Continuing operations at a pixel level

Cross - Correlation & Convolution

Correlation & Convolution

- Basic operation to extract information from an image.
- These operations have two key features:
  - shift invariant
  - linear
- Applicable to 1-D and multi dimensional images.

Correlation Example - 1D (Averaging)

Image I

2	3	6	5	5	1	8	9	7

$G = f(I)$

$I[2] = 3$

$$G[2] = \frac{2+3+6}{3} = \frac{11}{3}$$

2	$\frac{11}{3}$	6	5	5	1	8	9	7

$I[3] = 6$

$$G[3] = \frac{3+6+5}{3} = \frac{14}{3}$$

2	$\frac{11}{3}$	$\frac{14}{3}$	5	5	1	8	9	7

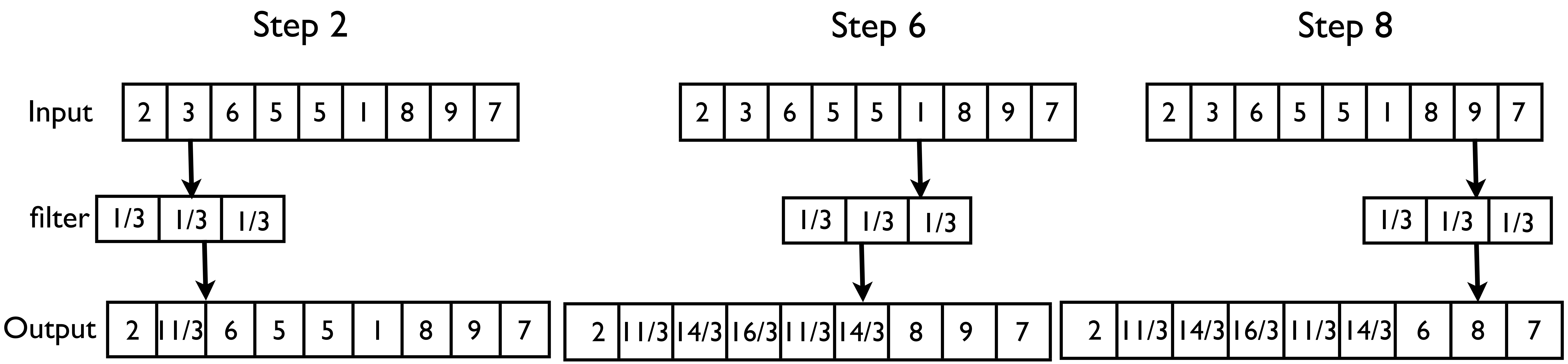
⋮

$I[8] = 9$

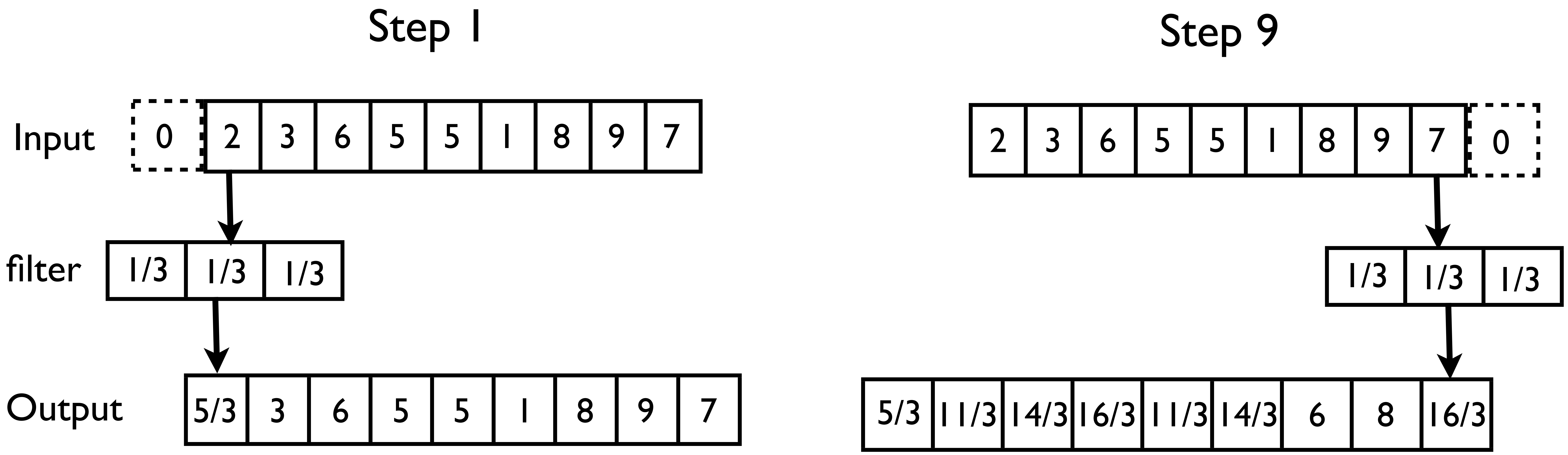
$$G[8] = \frac{8+9+7}{3} = 8$$

2	$\frac{11}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	6	8	7

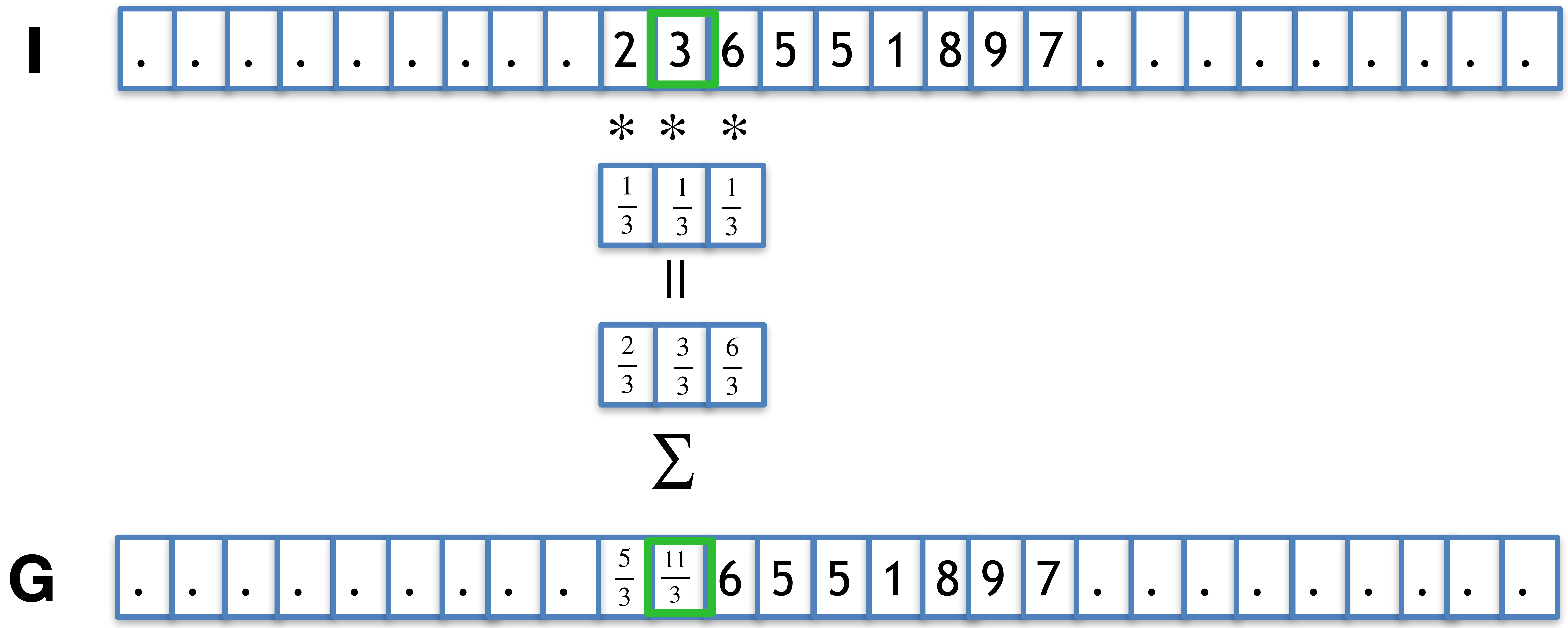
Correlation Example - 1D



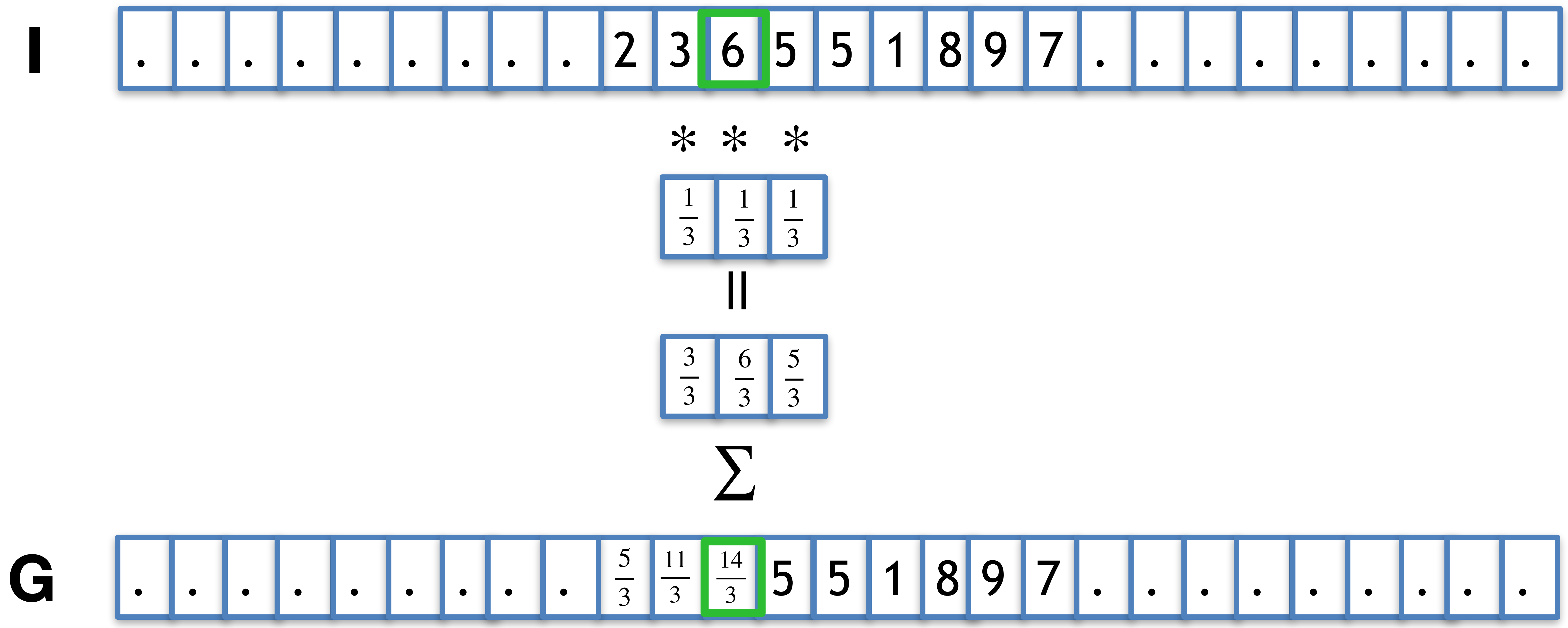
Correlation Example - 1D



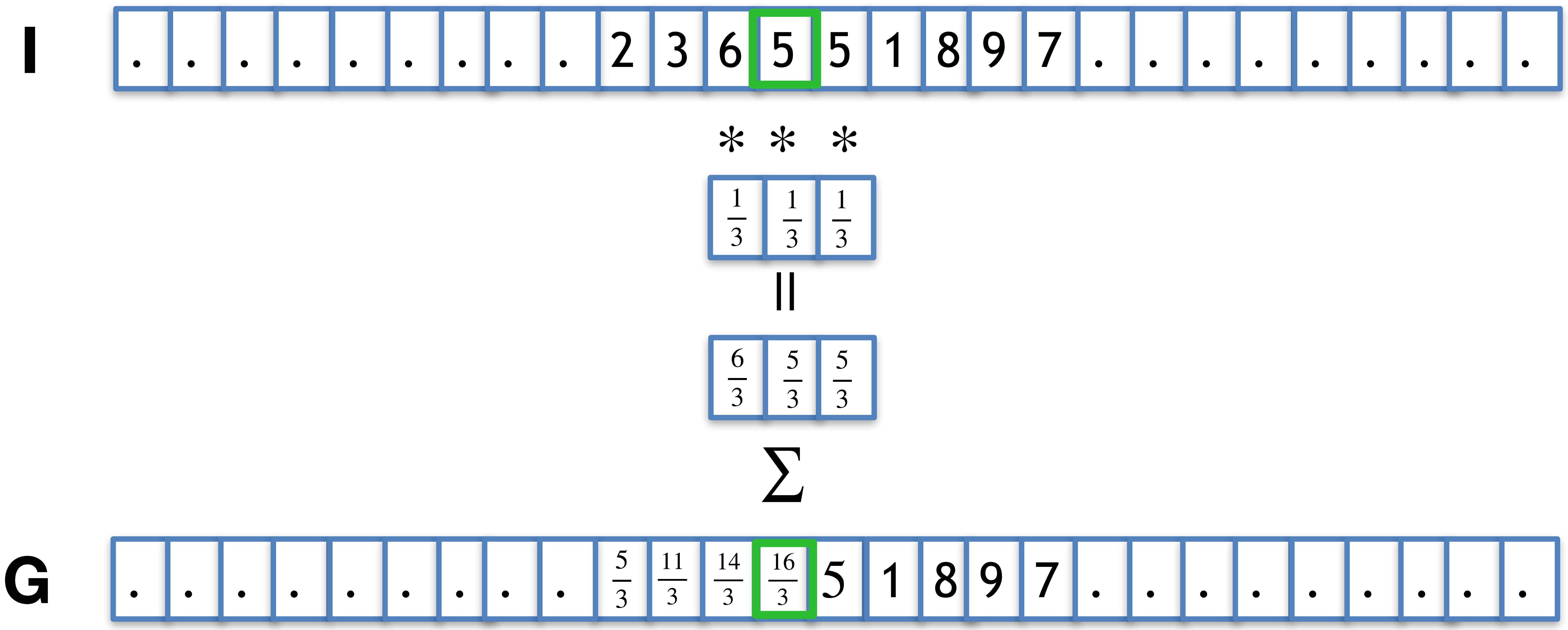
Correlation Example - 1D



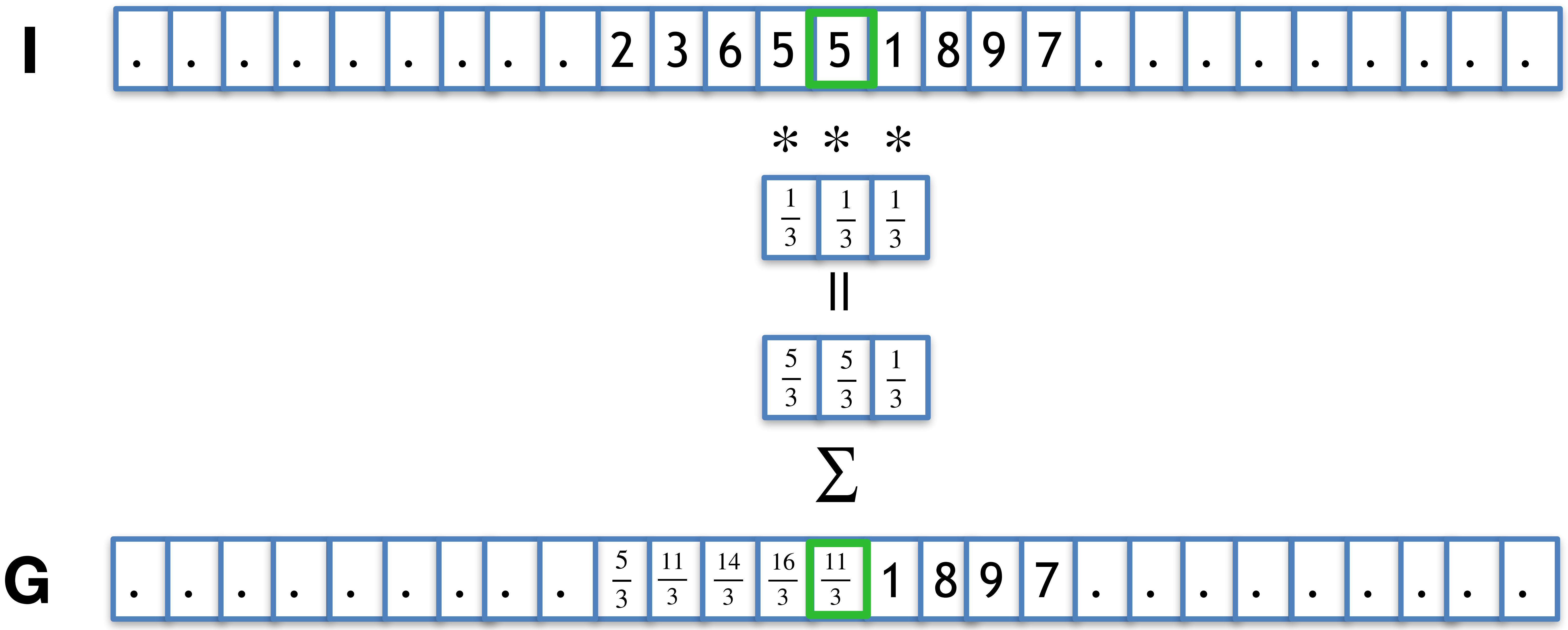
Correlation Example - 1D



Correlation Example - 1D



Correlation Example - 1D



Correlation Example - 1D

I

.	.	.	.	.	.	.	.	.	2	3	6	5	5	**1**	8	9	7	.	.	.	.	.	.	.	.	.

$* \quad * \quad *$

$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

||

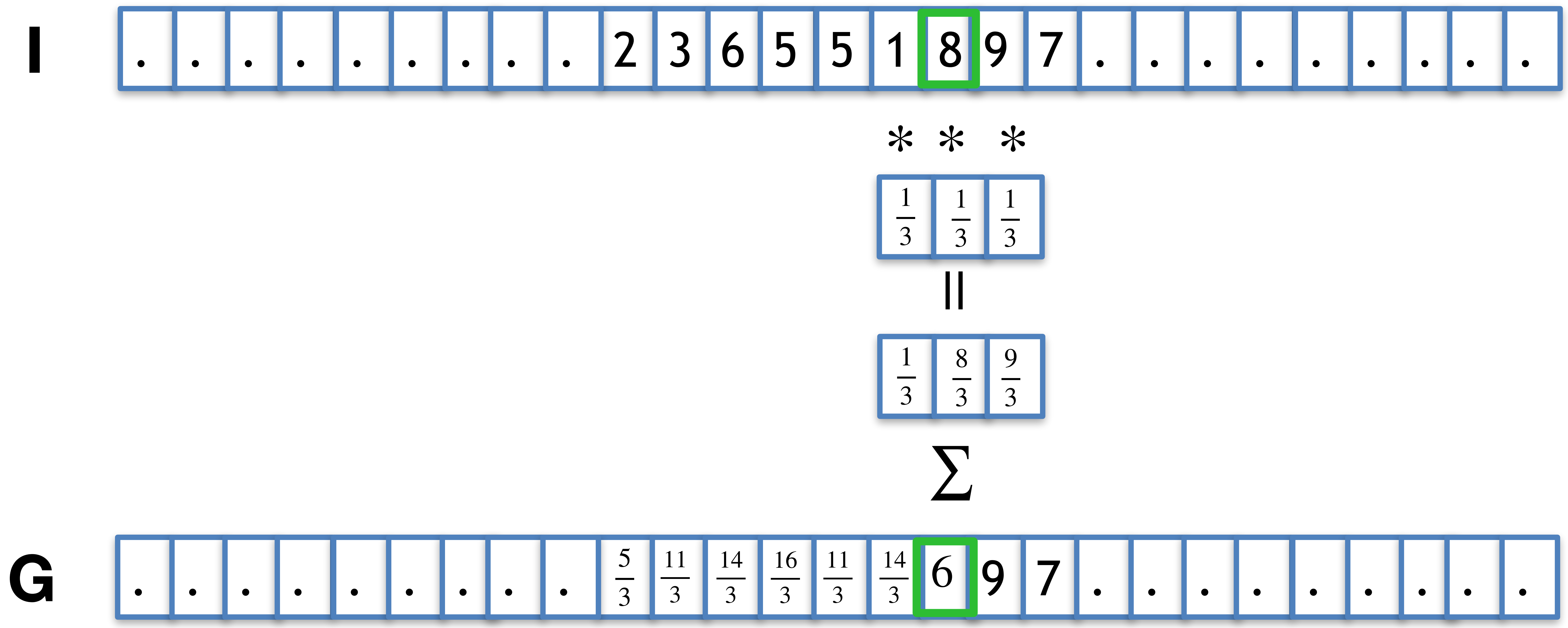
$\frac{5}{3}$	$\frac{1}{3}$	$\frac{8}{3}$

Σ

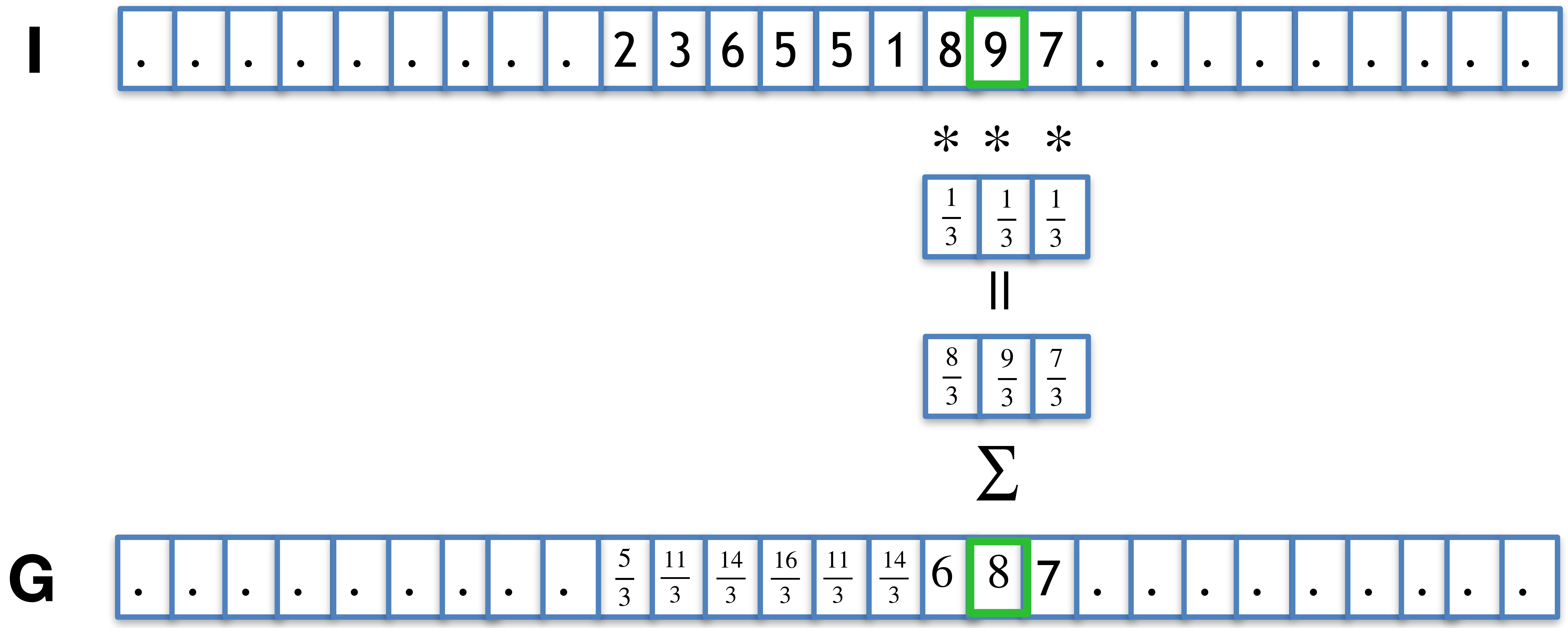
G

.	.	.	.	.	.	.	.	.	$\frac{5}{3}$	$\frac{11}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	$\frac{11}{3}$	**$\frac{14}{3}$**	8	9	7	.	.	.	.	.	.	.	.	.

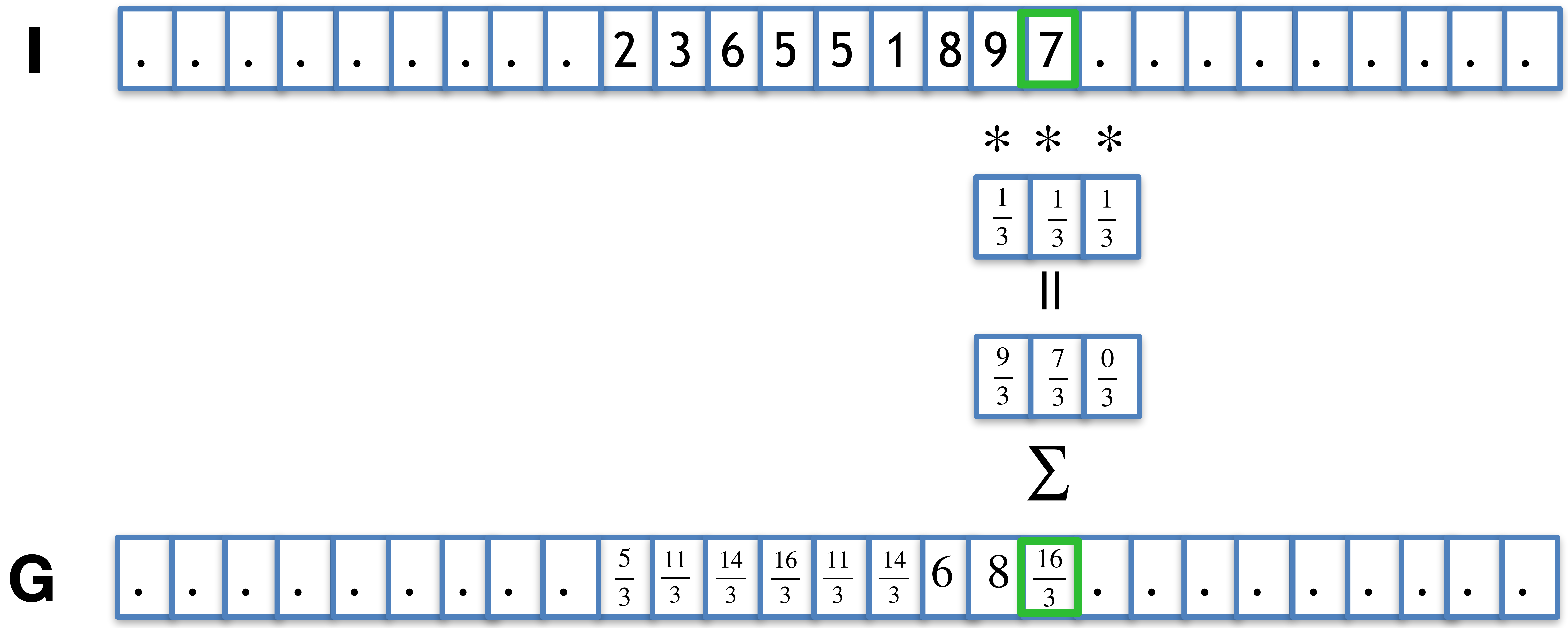
Correlation Example - 1D



Correlation Example - 1D



Correlation Example - 1D



Cross-Correlation and Convolution

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

Cross-Correlation and Convolution

$$2 \times 15 + 1 \times 1 = 31$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31				

Cross-Correlation and Convolution

$$-2 \times 5 + 2 \times 4 - 1 \times 10 + 5 \times 1 = 7$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7			

Cross-Correlation and Convolution

$$-2 \times 15 - 1 \times 1 + 1 \times 1 = -30$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30		

Cross-Correlation and Convolution

$-2 \times 4 - 1 \times 2 - 5 \times 1 = -15$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	

Cross-Correlation and Convolution

$$-1 \times 1 = -1$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1

Cross-Correlation and Convolution

$$15 \times 1 + 2 \times 1 + 9 \times 1 = 26$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26				

Cross-Correlation and Convolution

$$5 \times (-1) + 10 \times (-2) + 6 \times (-1) + 4 \times 1 + 5 \times 2 + 11 \times 1 = -6$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6			

Cross-Correlation and Convolution

$$15 \times (-1) + 1 \times (-2) + 9 \times (-1) + 0 \times 1 + 1 \times 2 + 1 \times 1 = -23$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23		

Cross-Correlation and Convolution

$$4 \times (-1) + 5 \times (-2) + 11 \times (-1) + (-1) \times 1 + 0 \times 2 + (-1) \times 1 = -27$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	

Cross-Correlation and Convolution

$$0 \times (-1) + 1 \times (-2) + 1 \times (-1) = -3$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3

Cross-Correlation and Convolution

$$1 \times 1 + 9 \times 2 + -1 \times 1 = 18$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18				

Cross-Correlation and Convolution

$$10 \times (-1) + 6 \times (-2) + 5 \times 1 + 11 \times 2 + 5 \times 1 = 10$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10			

Cross-Correlation and Convolution

$$1 \times (-1) + 9 \times (-2) + (-1) \times (-1) + 1 \times 1 + 2 \times 1 + 15 \times 1 = 0$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0		

Cross-Correlation and Convolution

$$5 \times (-1) + 11 \times (-2) + 5 \times (-1) + 0 \times 1 + (-1) \times 2 + 4 \times 1 = -30$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	

Cross-Correlation and Convolution

$$1 \times (-1) + 1 \times (-2) + 15 \times (-1) + 0 \times 1 + 0 \times 2 + 0 \times 1 = -18$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18

Cross-Correlation and Convolution

$$0\times(-1)+0\times(-2)+0\times(-1)+9\times 1+(-1)\times 2+0\times 1=7$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7				

Cross-Correlation and Convolution

$$6 \times (-1) + 0 \times (-2) + 1 \times (-1) + 11 \times 1 + 5 \times 2 + 10 \times 1 = 24$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24			

Cross-Correlation and Convolution

$$9 \times (-1) + -1 \times (-2) + 0 \times (-1) + 1 \times 1 + 15 \times 2 + 1 \times 1 = 25$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25		

Cross-Correlation and Convolution

$$11 \times (-1) + 5 \times (-2) + 10 \times (-1) + (-1) \times 1 + 4 \times 2 + 5 \times 1 = -19$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	

Cross-Correlation and Convolution

$$1 \times (-1) + 15 \times (-2) + 1 \times (-1) + 0 \times 1 + 0 \times 2 + 0 \times 1 = -32$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32

Cross-Correlation and Convolution

$$0 \times (-1) + 0 \times (-2) + 0 \times (-1) + (-1) \times 1 + 0 \times 2 + 0 \times 1 = -1$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32
-1				

Cross-Correlation and Convolution

$$0 \times (-1) + 1 \times (-2) + 0 \times (-1) + 5 \times 1 + 10 \times 2 + 0 \times 1 = 23$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32
-1	23			

Cross-Correlation and Convolution

$$-1 \times (-1) + 0 \times (-2) + 0 \times (-1) + 15 \times 1 + 01 \times 2 + 0 \times 1 = 16$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32
-1	23	16		

Cross-Correlation and Convolution

$$5 \times (-1) + 10 \times (-2) + 0 \times (-1) + 4 \times 1 + 5 \times 2 + 0 \times 1 = -11$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32
-1	23	16	-11	

Cross-Correlation and Convolution

$$15 \times (-1) + 1 \times (-2) + 0 \times (-1) + 0 \times 1 + 0 \times 2 + 0 \times 1 = -17$$

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32
-1	23	16	-11	-17

Cross-Correlation and Convolution

5	15	4	0	-1
10	1	5	1	0
6	9	11	1	-1
0	-1	5	15	4
1	0	10	1	5

$\circ$

-1	0	1
-2	0	2
-1	0	1

=

31	-7	-30	-15	-1
26	-6	-23	-27	-3
18	10	0	-30	-18
7	24	25	-19	-32
-1	23	16	-11	-17

Image, I　　Filter/template　　Output image

Cross-Correlation - Mathematically

$1D$

$$G = F \circ I[i] = \sum_{u=-k}^{k} F[u]I[i+u]$$ F has $2k+1$ elements

Box filter $F[u] = \frac{1}{3}$ for $u = -1,0,1$ and 0 otherwise

Cross-correlation filtering - 2D

Let's write this down as an equation. Assume the averaging window is (2k+1)x(2k+1):

$$G[i,j] = \frac{1}{(2k+1)^2} \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[i+u, j+v]$$

We can generalize this idea by allowing different weights for different neighboring pixels:

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[u,v] I[i+u, j+v]$$

This is called a **cross-correlation** operation and written:

$$G = F \circ I$$

F is called the "filter," "kernel," or "mask."

Convolution

Filter is flipped before correlating

$1D$ F has $2k+1$ elements

$$G = F * I[i] = \sum_{u=-k}^{k} F[u]I[i-u]$$

Box filter $F[u] = \frac{1}{3}$ for $u = -1,0,1$ and 0 otherwise

for example, convolution of 1D image with the filter [3,5,2]

is exactly the same as correlation with the filter [2,5,3]

Convolution filtering - 2D

For 2D the filter is flipped both horizontally and vertically

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[u,v] I[i-u, j-v]$$

Convolution with the filter

1	2	1
0	0	0
-1	-2	-1

is the same as Correlation with the filter

-1	-2	-1
0	0	0
1	2	1

Correlation and convolution are identical for symmetrical filters

Correlation and Convolution

- Convolution is associative:
  if F and G are filters, then $F * (G * I) = (F * G) * I$

- Generally, convolution is used for image processing operations like smoothing

- Correlation is used for template matching to an image.

- We combine two convolution filters but not two correlation filters.

Correlation and Convolution Terminology

We used

G for correlation/convolution output

I for image - In literature sometimes F is used for image

F for filter - In literature sometimes H is used for filter

$$G = H \circ F$$

$$G = H * F$$

Filter Image

Mean kernel

What's the kernel for a 3x3 mean filter?

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$F[i, j]$

1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$H[u, v]$

Box filter

Mean filtering (average over a neighborhood)

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$H[u, v]$

$G[x, y]$

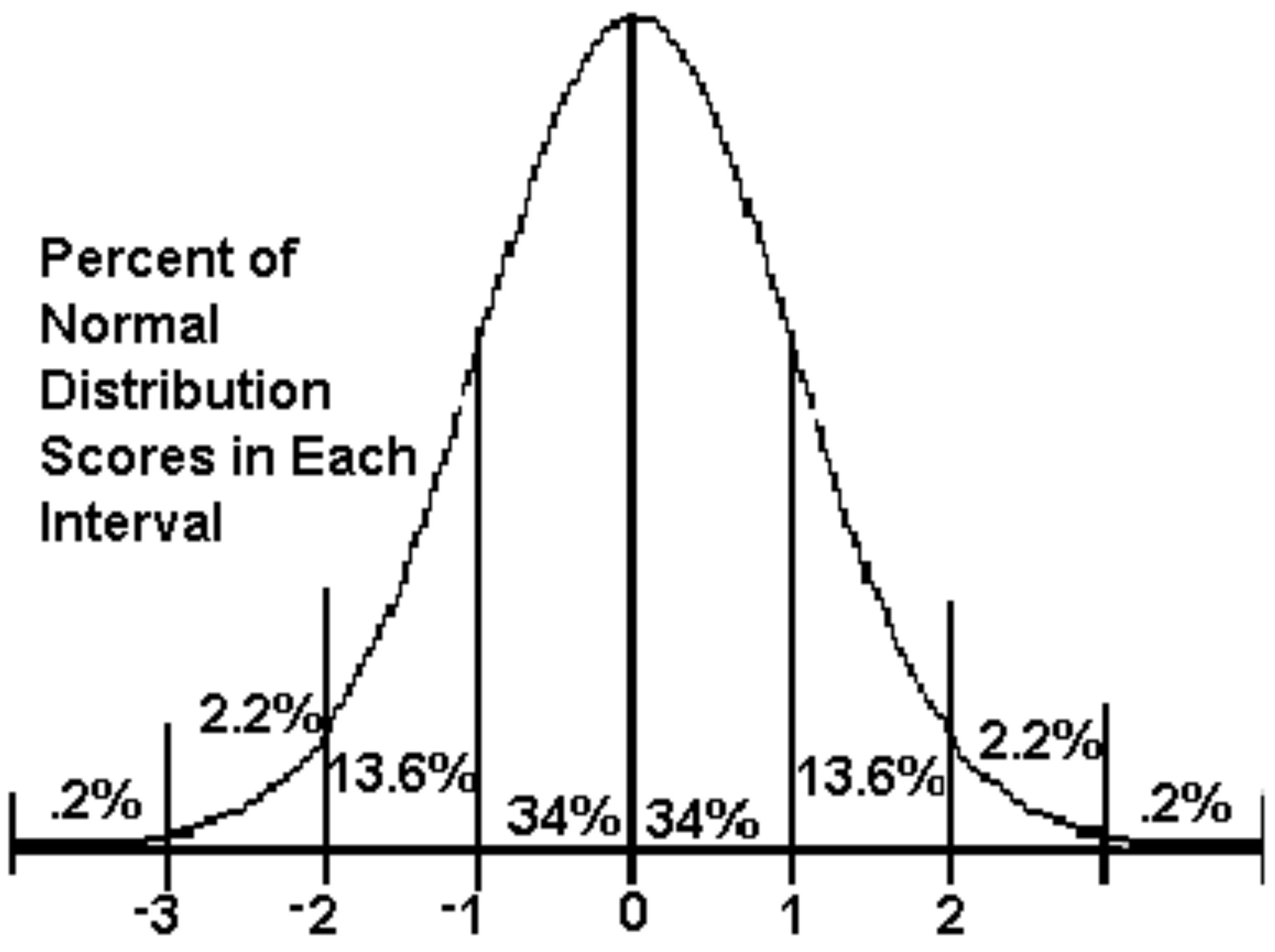
	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Gaussian Averaging

Rotationally symmetric.

Weights nearby pixels more than distant ones.

- This makes sense as probabilistic inference.

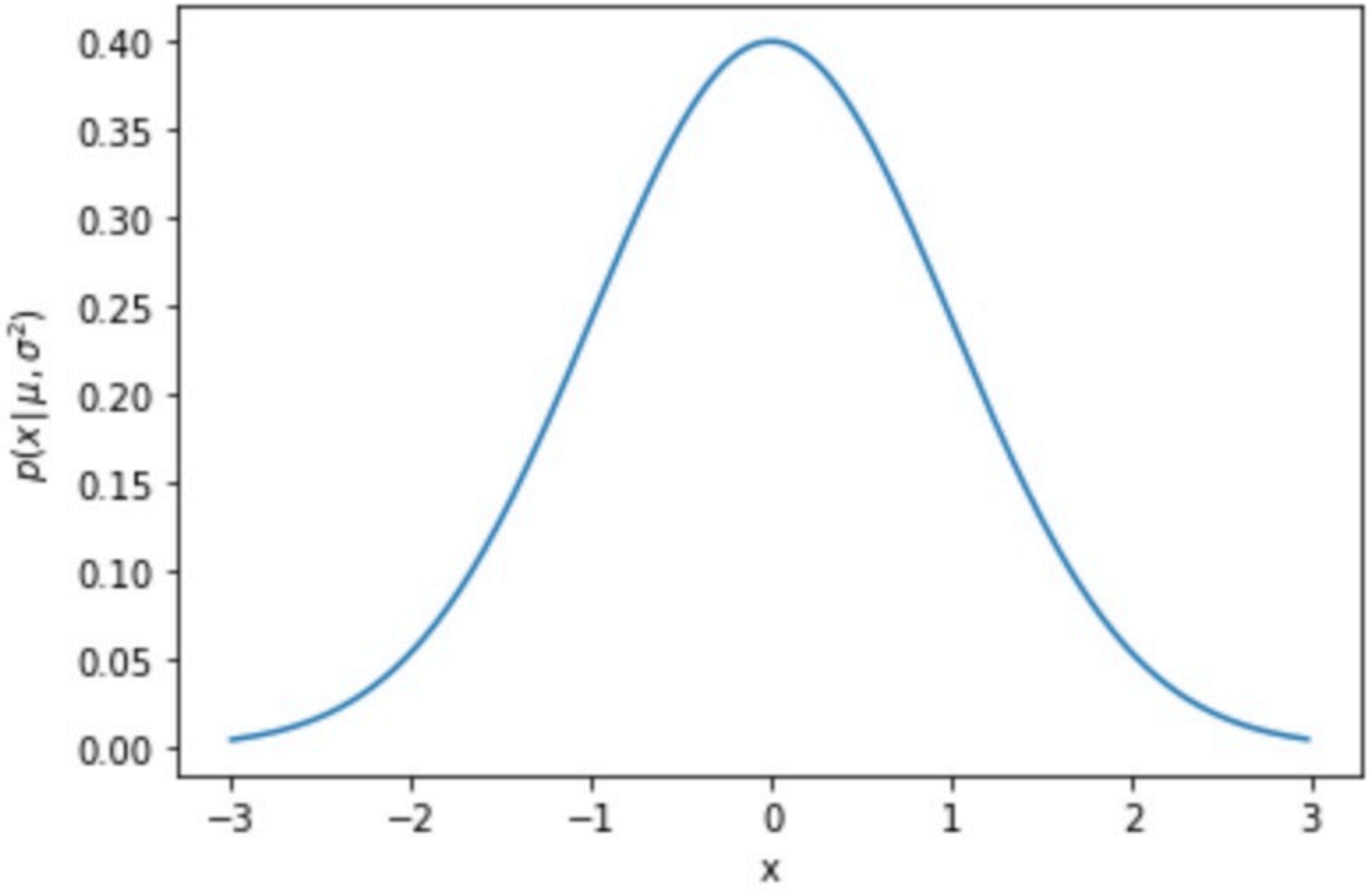


A Gaussian gives a good model of a fuzzy blob

Notation: Normal distribution 1D case

N(μ , σ) is a 1D normal (Gaussian) distribution with mean μ and standard deviation σ (so the variance is σ^2.

$$\mathcal{N}(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



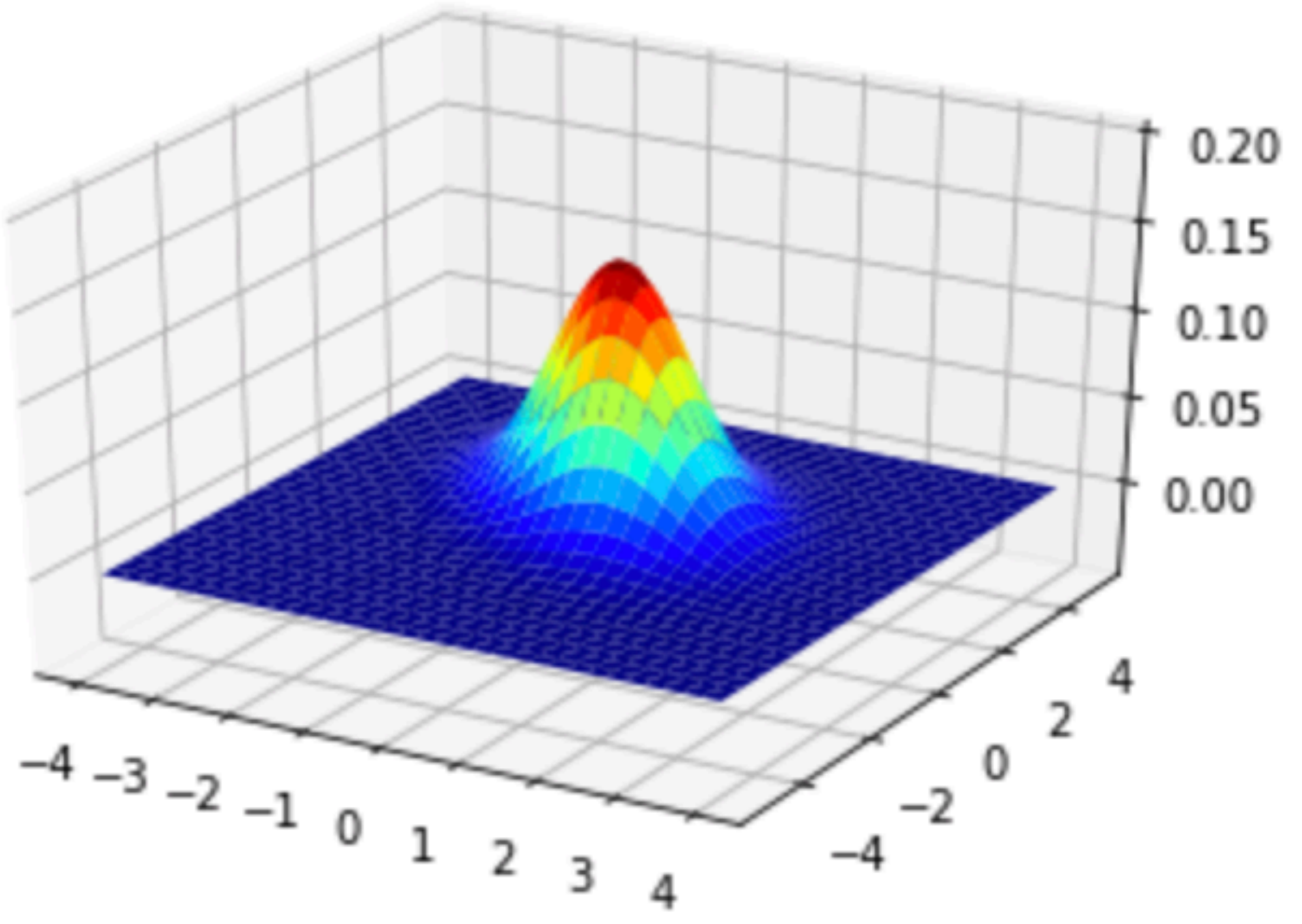
Multivariate Normal distribution

$$\mathcal{N}(z \mid \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{D}{2}}} \frac{1}{|\Sigma|^{\frac{1}{2}}} exp\left\{ -\frac{1}{2}(z-\mu)^T \Sigma^{-1} (z-\mu) \right\}$$

z is a D dimensional vector

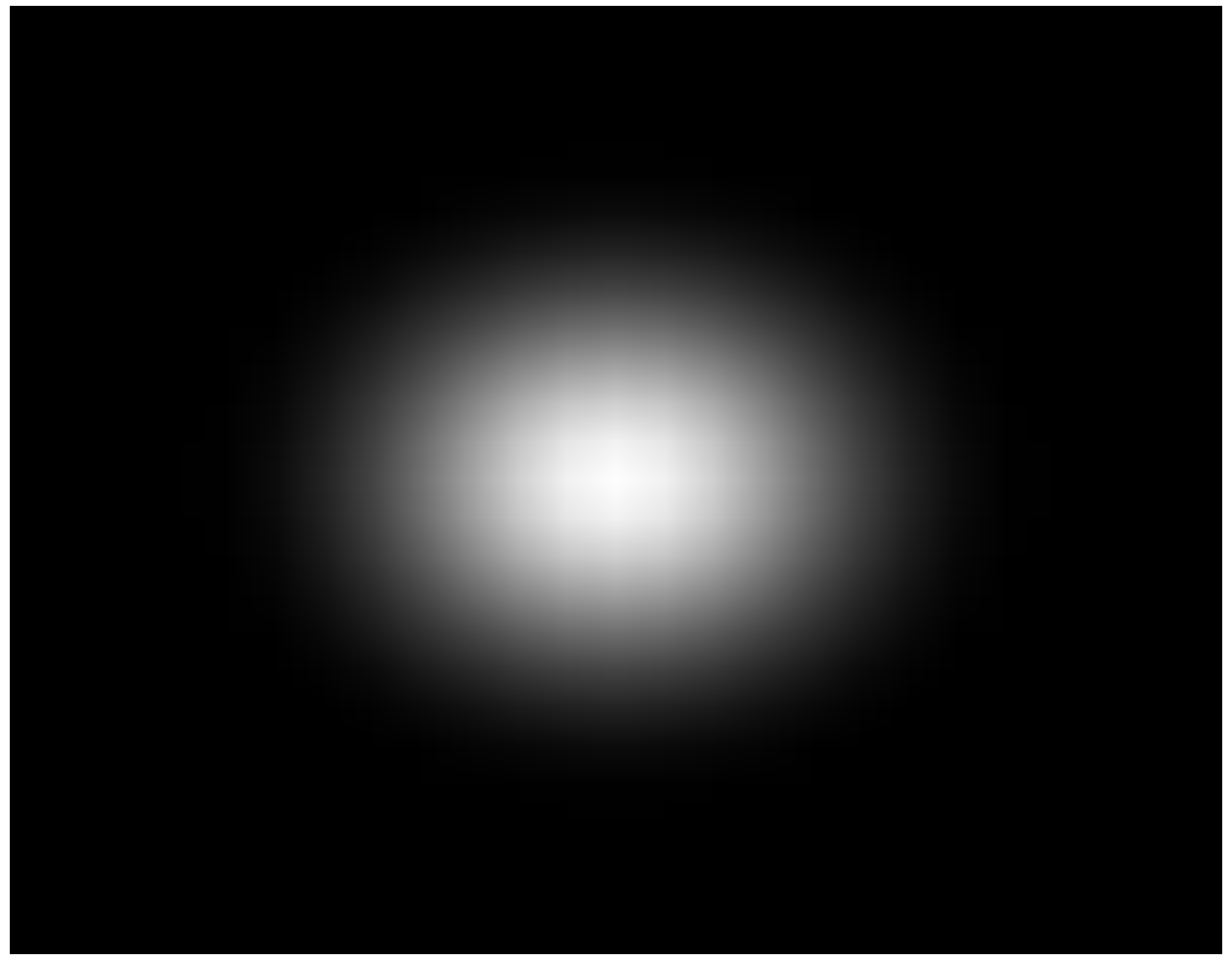
μ is a D-dimensinal mean vector

Σ is a D x D covariance matrix



Example: when D =2 $\quad z = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$

An Isotropic Gaussian



The picture shows a smoothing kernel proportional to

$$\exp\left(-\left(\frac{x^2+y^2}{2\sigma^2}\right)\right)$$

(which is a reasonable model of a circularly symmetric fuzzy blob)

Gaussian Filtering

A Gaussian kernel gives less weight to pixels further from the center of the window

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$F[x, y]$

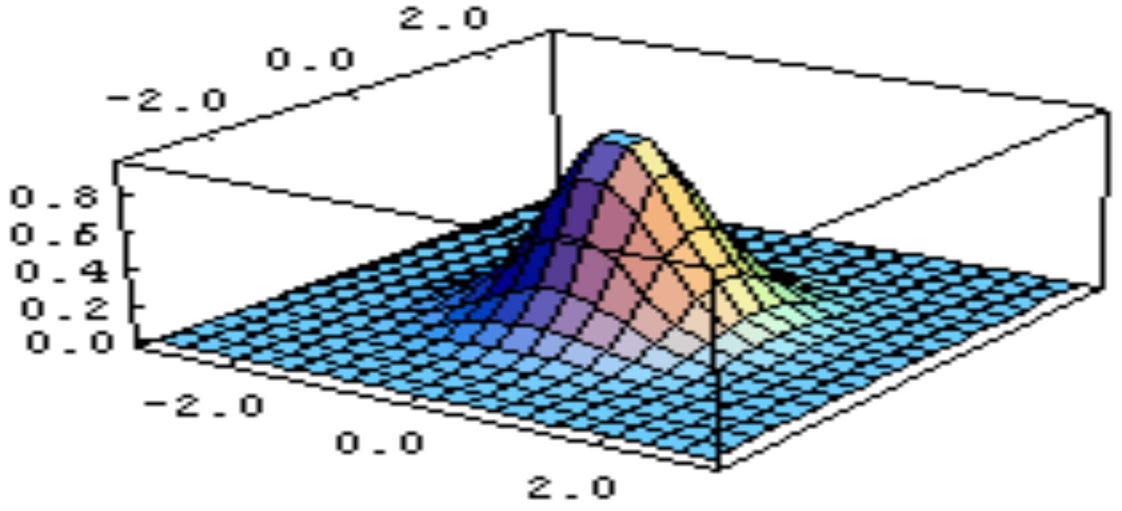
$\frac{1}{16}$

1	2	1
2	4	2
1	2	1

$H[u, v]$

This kernel is an approximation of

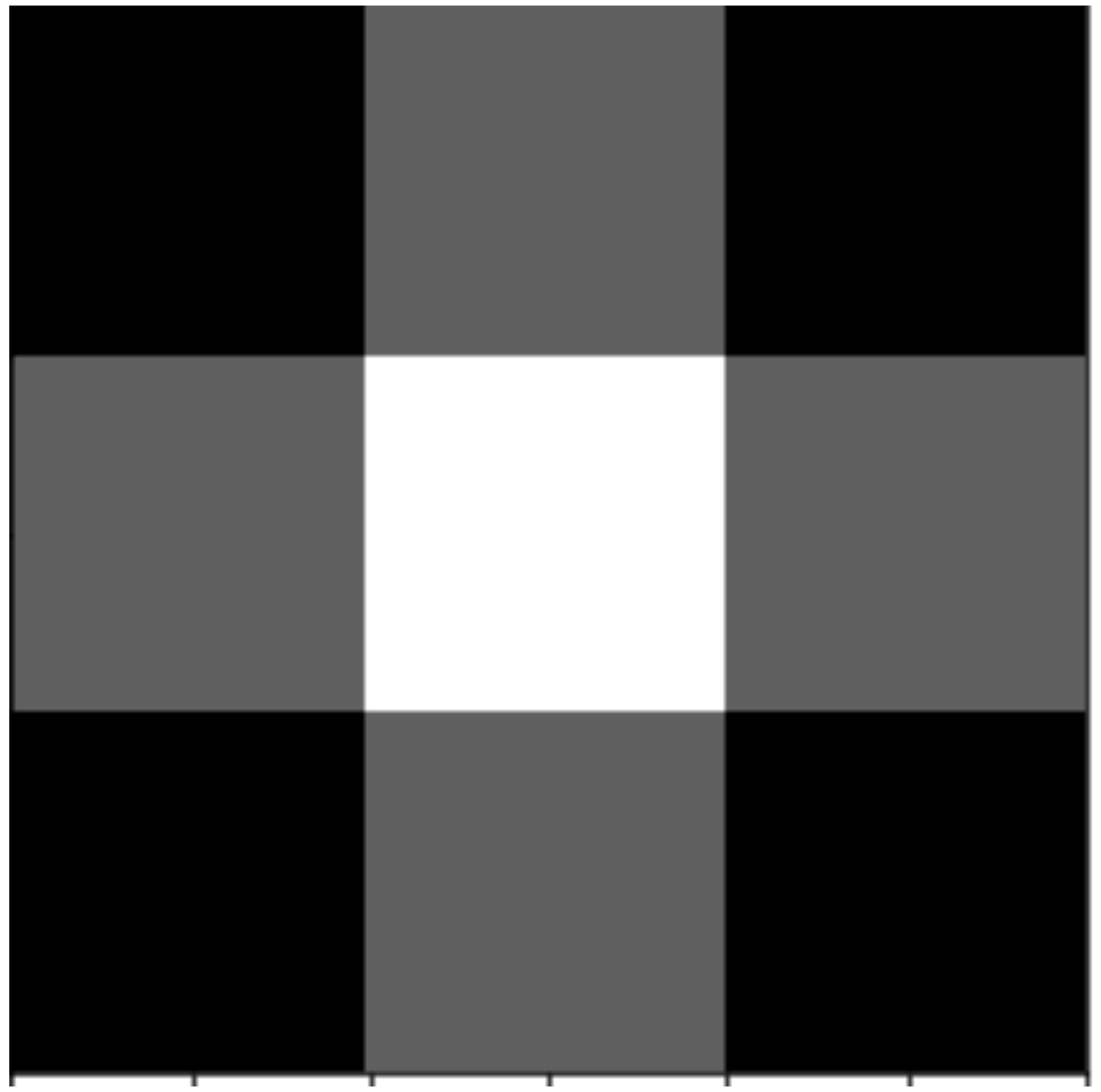
$$h(u, v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{\sigma^2}}$$

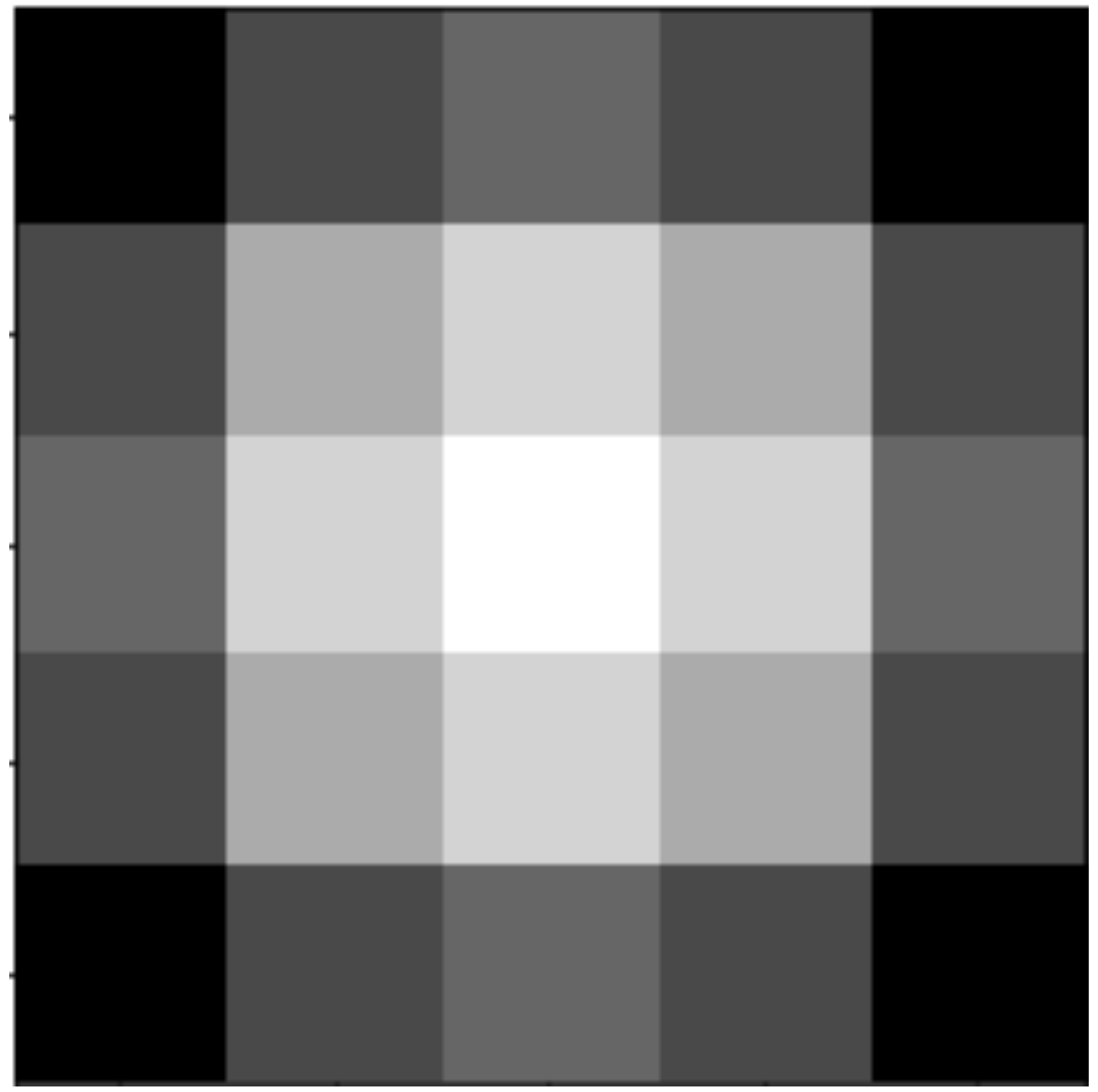


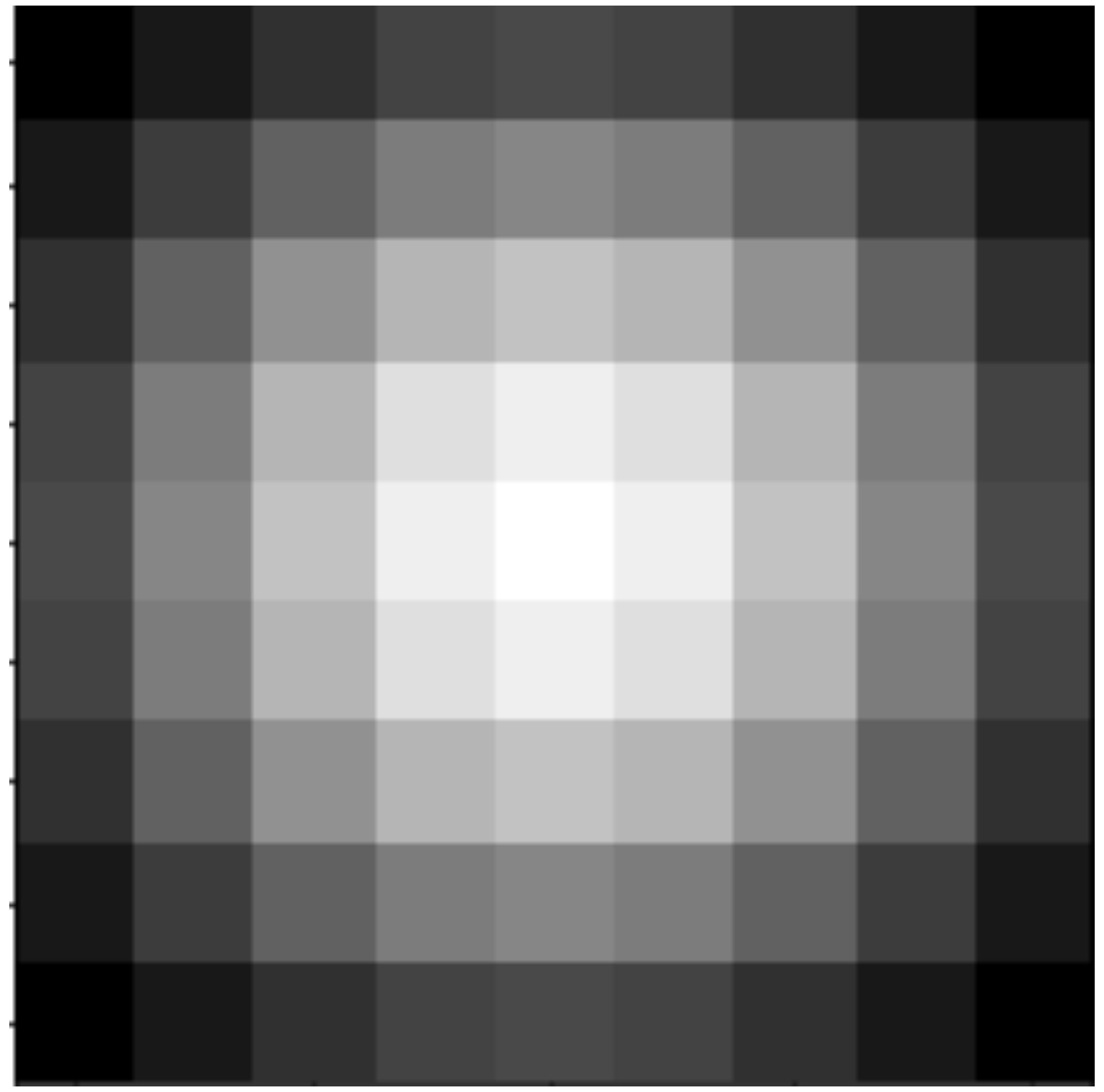
The size of the mask

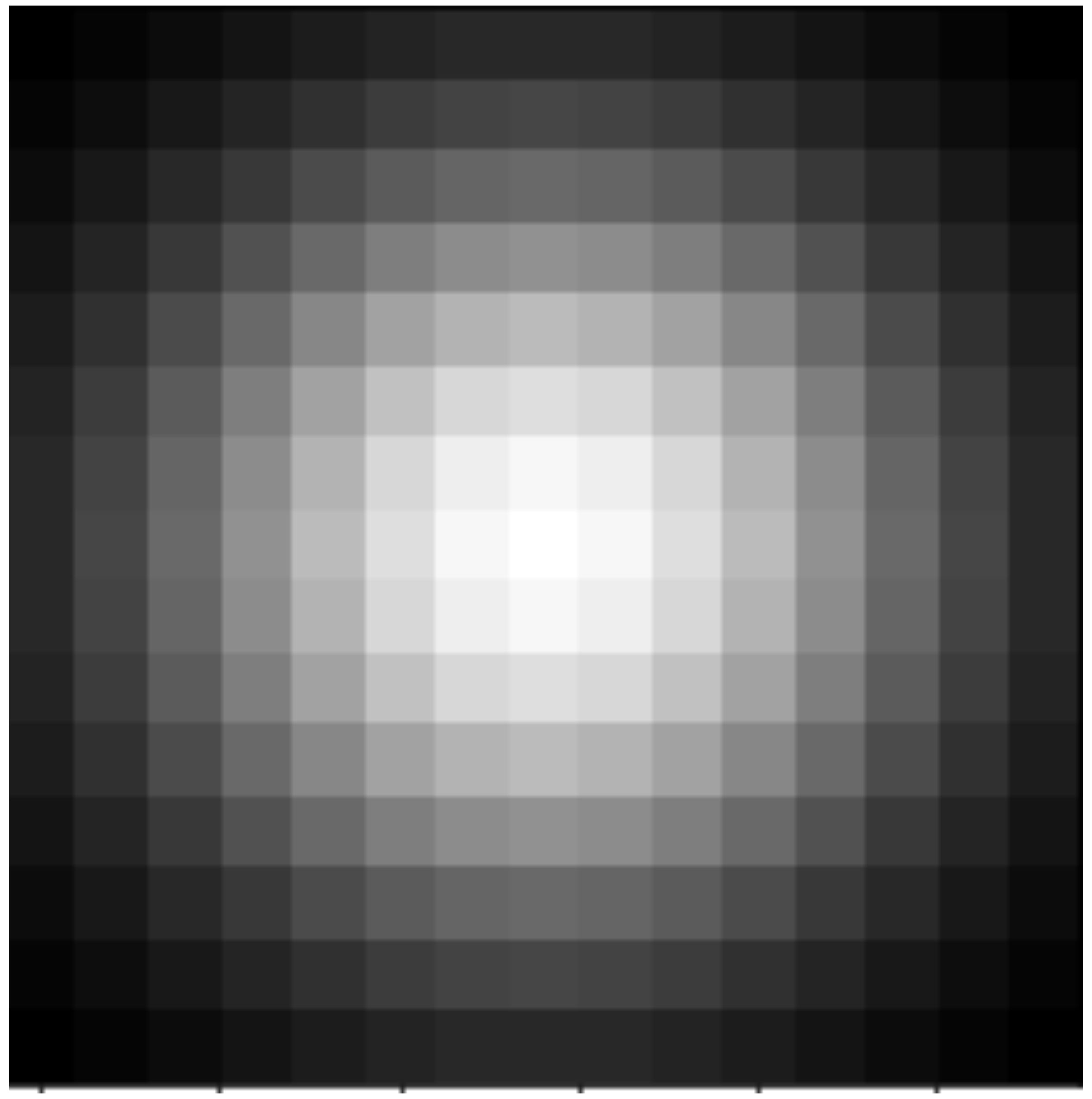
- Bigger mask:
  - more neighbors contribute.
  - smaller noise variance of the output.
  - bigger noise spread.
  - more blurring.
  - more expensive to compute.

Gaussians masks of different sizes



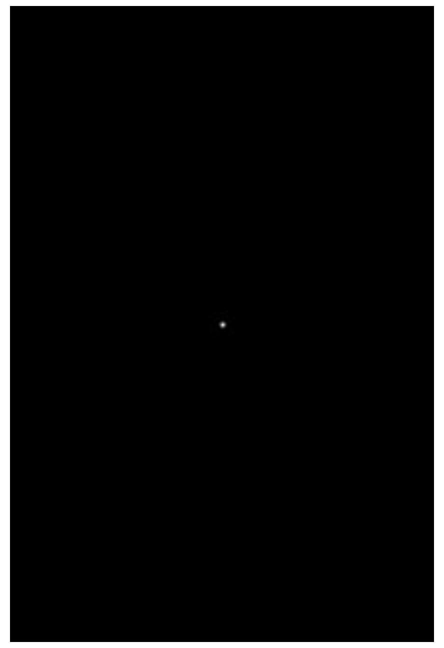






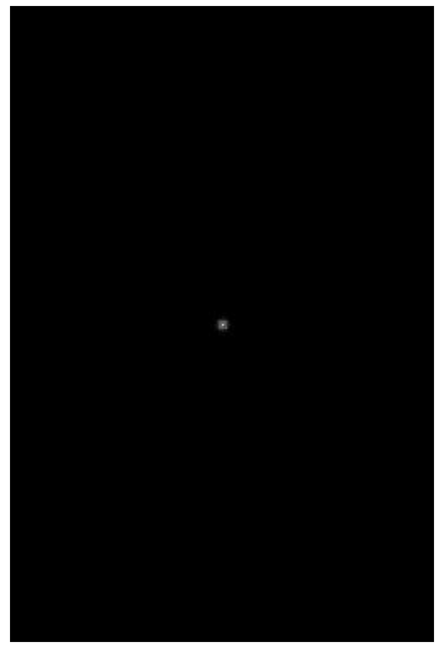
Convolution with masks of different sizes

*



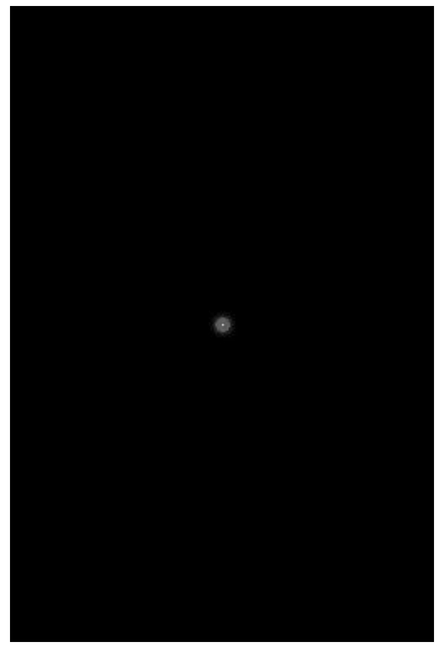
$\sigma = 1$

*



$\sigma = 2$

*



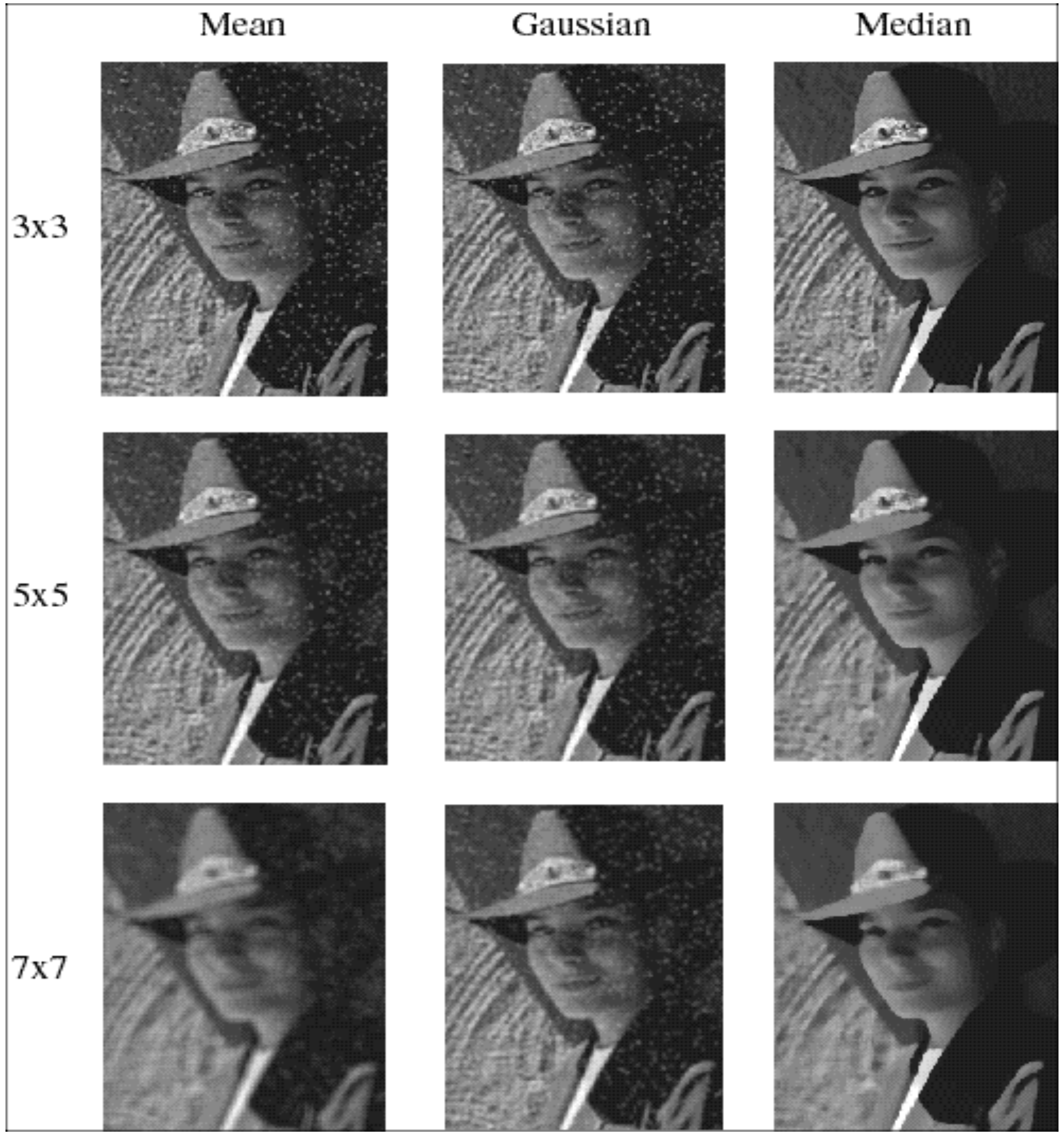
$\sigma = 3$

Gaussian filters

- Remove “high-frequency” components from the image (low-pass filter)
- Convolution with self is another Gaussian
- Separable kernel
  - Factors into product of two 1D Gaussians

Median filter

Comparison: salt and pepper noise



Comparison: Gaussian noise