CMSC/PHYS 457, Introduction to Quantum Computing
Spring 2021, University of Maryland

Due: March 5th, 2021
Date: February 20, 2021

Assignment 3

Please submit it electronically to ELMS. This assignment is 7% in your final grade. For the simplicity of the grading, the total number of points for the assignment is 70.

Problem 1. *The Bernstein-Vazirani problem.*

1. *(3 points)* Suppose $f : \{0,1\}^n \to \{0,1\}$ is a function of the form

$$f(\underline{x}) = x_1 s_1 + x_2 s_2 + \cdots + x_n s_n \bmod 2$$

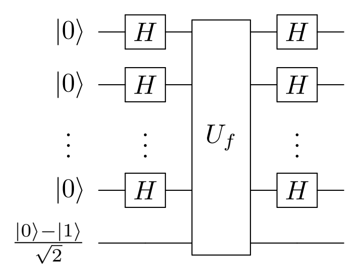
for some unknown $\underline{s} \in \{0,1\}^n$. Given a black box for f, how many classical queries are required to learn s with certainty?

2. *(4 points)* Prove that for any n-bit string $\underline{u} \in \{0,1\}^n$,

$$\sum_{\underline{v} \in \{0,1\}^n} (-1)^{\underline{u} \cdot \underline{v}} = \begin{cases} 2^n & \text{if } \underline{u} = \underline{0} \\ 0 & \text{otherwise} \end{cases}$$

where $\underline{0}$ denotes the n-bit string $00\ldots0$.

3. *(4 points)* Let U_f denote a quantum black box for f, acting as $U_f|\underline{x}\rangle|y\rangle = |\underline{x}\rangle|y \oplus f(\underline{x})\rangle$ for any $\underline{x} \in \{0,1\}^n$ and $y \in \{0,1\}$. Show that the output of the following circuit is the state $|\underline{s}\rangle(|0\rangle - |1\rangle)/\sqrt{2}$.



4. *(1 points)* What can you conclude about the quantum query complexity of learning s?

Problem 2. *One-out-of-four search.* Let $f\colon \{0,1\}^2 \to \{0,1\}$ be a black-box function taking the value 1 on exactly one input. The goal of the one-out-of-four search problem is to find the unique $(x_1, x_2) \in \{0,1\}^2$ such that $f(x_1, x_2) = 1$.

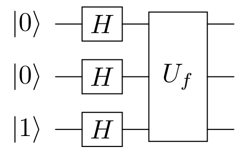
1. *(2 points)* Write the truth tables of the four possible functions f.

2. *(3 points)* How many classical queries are needed to solve one-out-of-four search?

3. *(7 points)* Suppose f is given as a quantum black box U_f acting as

$$|x_1, x_2, y\rangle \overset{U_f}{\mapsto} |x_1, x_2, y \oplus f(x_1, x_2)\rangle.$$

Determine the output of the following quantum circuit for each of the possible black-box functions f:



4. *(3 points)* Show that the four possible outputs obtained in the previous part are pairwise orthogonal. What can you conclude about the quantum query complexity of one-out-of-four search?

Problem 3. *Implementing the square root of a unitary.*

1. *(3 points)* Let U be a unitary operation with eigenvalues ± 1. Let P_0 be the projection onto the $+1$ eigenspace of U and let P_1 be the projection onto the -1 eigenspace of U. Let $V = P_0 + iP_1$. Show that $V^2 = U$.

2. *(3 points)* Give a circuit of 1- and 2-qubit gates and controlled-U gates with the following behavior (where the first register is a single qubit):

$$|0\rangle|\psi\rangle \mapsto \begin{cases} |0\rangle|\psi\rangle & \text{if } U|\psi\rangle = |\psi\rangle \\ |1\rangle|\psi\rangle & \text{if } U|\psi\rangle = -|\psi\rangle. \end{cases}$$

3. *(4 points)* Give a circuit of 1- and 2-qubit gates and controlled-U gates that implements V. Your circuit may use ancilla qubits that begin and end in the $|0\rangle$ state.

Problem 4. *Determining the "slope" of a linear function over $\mathbb{Z}_4$.* Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, with arithmetic operations of addition and multiplication defined with respect to modulo 4 arithmetic on this set. Suppose that we are given a black-box computing a linear function $f : \mathbb{Z}_4 \to \mathbb{Z}_4$, which of the form $f(x) = ax+b$, with unknown coefficients $a, b \in \mathbb{Z}_4$ (throughout this question, multiplication and addition mean these operations in modulo 4 arithmetic). Let our goal be to determine the coefficient a (the "slope" of the function). We will consider the number of quantum and classical queries needed to solve this problem.

Assume that what we are given is a black box for the function f that is in reversible form in the following sense. For each $x, y \in \mathbb{Z}_4$, the black box maps (x, y) to $(x, y + f(x))$ in the classical case; and $|x\rangle|y\rangle$ to $|x\rangle|y + f(x)\rangle$ in the quantum case (which is unitary).

Also, note that we can encode the elements of $\mathbb{Z}_4$ into 2-bit strings, using the usual representation of integers as a binary strings ($00 = 0$, $01 = 1$, $10 = 2$, $11 = 3$). With this encoding, we can view f as a function on 2-bit strings $f : \{0,1\}^2 \to \{0,1\}^2$. When refering to the elements of $\mathbb{Z}_4$, we use the notation $\{0, 1, 2, 3\}$ and $\{00, 01, 10, 11\}$ interchangeably.

(1) *(5 points)* Prove that every classical algorithm for solving this problem must make two queries.

(2) *(5 points)* Consider the 2-qubit unitary operation A corresponding to "add 1", such that $A|x\rangle = |x + 1\rangle$ for all $x \in \mathbb{Z}_4$. It is easy to check that

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $|\psi\rangle = \frac{1}{2}(|00\rangle + i|01\rangle + i^2|10\rangle + i^3|11\rangle)$, where $i = \sqrt{-1}$. Prove that $A|\psi\rangle = -i|\psi\rangle$.

(3) *(5 points)* Show how to create the state $\frac{1}{2}((-i)^{f(00)}|00\rangle + (-i)^{f(01)}|01\rangle + (-i)^{f(10)}|10\rangle + (-i)^{f(11)}|11\rangle)$ with a single query to U_f. (Hint: you may use the result in part (2) for this.)

(4) *(5 points)* Show how to solve the problem (i.e., determine the coefficient $a \in \mathbb{Z}_4$) with a single quantum query to f. (Hint: you may use the result in part (3) for this.)

Problem 5. *Searching for a quantum state.*

Suppose you are given a black box U_ϕ that identifies an unknown quantum state $|\phi\rangle$ (which may not be a computational basis state). Specifically, $U_\phi|\phi\rangle = -|\phi\rangle$, and $U_\phi|\xi\rangle = |\xi\rangle$ for any state $|\xi\rangle$ satisfying $\langle\phi|\xi\rangle = 0$.

Consider an algorithm for preparing $|\phi\rangle$ that starts from some fixed state $|\psi\rangle$ and repeatedly applies the unitary transformation VU_ϕ, where $V = 2|\psi\rangle\langle\psi| - I$ is a reflection about $|\psi\rangle$.

Let $|\phi^\perp\rangle = \frac{e^{-i\lambda}|\psi\rangle - \sin(\theta)|\phi\rangle}{\cos(\theta)}$ denote a state orthogonal to $|\phi\rangle$ in $\mathrm{span}\{|\phi\rangle, |\psi\rangle\}$, where $\langle\phi|\psi\rangle = e^{i\lambda}\sin(\theta)$ for some $\lambda, \theta \in \mathcal{R}$.

1. *(2 points)* Write the initial state $|\psi\rangle$ in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$.
2. *(3 points)* Write U_ϕ and V as matrices in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$.
3. *(3 points)* Let k be a positive integer. Compute $(VU_\phi)^k$.
4. *(3 points)* Compute $\langle\phi|(VU_\phi)^k|\psi\rangle$.
5. *(2 points)* Suppose that $|\langle\phi|\psi\rangle|$ is small. Approximately what value of k should you choose in order for the algorithm to prepare a state close to $|\phi\rangle$, up to a global phase? Express your answer in terms of $|\langle\phi|\psi\rangle|$.
