EIGEN VALUES AND VECTORS

MATRICES AND EIGEN VECTORS

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \times \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- **Scale**

$$2 \times \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \end{bmatrix} = 4 \times \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

EIGEN VECTOR - PROPERTIES

- Eigen vectors can only be found for square matrices
- Not every square matrix has eigen vectors.
- Given an n x n matrix that does have eigenvectors, there are n of them

  for example, given a 3 x 3 matrix, there are 3 eigenvectors.
- Even if we scale the vector by some amount, we still get the same multiple

EIGEN VECTOR - PROPERTIES

- Even if we scale the vector by some amount, we still get the same multiple
- Because all you're doing is making it longer, not changing its direction.
- All the eigenvectors of a matrix are perpendicular or orthogonal.
- This means you can express the data in terms of these perpendicular eigenvectors.
- Also, when we find eigenvectors we usually normalize them to length one.

EIGEN VALUES - PROPERTIES

- Eigenvalues are closely related to eigenvectors.
- These scale the eigenvectors
- eigenvalues and eigenvectors always come in pairs.

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \end{bmatrix} = 4 \times \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

SPECTRAL THEOREM

Theorem: If $A \in \mathbb{R}^{m \times n}$ is symmetric matrix (meaning $A^T = A$),

then, there exist real numbers $\lambda_1, \ldots, \lambda_n$ (the eigenvalues)

and orthogonal, non-zero real vectors $\phi_1, \phi_2, \ldots, \phi_n$

(the eigenvectors) such that for each $i = 1, 2, \ldots, n$:

$$A\phi_i = \lambda_i \phi_i$$

EXAMPLE

$$A = \begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix}$$

From spectral theorem:

$$A\phi = \lambda\phi$$

EXAMPLE

$$A = \begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix}$$

From spectral theorem:

$$A\phi = \lambda\phi \implies A\phi - \lambda I\phi = 0$$

$$(A - \lambda I)\phi = 0$$

$$\begin{bmatrix} 30-\lambda & 28 \\ 28 & 30-\lambda \end{bmatrix} = 0 \implies \lambda = 58 \text{ and } \lambda = 2$$

EXAMPLE

$$A = \begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix}$$

From spectral theorem:

$$A\phi = \lambda\phi$$

$$\begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 58 \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} \implies \phi_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

EXAMPLE

$$A = \begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix}$$

From spectral theorem: $A\phi = \lambda\phi$

$$\begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = 58 \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} \implies \phi_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = 2 \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} \implies \phi_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

EXAMPLE

$$A = \begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix}$$

From spectral theorem: $A\phi = \lambda\phi$

$$\phi_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \lambda_1 = 58 \qquad \phi_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \quad \lambda_2 = 2$$

$$\phi = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

SINGULAR VALUE DECOMPOSITION

- Every matrix, A, can be written as

$$A = U\Sigma V^T$$

where U and V are orthonormal matrices and Σ is a diagonal matrix.

- This is known as Singular Value Decomposition (SVD) of matrix A.

- The values in the diagonal of Σ are called the singular values of the matrix A

SINGULAR VALUE DECOMPOSITION

- Thus, $Ax = U(\Sigma(V^T x))$

- Applying A to a matrix x amounts to rotating using V^T, scaling using Σ and rotating again using U
- The rank of the matrix A is the number of non-zero singular values.
- Given a matrix $A = U\Sigma V^T$ of rank r, if we want the closest matrix A' of rank $r - k$, then one can simply zero out the k smallest singular values (smallest in absolute value) in Σ to produce Σ'. A' is then $U\Sigma' V^T$.
- If u_i is the i-th column of U, v_i is the i-th column of V and σ_i is the i-th diagonal element of Σ, then $A = \sum_i \sigma_i u_i v_i^T$

SINGULAR VALUE DECOMPOSITION

Theorem : $$A_{nm} = U_{nn}\Sigma_{nm}V_{mm}^T$$

A - Rectangular matrix, $n \times m$

Columns of U are orthonormal eigenvectors of AA^T

Columns of V are orthonormal eigenvectors of A^TA

Σ is a diagonal matrix containing the square roots of eigenvalues from U or V in descending order called singular values

$Av_i = \sigma_i u_i$ **and** $A^T u_i = \sigma_i v_i$

Where σ is the singular value value

SVD - EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{3\times 2} = U_{3\times 3}\Sigma_{3\times 2}V^T_{2\times 2}$$

Columns of U are orthonormal eigenvectors of AA^T

$$U = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

SVD - EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{3\times 2} = U_{3\times 3}\Sigma_{3\times 2}V_{2\times 2}^T$$

Columns of V are orthonormal eigenvectors of A^TA

$$V^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

SVD - EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{3\times2} = U_{3\times3}\Sigma_{3\times2}V_{2\times2}^T$$

Σ is a diagonal matrix containing the square roots of eigenvalues from U or V in descending order

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

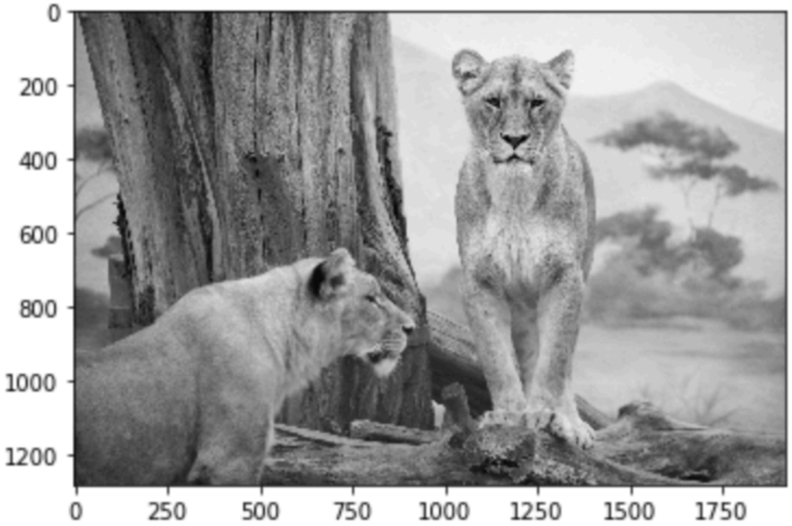
SVD - EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{3\times2} = U_{3\times3}\Sigma_{3\times2}V_{2\times2}^{T}$$

$$A = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

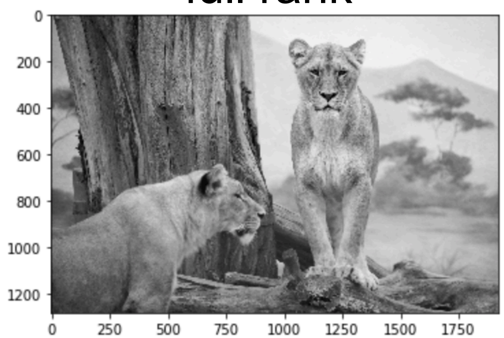
SVD - EXAMPLE



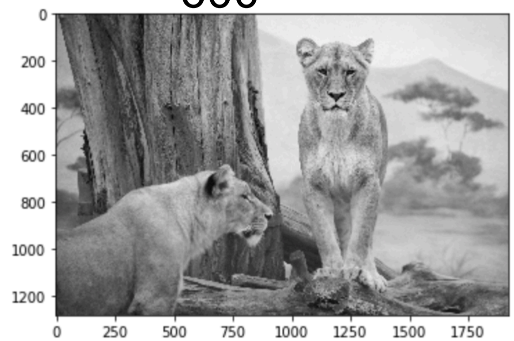
$$U, S, V^T = numpy.linalg.svd(img)$$

SVD - EXAMPLE

full rank



600



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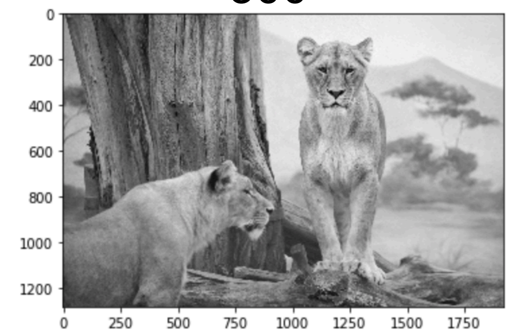

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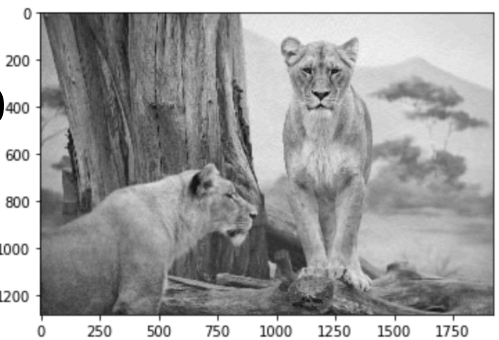

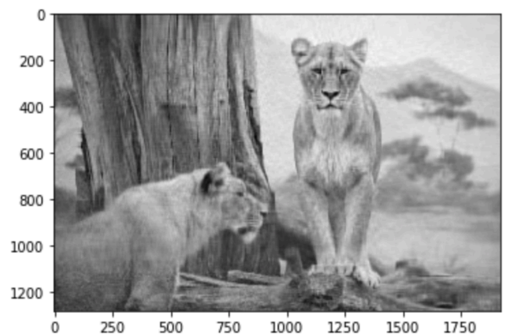

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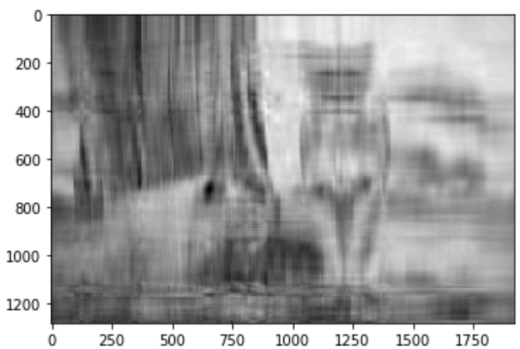
20

10



$$U[:,k]S[:k]V^T[:k,:]$$