CMSC 330: Organization of Programming Languages

Tail Recursion

Factorial

$$\texttt{fact n} = \begin{cases} \texttt{1} & \texttt{n=0} \\ \texttt{n * fact (n-1)} & \texttt{n>0} \end{cases}$$

```
let rec fact n =
    if n = 0 then 1
    else n * fact (n-1)
```

Factorial

```
fact n = { 1                 n=0
         { n * fact (n-1)    n>0
```

```
fact 3 = 3 * fact 2
       = 3 * 2 * fact 1
       = 3 * 2 * 1 * fact 0
       = 3 * 2 * 1 * 1
     = 3 * 2 * 1
       = 3 * 2
       = 6
```

Stack

fact 0		1
fact 1	1	1 * fact 0
fact 2	2	2 * fact 1
fact 3	3	3 * fact 2

Stack Overflow

```
# let rec fact n = if n = 0 then 1 else n * fact (n-1);;
val fact : int -> int = <fun>
# fact 1000000 ;;
Stack overflow during evaluation (looping recursion?).
```

Yet Another Factorial

$$\texttt{aux x a} = \begin{cases} \texttt{a} & \texttt{x=0} \\ \texttt{aux (x-1) x*a} & \texttt{x>0} \end{cases}$$

```
fact n  = aux n 1
```

```
let fact n =
  let rec aux x a =
    if x = 0 then a
    else aux (x-1) x*a in
  aux n 1
```

Yet Another Factorial

$$\texttt{aux x a} = \begin{cases} \texttt{a} & x=0 \\ \texttt{aux (x-1) x*a} & x>0 \end{cases}$$

```
fact n  = aux n 1
```

```
fact 3 = aux 3 1
       = aux (3-1) 3*1 = aux 2 3
       = aux (2-1) 2*3 = aux 1 6
       = aux (1-1) 1*6 = aux 0 6
       = 6
```

Look, Ma! No Stack!

No need to push a new frame on each call

- The result of the evaluation is *exactly* the result of the recursive call – nothing to remember
- *So: Reuse the current frame*

Tail Recursion

- Whenever a function's result is completely computed by its recursive call, it is called **tail recursive**
  - Its "tail" – the last thing it does – is recursive
- Tail recursive functions can be implemented without requiring a stack frame for each call
  - No intermediate variables need to be saved, so the compiler overwrites them
- Typical pattern is to use an accumulator to build up the result, and return it in the base case

Compare fact and aux

```
let rec fact n =
  if n = 0 then 1
  else n * fact (n-1)
```

1. Compute **`fact (n-1)`**
2. Multiply it by **`n`**
3. Return result

Waits for recursive call's result to compute final result

```
let fact n =
  let rec aux x acc =
    if x = 1 then acc
    else aux (x-1) (acc*x) in
  aux n 1
```

1. Compute **`aux (x-1)`** …
2. Return it

*Final result **is** the result of the recursive call*

Quiz #1

True/false: map is tail-recursive.

```
let rec map f = function
| [] -> []
| (h::t) -> (f h)::(map f t)
```

A. True
B. False

Quiz #1

True/false: map is tail-recursive.

```
let rec map f = function
| [] -> []
| (h::t) -> (f h)::(map f t)
```

A. True

B. False

Quiz #2

True/false: fold is tail-recursive

```
let rec fold f a = function
| [] -> a
| (h::t) -> fold f (f a h) t
```

A. True
B. False

Quiz #2

True/false: fold is tail-recursive

```
let rec fold f a = function
| [] -> a
| (h::t) -> fold f (f a h) t
```

A. **True**
B. False

Quiz #3

True/false: fold_right is tail-recursive

```
let rec fold_right f l a =
  match l with
  | [] -> a
  | (h::t) -> f h (fold_right f t a)
```

A. True
B. False

Quiz #3

True/false: fold_right is tail-recursive

```
let rec fold_right f l a =
  match l with
  | [] -> a
  | (h::t) -> f h (fold_right f t a)
```

A. True
B. False

Exercise: Finish Tail-recursive Version

```
let rec sumlist l =
  match l with
  | [] -> 0
  | (x::xs) -> (sumlist xs) + x
```

Tail-recursive version:

```
let sumlist l =
  let rec helper l a =
    match l with
    | [] -> ______
    | (x::xs) -> __________ in
  helper l 0
```

Exercise: Finish Tail-recursive Version

```
let rec sumlist l =
  match l with
    [] -> 0
  | (x::xs) -> (sumlist xs) + x
```

Tail-recursive version:

```
let sumlist l =
  let rec helper l a =
    match l with
      [] -> a
    | (x::xs) -> helper xs (x+a) in
  helper l 0
```

Tail Recursion Pattern (1 argument)

```
let func x =
  let rec helper arg acc =
    if (base case) then acc
    else
      let arg’ = (argument to recursive call)
      let acc’ = (updated accumulator)
      helper arg’ acc’ in (* end of helper fun *)
  helper x (initial val of accumulator -- used for base case)
;;
```

Tail Recursion Pattern with fact

```
let fact x =
  let rec helper arg acc =
    if arg = 0 then acc
    else
      let arg’ = arg – 1 in
      let acc’ = acc * arg in
      helper arg’ acc’ in (* end of helper fun *)
  helper x 1
;;
```

Tail Recursion Pattern with rev

```
let rev x =
  let rec rev_helper arg acc =
    match arg with [] -> acc
    | h::t ->
      let arg’ = t in
      let acc’ = h::acc in
      rev_helper arg’ acc’ in (* end of helper fun *)
  rev_helper x []
;;
```

Can generalize to more than one argument, and multiple cases for each recursive call

Quiz #4

True/false: this is a tail-recursive map

```
let map f l =
  let rec helper l a =
    match l with
    | [] -> a
    | h::t -> helper t ((f h)::a)
  in helper l []
```

A. True
B. False

Quiz #4

True/false: this is a tail-recursive map

```
let map f l =
  let rec helper l a =
    match l with
    | [] -> a
    | h::t -> helper t ((f h)::a)
  in helper l []
```

A. True

B. False (elements are reversed)

A Tail Recursive **map**

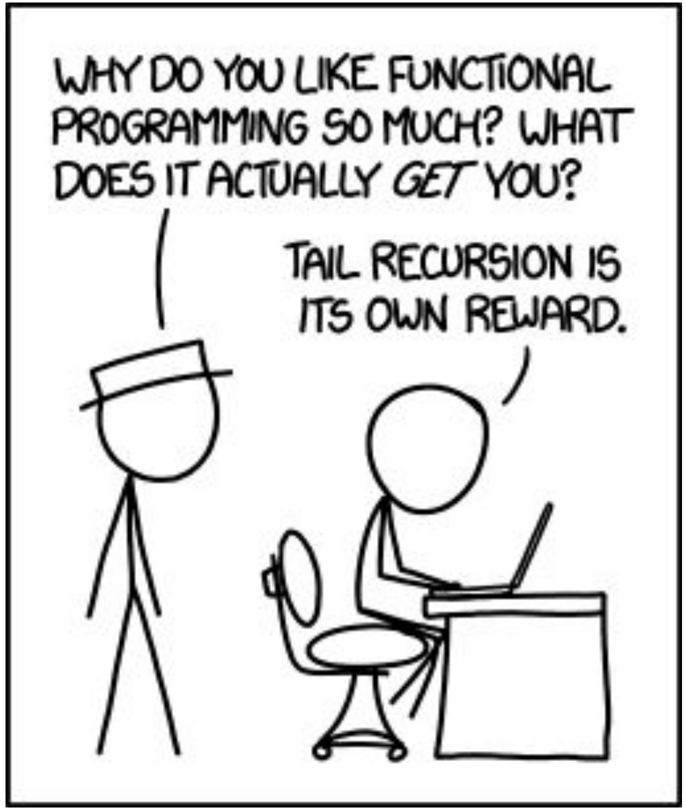
```
let map f l =
  let rec helper l a =
    match l with
      [] -> a
    | h::t -> helper t ((f h)::a)
  in rev (helper l [])
```

Could instead change `(f h)::a` to be `a@[f h]`

Q: Why is the above implementation a better choice?

A: $O(n)$ running time, not $O(n^2)$ (where n is length of list)

https://xkcd.com/1270/



Tail Recursion is Important

- Pushing a call frame for each recursive call when operating on a list is dangerous
  - One stack frame for each list element
  - Big list = stack overflow!

- So: favor tail recursion when inputs could be large (i.e., recursion could be deep). E.g.,
  - Prefer `List.fold_left` to `List.fold_right`
    - Library documentation should indicate tail recursion, or not
  - Convert recursive functions to be tail recursive

Outlook: Tail Recursion is General, too

- A function that is tail-recursive returns **at most once** (to its caller) when completely finished
  - The final result is exactly the result of a recursive call; no stack frame needed to remember the current call
- Is it possible to convert an *arbitrary program* into an equivalent one, except where **no call ever returns**?
  - Yes. This is called **continuation-passing style**
  - More later!