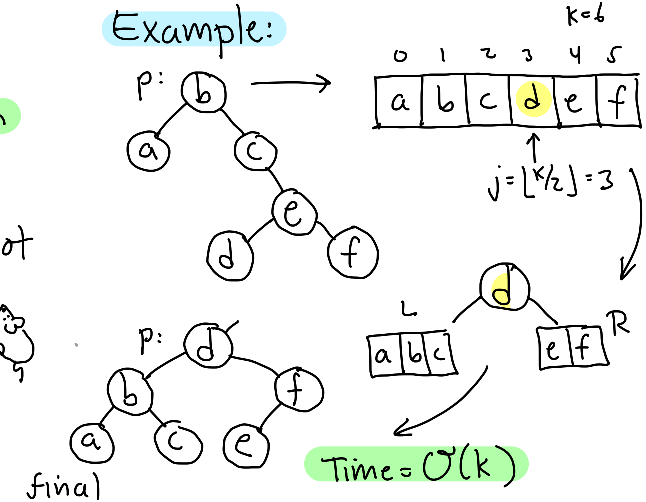
Find: Same as std BST
- Tree height $\leq \log_{3/2} n \approx 1.71 \lg n$

Delete:
- Same as std. BST
- If num. of deletes is large rel. to n – rebuild entire tree!

How? Maintain $n, m \leftarrow 0$

Insert: $n{+}{+}, m{+}{+}$

Delete: $n{-}{-}$ ⋯→ If $m > 2n$ rebuild

Example:


Time = $O(k)$

How to rebuild?
rebuild(p):
- inorder traverse p's subtree → array A[ ]
- buildSubtree(A)

buildSubtree(A[0..k-1]):
- if $k = 0$ return null
- $j \leftarrow \lfloor k/2 \rfloor$; $x \leftarrow A[j]$ median
- $L \leftarrow$ buildSubtree(A[0..j-1])
- $R \leftarrow$ buildSubtree(A[j+1..k-1])
- return Node(x, L, R)

Scapegoat Trees II

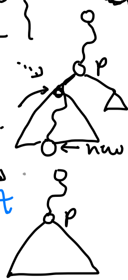
Details of Operations:

Init: $n \leftarrow m \leftarrow 0$ root $\leftarrow$ null

Delete:
- Same as std BST
- $n--$
- if $m > 2n$, rebuild(root)

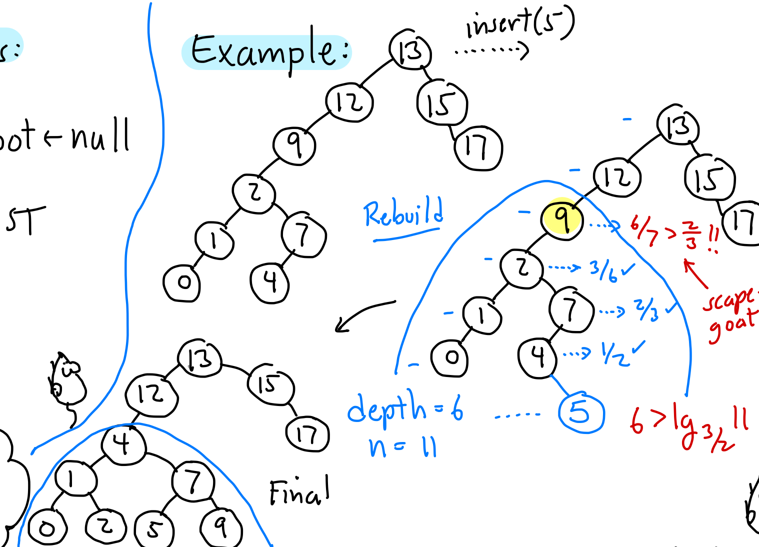
Time: $O(n)$

Insert:
- $n++$; $m++$
- Same as std BST but keep track of inserted node's depth $\rightarrow d$
- if ($d > \log_{3/2} m$) {
  /* rebuild event */
  - trace path back to root
  - for each node p visited, size(p) = no. of nodes in p's subtree
  - if $\frac{\text{size(p.child)}}{\text{size(p)}} > \frac{2}{3}$
    p ← rebuild(p)
  - break



How to compute size(p)?
- Can compute it on the fly
- While backing out, traverse "other sibling"
- Too slow? No!
  → Charge to rebuild.

Example:

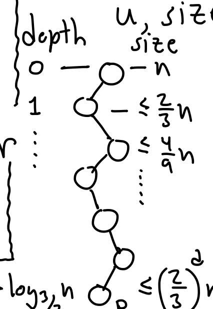


Must there be a scapegoat? yes!

Lemma: Given a binary tree with n nodes, if $\exists$ node p of depth $> \log_{3/2} n$, then $\exists$ ancestor of p that satisfies scapegoat condition

Proof: By contradiction

- Suppose p's depth $> \log_{3/2} n$ but $\forall$ ancestors u, $\text{size(u.child)} \le 2/3 \cdot \text{size}(u)$



$\Rightarrow$ Since p has 1 node:

$1 \le \text{size}(p) \le \left(\frac{2}{3}\right)^d n$

$\Rightarrow (3/2)^d \le n$

$\Rightarrow d \le \log_{3/2} n$ □

Scapegoat Trees III

Theorem: Starting with an empty tree, any sequence of m dictionary operations on a scapegoat tree take time $O(m \log m)$ [Amortized: $O(\log m)$]

Proof: (Sketch)

Find: $O(\log n)$ guaranteed [Height $= O(\log n)$]

Delete: In order to induce a rebuild, number of deletes $\sim$ number of nodes in tree

→ Amortize rebuild time against delete ops

Insert: Based on potential argument

→ It takes $\sim k$ ops to cause a subtree to size k to be unbalanced.

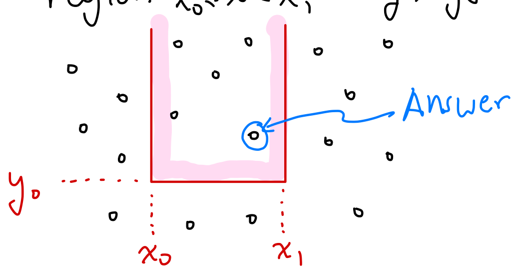
→ Charge rebuild time to these operations

Range Tree Applications:

- Range trees can be applied to a variety of query problems
- Methods:
  - Minimization/Maximization
  - Transform coordinates
  - Adding new coordinates

Minimization/Maximization - 3-Sided Min Query

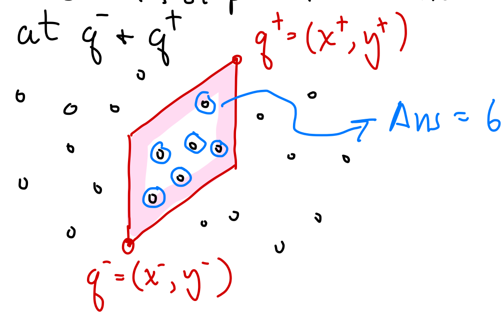
Given a set P of n pts in $\mathbb{R}^2$, a query consists of x-interval $[x_0, x_1]$ and y value y_0. Return the lowest pt in 3-sided region $x_0 \le x \le x_1$ & $y \ge y_0$



Transforming coordinates:

Skewed rectangle query:

Given a set P of n pts in $\mathbb{R}^2$, a skewed rectangle is given by 2 pts $q^- = (x^-, y^-)$ and $q^+ = (x^+, y^+)$ and consists of pts in parallelogram with two vertical sides and two with slope +1 & corners at q^- & q^+

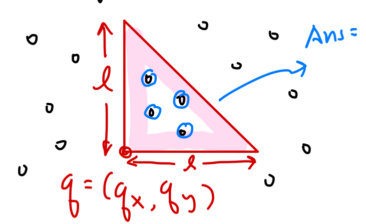


Return a count of the number of pts of P inside the skewed rectangle.

Adding New Coordinates:

NE Right Triangle Query

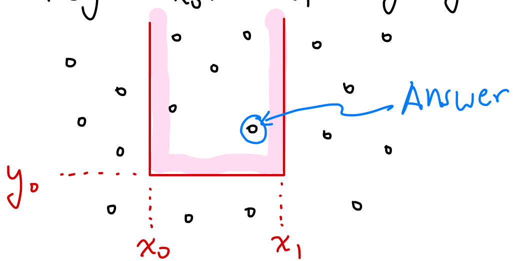
Given a set P of n pts in $\mathbb{R}^2$ and scalar $\ell > 0$, a NE triangle is a 45-45 right triang with lower left corner at q and side length ℓ.



Return a count of the number of pts of P lying within the triangle.

3-Sided Min Query

Return lowest in region region $x_0 \le x \le x_1$ + $y \ge y_0$



Data structure:

- Build a range tree for x
- Aux. trees are range trees for y that support findLarger

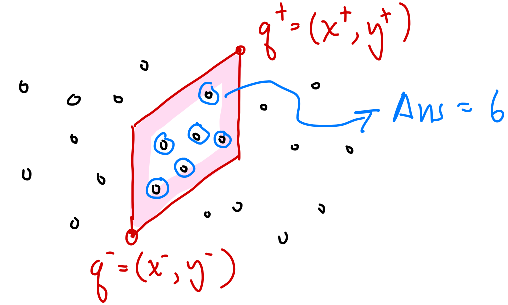
Query Processing:

- Do 1D range search in main tree for interval $[x_0, x_1]$.
- For each maximal subtree in range, do findLarger(y_0)
- Return smallest of these.

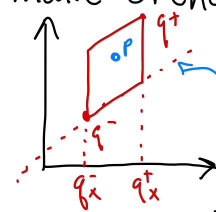
Analysis:

- Same as 2D range tree
- Space: $O(n \log n)$ Time: $O(\log^2 n)$

Skewed rectangle query:



Transform coordinates to make orthog range query



$$q_x^- \le p_x \le q_x^+$$

Line equation: $y = x + (q_y^- - q_x^-)$

$$p_x + (q_y^- - q_x^-) \le p_y \le p_x + (q_y^+ - q_x^+)$$

$$\iff \underbrace{q_y^- - q_x^- \le p_y - p_x}_{p_y' = p_y - p_x} \le q_y^+ - q_x^+$$

Map each $p = (p_x, p_y) \in P$ to $p' = (p_x', p_y') \triangleq (p_x, p_y - p_x)$

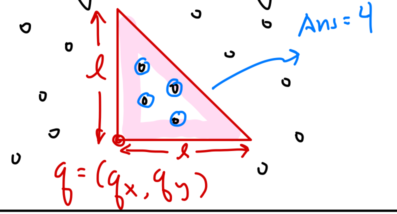
Let P' be resulting set.

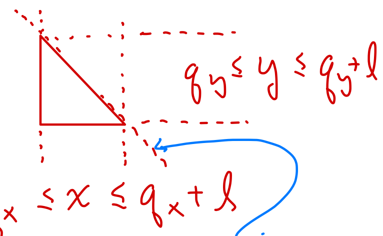
Build std. range tree for P'. Return ans. to query

$$q_x^- \le x \le q_x^+$$

$$q_y^- - q_x^- \le y \le q_y^+ - q_x^+$$

NE Right Triangle Query





- Add new coord: $z = x + y$
- Map pts: $p = (p_x, p_y) \rightarrow p' = (p_x, p_y, p_x + p_y)$
- Let P' be resulting set

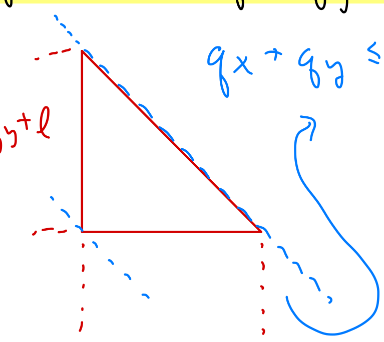
Build a 3D range tree on P'

NE triangle query becomes:

$$q_x \le x \le q_x + \ell$$

$$q_y \le y \le q_y + \ell$$

$$q_x + q_y \le z \le q_x + q_y + \ell$$



Space:

$O(n \log^2 n)$

Query time:

$O(\log^3 n)$

Range Trees I

Can we do better?

Range Trees:

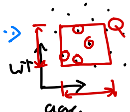
- Space is $O(n \log^{d-1} n)$
- Query time:
  - **Counting:** $O(\log^d n)$
  - **Reporting:** $O(k + \log^d n)$

→ In $\mathbb{R}^2$: $\log^2 n$ **much better** than $\sqrt{n}$ for large n

→ Range trees are **more limited**

Layering: Combining search structures

- Suppose you want to answer a **composite query** w. multiple criteria:
- Medical data: Count subjects
  - **Age range:** $a_{lo} \le age \le a_{hi}$
  - **Weight range:** $w_{lo} \le weight \le w_{hi}$



- Design a data structure for each criterion **individually**
- **Layer** these structures together to answer full query

→ **Multi-Layer Data Structures**

Recap:

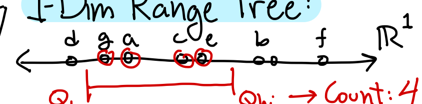
- **kd-Tree:** General-purpose data structure for pts in $\mathbb{R}^d$
- **Orthogonal range query:** Count/report pts in axis-aligned rect.

→ Ans = 4

- kd-Tree: **Counting:** $O(\sqrt{n})$ time
  - **Report:** $O(k + \sqrt{n})$ time

No. of pts reported

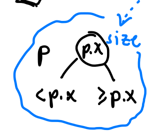
1-Dim Range Tree:



→ Count: 4
Rept: {g, a, c, e}

Approach:

- Balanced BST (eg. AVL, RB, ...)
- Assume **extended tree**
- Each node p stores no. of entrie in subtree: **p.size**

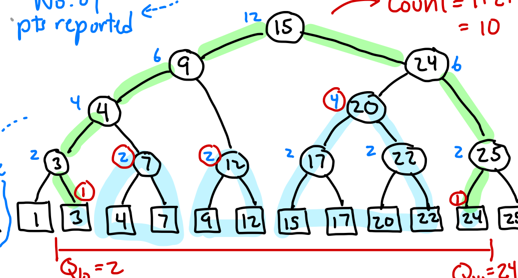


Call this a 1-Dim Range Tree:

Claim: A 1-Dim range tree with n pts has space $O(n)$ and answers 1-D range count/rept queries in time $O(\log n)$ (or $O(k + \log n)$)

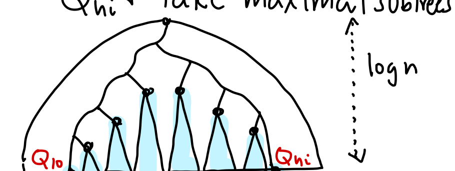
No. of pts reported

→ Count = 1+2+2+4+1 = 10



Canonical Subsets:

- **Goal:** Express answer as disjoint union of subsets
- **Method:** Search for Q_{lo} + Q_{hi} + take maximal subtrees

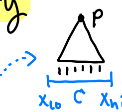


Range Trees II

More details:

Given a 1-D range tree T:

- Let $Q = [Q_{lo}, Q_{hi}]$ be **query interval**
- For each node p, define interval **cell** $C = [x_0, x_1]$ s.t. all pts of p's subtree lie in C



- **Root cell**: $C_0 = [-\infty, +\infty]$

Recursive helper:

int range1Dx(Node p, Intv $Q = [Q_{lo}, Q_{hi}]$, Intv $C = [x_0, x_1]$)

initial call: range1Dx(root, Q, C_0)

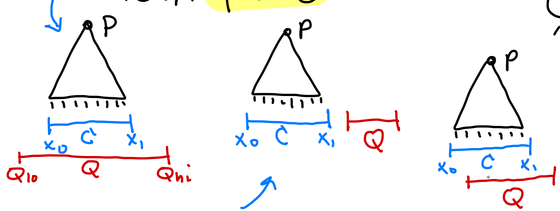
Cases:

p is external:

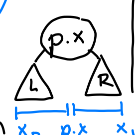
- if $p.pt.x \in Q \to 1$ else $\to 0$

p is internal:

- **$C \subseteq Q$** $\Rightarrow$ all of p's pts lie within query
  $\to$ return **p.size**



- **C is disjoint from Q** $\Rightarrow$ none of p's pts lie in Q
  $\to$ return **0**
- Else **partial overlap**
  $\to$ Recurse on p's children + trim the cell




```
int range1Dx(Node p,
   Intv Q, Intv C=[x0,x1]) {
  if(p is external)                T -> 1
    return p.pt.x ∈ Q              F -> 0
  else if (C ⊆ Q) return p.size
  else if (Q+C disjoint) return 0
  else return:
      range1Dx(p.left, Q, [x0, p.x])
    + range1Dx(p.right, Q, [p.x, x1])
```

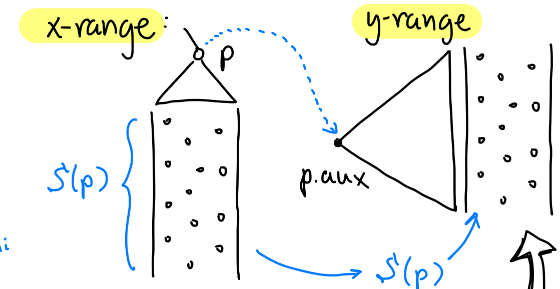
Analysis:

Lemma: Given a 1-D range tree with n pts, given any interval Q, can compute $O(\log n)$ subtrees whose union is answer to query.

Thm: Given 1-D range tree... can answer range queries in time $O(\log n)$ (+k to report)

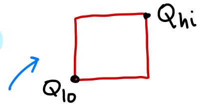
2-D Range Searching:

- "**Layer**" a range tree for x with range tree for y
- For each node $p \in$ 1D-x tree, let **S(p)** = set of pts in p's subtree
- Def: **p.aux**: A 1D-y tree for S(p)



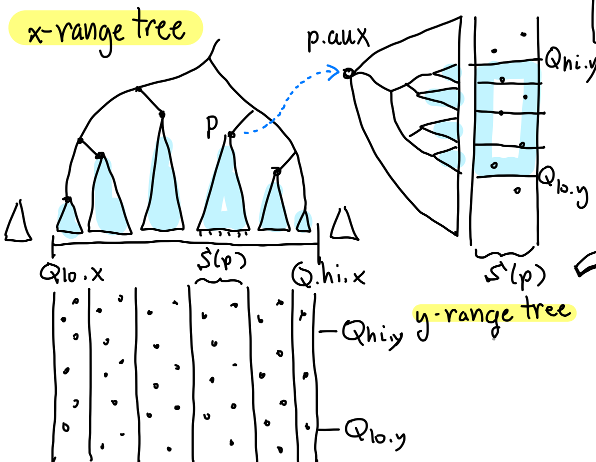
Range Trees III

Answering Queries?



Given query range
$Q = [Q_{lo}.x, Q_{hi}.x] \times [Q_{lo}.y, Q_{hi}.y]$

- Run range1Dx to find all subtrees that contribute
- For each such node p, run range1Dy on p.aux
- Return sum of all result



Intuition: The x-layer finds subtrees p contained in x-range + each aux tree filters based on y.

2D Range Tree:

- Construct 1D range tree based on x coords for all pts
- For each node p:
  - Let S(p) be pts of p's tree
  - Build 1D range tree for S(p) based on y → p.aux
- Final structure is union of x-tree + (n-1) y-trees

Higher Dimensions?

- In d-dim space, we create d-layers
- Each recurses one dim lower until we reach 1-d search
- Time is the product:
  $\underbrace{\log n \cdot \log n \cdots \log n}_{d} = O(\log^d n)$

Analysis: The 1D x search takes of $O(\log n)$ time + generates $O(\log n)$ calls to 1D y search

⇒ Total: $O(\log n \cdot \log n) = O(\log^2 n)$

```
int range2D(Node p, Rect Q, Intv C=[x0,x1]){
    if (p is external) return p.pt ∈ Q ? (T → 1, F → 0)
    else if (Q.x contains C){      // C ⊆ Q's x-projection
        [y0,y1] = [-∞,+∞]           // init y-cell
        return range1Dy(p.aux, Q, [y0,y1])
    } else if (Q.x is disjoint of C) return 0
    else                            // partial x-overlap
        return range2D(p.left, Q, [x0, p.x])
             + range2D(p.right, Q, [p.x, x1])
}
```

Analysis:

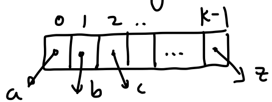
Invoked $O(\log n)$ times - once per maximal subtree

Invoked $O(\log n)$ times - once for each ancestor of max subtree

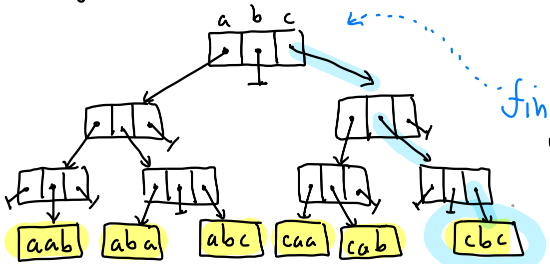
Tries and Digital Search Trees I

Tries: History
- de la Briandais (1959)
- Fredkin - "trie" from "retrieval"
- Pronounced like "try"

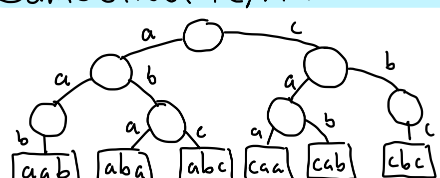
Node: Multiway of order k



Example: $\Sigma = \{a=0, b=1, c=2\}$
Keys: {aab, aba, abc, caa, cab, cbc}



Same structure/Alt. Drawing



Digital Search:
- Keys are strings over some alphabet Σ
- E.g. $\Sigma_1 = \{a, b, c, \ldots\}$
  $\Sigma_1 = \{0, 1\}$ Let $k = |\Sigma_1|$
- Assume chars coded as ints: a=0, b=1, ... z=k-1

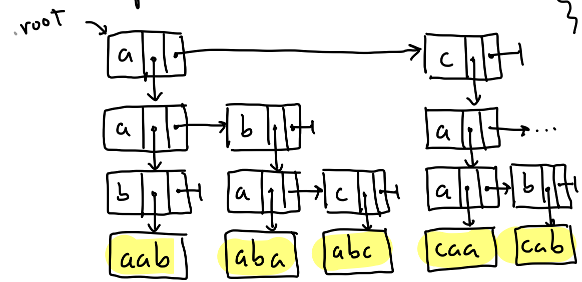
Analysis:
Search: ~ length of query string [O(1) time per node]

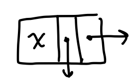
Space:
- No. of nodes ~ total no. of chars in all strings
- Space ~ k·(no. of nodes)

Large!

Analysis:
- **Space**: Smaller by factor k
- **Search Time**: Larger by factor of k

Example:



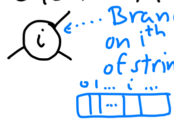
How to save space?
de la Briandais trees:
- Store 1 char. per node
-  $\neq x \Rightarrow$ try next char in Σ

  $== x \Rightarrow$ advance to next character of search string
- First-child/next-sibling

Tries and Digital Search Trees II

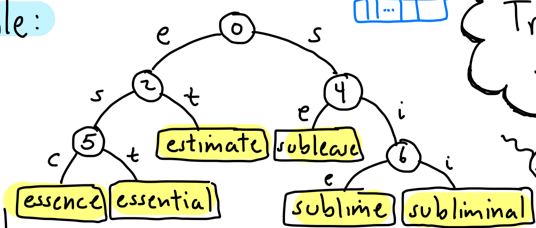
Patricia Tries:

- Improves trie by compressing degenerate paths
- PATRICIA = Practical Alg. to Retrieve Info. Coded in Alpha...
- Late 1960's: Morrison + Gwehenberger
- Each node has index field, indicates which char to check next (Increase with depth)

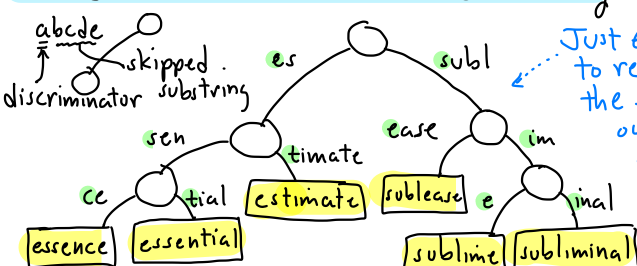


Example:

0 1 2 3 ...
essence
essential
estimate
sublease
sublime
subliminal



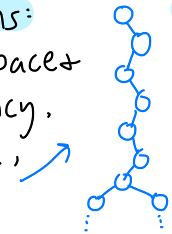
Same data structure – Drawn differently



Just easier to read the strings out.... same data struct.

Dealing with long Paths:

- To get both good space + query time efficiency, need to avoid long, degenerate paths.



- Path compression!

Example:

	ID		ID
S_{10}: $	$	S_5: ajam...	aj
S_9: a$	a$	S_4: pajam...	paj
S_8: ma$	ma$	S_3: apaja...	ap
S_7: ama$	ama$	S_2: mapaj...	map
S_6: jama$	j	S_1: amapaj...	amap
		S_0: pamapa...	pam

Example:

S = pamapajama$ (positions 0 1 2 3 4 5 6 7 8 9 10)

S_{10} - $
S_9 - a$
S_8 - ma$
S_7 - ama$
⋮

Def: Substring identifier for S_i is shortest prefix of S_i unique to this string

E.g. ID(S_1) = "amap"
ID(S_7) = "ama$"

Analysis:

- Query time: (Same as std trie) ~ search string length (may be less)
- Space:
  - No. nodes: ~ No. of strings (irresp. of length)
  - Total space: k·(No. of nodes) + (Storage for strings)

Suffix Trees:

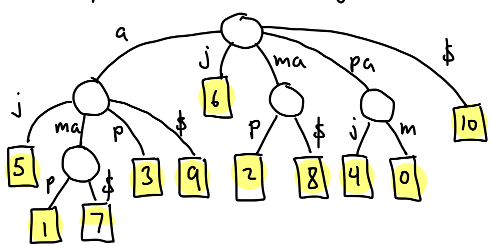
- Given single large text S
- Substring queries: "How many occurrences of "tree" in CMSC 420 notes"

Notation: $S = a_0 a_1 a_2 \ldots a_{n-1}\$$ (special terminal)

- Suffix: $S_i = a_i a_{i+1} \ldots a_{n-1}\$$
- Q: What is minimum substring needed to identify suffix S_i?

Tries and Digital Search Trees III

Example: S = pamapajama\$ (positions 0 1 2 3 4 5 6 7 8 9 10)



E.g. $ID(S_1)$ = amap $ID(S_7)$ = ama\$

Substring Queries:

How many occurrences of t in text?

- Search for target string t in trie
- if we end in internal node (or midway on edge) – return no. of extern. nodes in this subtree
- else (fall of on extern node)
  - compare target with string
    - if matches – found 1 occurrence
    - else – no occurrences

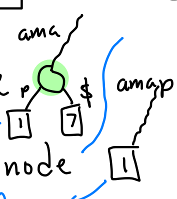
Example:

Search("ama") → End at intern node

Report: 2 occ's.

Search("amapaj") → End at extern node

Go to S_1 + verify



Suffix Trees (cont.)

S - text string $|S| = n$

S_i = i^{th} suffix

Substring ID = min substr. needed to identify S_i

A **suffix tree** is a Patricia trie of the n+1 substring identifiers

Analysis:

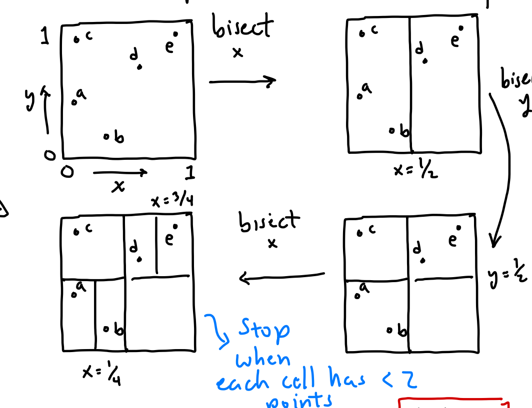
- **Space:** $O(n)$ nodes
  $O(n \cdot k)$ total space
  $(k = |\Sigma| = O(1))$
- **Search time:** ~ to length of target string
- **Construction time:**
  - $O(n \cdot k)$ [nontrivial]

PR k-d tree: Can be used for answering same queries as point kd-tree (orth. range, near. neigh)

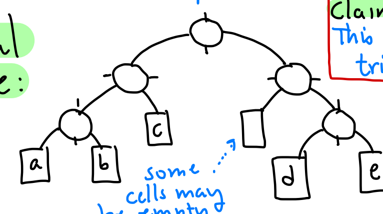
Geometric Applications:

PR kd-Tree: kd-tree based on midpoint subdivision

Assume points lie in unit square



Final tree:



Claim: This is a trie!

Tries and Digital Search Trees IV

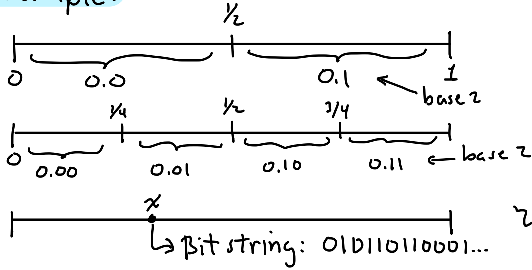
Binary Encoding:

- Assume our points are scaled to lie in **unit square** $0 \le x, y < 1$ (can always be done)
- Represent each coordinate as **binary fraction**:

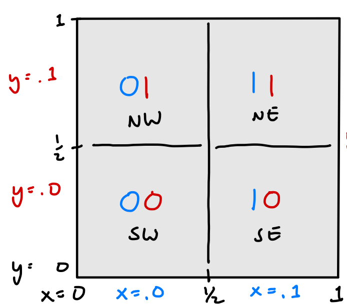
$x = 0.a_1 a_2 a_3 \ldots \qquad a_i \in \{0,1\}$

$x = \sum_i a_i \cdot 1/2^i$

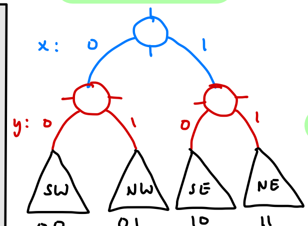
Example:



How do we extend to 2-D?



PR kd-tree



PR kd-Tree ≡ Trie ??

- Approach: Show how to map any point in $\mathbb{R}^2$ to bit string
- Store bit strings in a trie (alphabet $\Sigma = \{0,1\}$)
- Prove that this trie has same structure as kd-tree

Bit Interleaving:

Given a point $p = (x,y)$

$0 \le x, y < 1$

let: $x = 0.a_1 a_2 \ldots$ in binary

$y = 0.b_1 b_2 \ldots$

Define:

$\phi(x,y) = a_1 b_1 a_2 b_2 a_3 b_3 \ldots$

Called **Morton Code** of p

Further Remarks:

- Techniques for efficiently encoding, building, serializing, compressing... tries **apply immediately to PR kd-tree**
- Can generalize to **any dimension**

$$\left.\begin{aligned} x &= 0.a_1 a_2 \ldots \\ y &= 0.b_1 b_2 \ldots \\ z &= 0.c_1 c_2 \ldots \end{aligned}\right\} \phi = a_1 b_1 c_1 a_2 b_2 c_2 \ldots$$

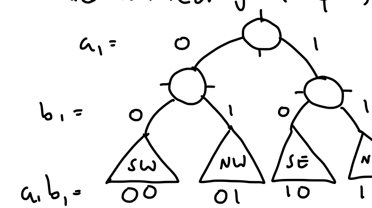
Lemma: Given a pt set $P \subseteq \mathbb{R}^2$ (in unit square $[0,1)^2$) let $P = \{p_1, \ldots, p_n\}$ where $p_i = (x_i, y_i)$

Let $\Phi(P) = \{\phi(p_1), \phi(p_2), \ldots, \phi(p_n)\}$ (n binary strings)

Then the PR kd-tree for P is equivalent to binary trie for $\Phi(P)$.

Proof: By induction on no. of bits

Let $x = 0.a_1 a_2 \ldots \quad y = 0.b_1 b_2 \ldots$ and consider just $\phi(x,y) = a_1 b_1 \ldots$



The PR kd-tree + binary trie assign pts to same subtrees.

(.... induction)