CMSC 330: Organization of Programming Languages

DFAs, and NFAs, and Regexps

The story so far, and what's next

- Goal: Develop an algorithm that determines whether a string *s* is matched by regex *R*
  - I.e., whether *s* is a member of *R*'s *language*

- Approach to come: Convert *R* to a finite automaton *FA* and see whether *s* is accepted by *FA*
  - Details: Convert *R* to a *nondeterministic FA* (NFA), which we then convert to a *deterministic FA* (DFA),
    - which enjoys a fast acceptance algorithm

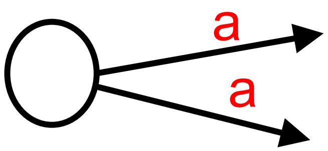
Two Types of Finite Automata

- Deterministic Finite Automata (DFA)
  - Exactly one sequence of steps for each string
    - Easy to implement acceptance check
  - (Almost) all examples so far

- Nondeterministic Finite Automata (NFA)
  - May have many sequences of steps for each string
  - Accepts if any path ends in final state at end of string
  - More compact than DFA
    - But more expensive to test whether a string matches

Comparing DFAs and NFAs

- NFAs can have more than one transition leaving a state on the same symbol



- DFAs allow only one transition per symbol
  - I.e., transition function must be a valid function
  - DFA is a special case of NFA

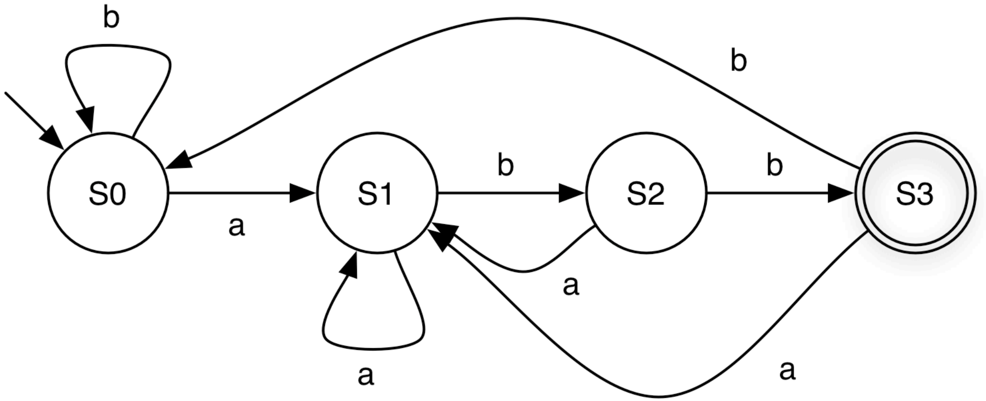
Comparing DFAs and NFAs (cont.)

- NFAs may have transitions with empty string label
  - May move to new state without consuming character

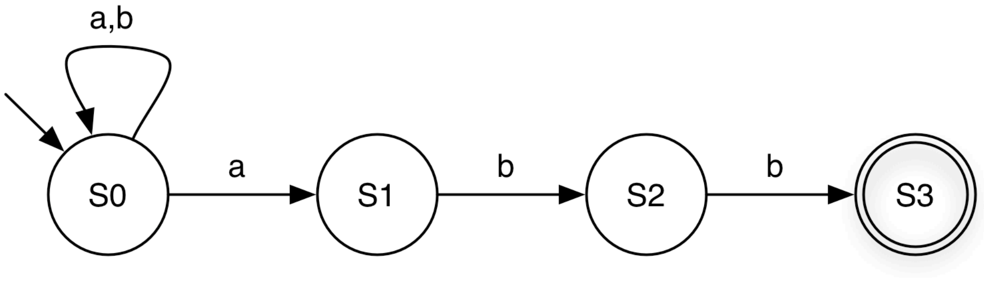
ε-transition

- DFA transition must be labeled with symbol
  - A DFA is a specific kind of NFA

DFA for (a|b)*abb

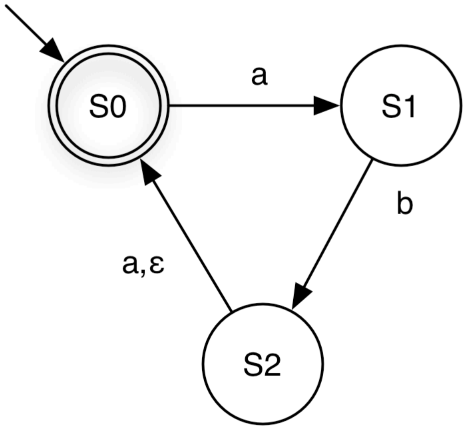


NFA for (a|b)*abb



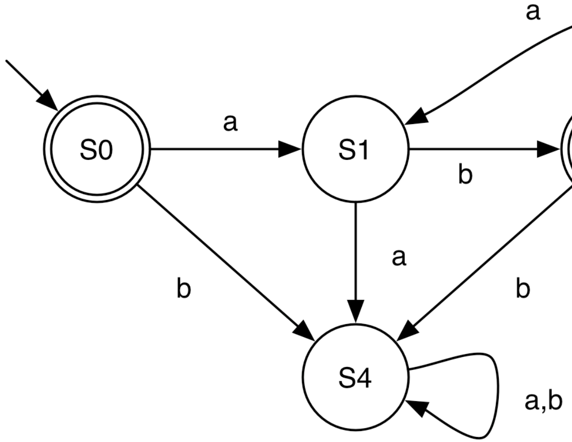
- ba
  - Has paths to either S0 or S1
  - Neither is final, so rejected
- babaabb
  - Has paths to different states
  - One path leads to S3, so accepts string

NFA for (ab|aba)*

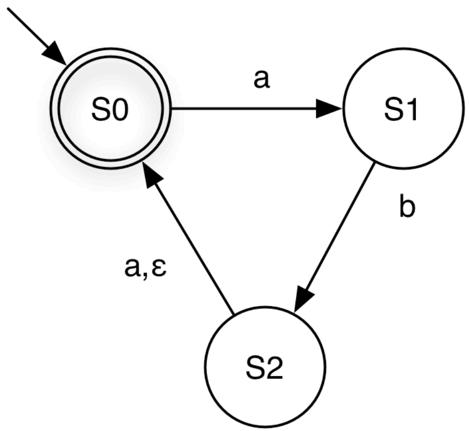


- aba
- ababa
  - Has paths to states S0, S1
  - Need to use ε-transition

NFA and DFA for (ab|aba)*



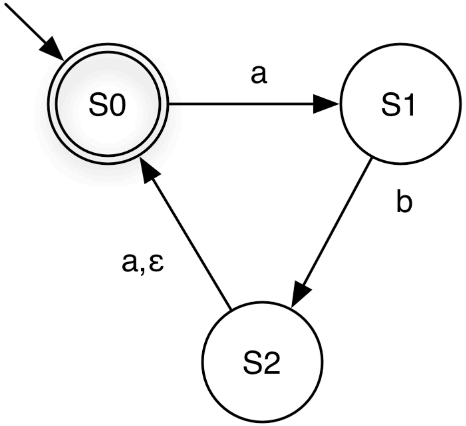
DFA



NFA

Quiz 1: Which string is NOT accepted by this NFA?

A. ab
B. abaa
C. abab
D. abaab



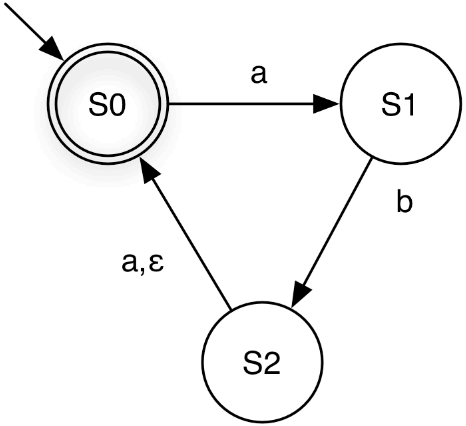
Quiz 1: Which string is NOT accepted by this NFA?

A. ab

B. abaa

C. abab

D. abaab

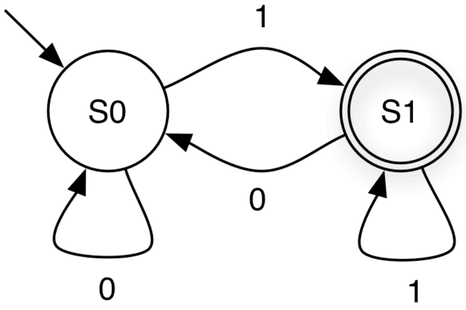


Formal Definition

- A deterministic finite automaton *(DFA)* is a 5-tuple $(\Sigma, Q, q_0, F, \delta)$ where
  - Σ is an alphabet
  - Q is a nonempty set of states
  - $q_0 \in Q$ is the start state
  - $F \subseteq Q$ is the set of final states
  - $\delta : Q \times \Sigma \rightarrow Q$ specifies the DFA's transitions
    - What's this definition saying that δ is?
- A DFA accepts s if it stops at a final state on s

Formal Definition: Example

- $\Sigma = \{0, 1\}$
- $Q = \{S0, S1\}$
- $q_0 = S0$
- $F = \{S1\}$
- $\delta =$

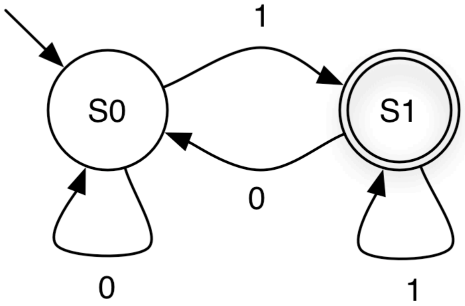


input state \ symbol	0	1
S0	S0	S1
S1	S0	S1

or as { (S0,0,S0),
(S0,1,S1),
(S1,0,S0),
(S1,1,S1) }

Implementing DFAs (one-off)

It's easy to build a program which mimics a DFA



```
cur_state = 0;
while (1) {

  symbol = getchar();

  switch (cur_state) {

    case 0: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("rejected\n"); return 0;
              default:   printf("rejected\n"); return 0;
            }
            break;

    case 1: switch (symbol) {
              case '0':  cur_state = 0; break;
              case '1':  cur_state = 1; break;
              case '\n': printf("accepted\n"); return 1;
              default:   printf("rejected\n"); return 0;
            }
            break;

    default: printf("unknown state; I'm confused\n");
             break;
  }
}
```

Implementing DFAs (generic)

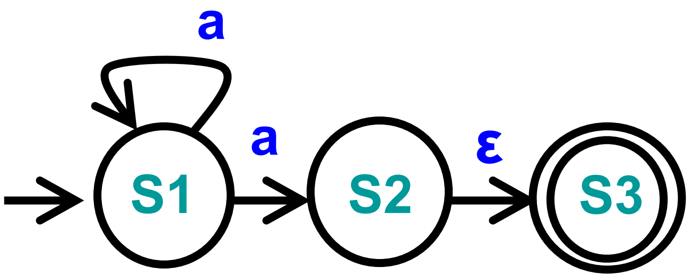
More generally, use generic table-driven DFA

```
given components ($\Sigma$, Q, $q_0$, F, $\delta$) of a DFA:
let q = $q_0$
while (there exists another symbol $\sigma$ of the input string)
    q := $\delta$(q, $\sigma$);
if q $\in$ F then
  accept
else reject
```

- q is just an integer
- Represent δ using arrays or hash tables
- Represent F as a set

Nondeterministic Finite Automata (NFA)

- An *NFA* is a 5-tuple (Σ, Q, q_0, F, δ) where
  - Σ, Q, q0, F as with DFAs
  - $\delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$ specifies the NFA's transitions



Example

- $\Sigma = \{a\}$
- Q = {S1, S2, S3}
- q_0 = S1
- F = {S3}
- δ = { (S1,a,S1), (S1,a,S2), (S2,ε,S3) }

- An NFA accepts s if there is at least one path via s from the NFA's start state to a final state

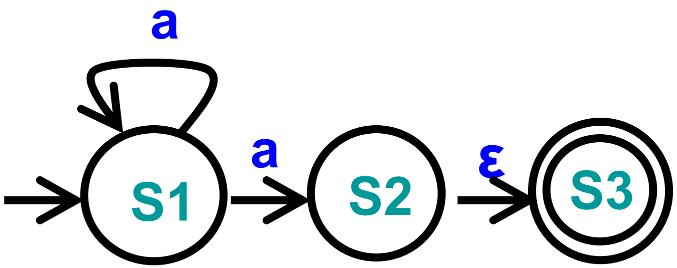
NFA Acceptance Algorithm (Sketch)

- When NFA processes a string s
  - NFA must keep track of several “current states”
    - Due to multiple transitions with same label, and ε-transitions
  - If any current state is final when done then accept s
- Example
  - After processing “a”
    - NFA may be in states

      S1

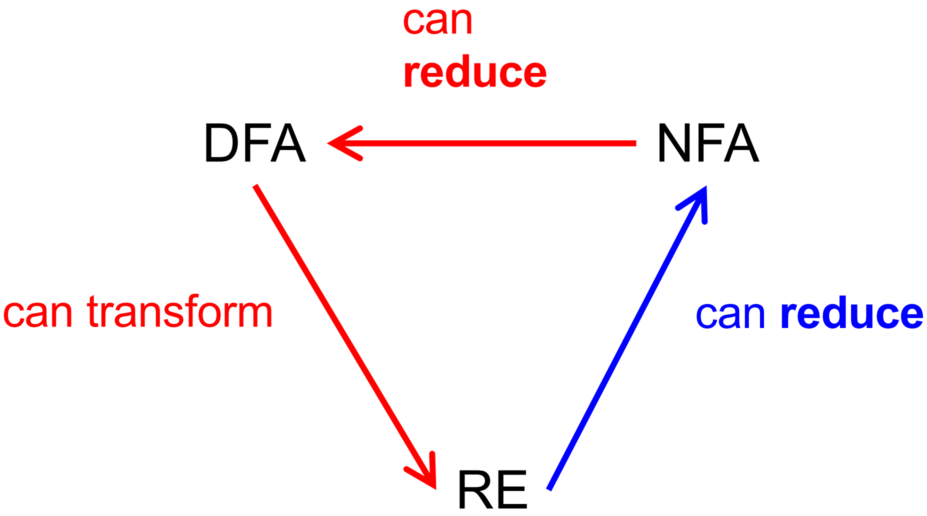
      S2

      S3
    - Since S3 is final, s is accepted
- Algorithm is slow, space-inefficient; prefer DFAs!



Relating REs to DFAs and NFAs

- Regular expressions, NFAs, and DFAs accept the same languages! *Can convert between them*



can **reduce**

DFA

NFA

can transform

can **reduce**

RE

NB. Both *transform* and *reduce* are historical terms; they mean “convert”

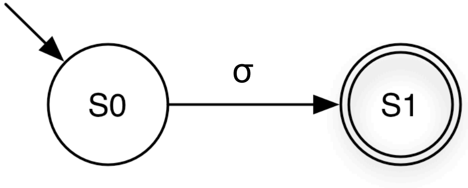
Reducing Regular Expressions to NFAs

- Goal: Given regular expression *A*, construct NFA: <*A*> = $(\Sigma, Q, q_0, F, \delta)$
  - Remember regular expressions are defined recursively from primitive RE languages
  - Invariant: $|F| = 1$ in our NFAs
    - Recall F = set of final states

- Will define <*A*> for base cases: σ , ε , $\emptyset$
  - Where σ is a symbol in Σ
- And for inductive cases: *AB*, *A*|*B*, *A**

Reducing Regular Expressions to NFAs

- Base case: σ



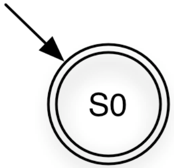
Recall: NFA is $(\Sigma, Q, q_0, F, \delta)$
where
Σ is the alphabet
Q is set of states
q_0 is starting state
F is set of final states
δ is transition relation

<σ> = ({σ}, {S0, S1}, S0, {S1}, {(S0, σ, S1)})

$(\Sigma,$ $Q,$ $q_0,$ $F,$ δ $)$

Reduction

- Base case: ε



<ε> = (∅, {S0}, S0, {S0}, ∅)

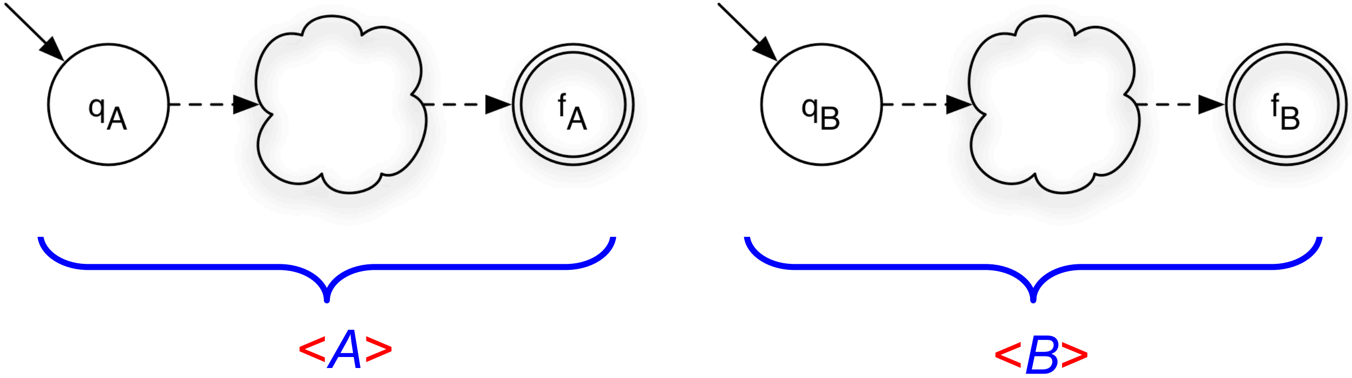
Recall: NFA is (Σ, Q, q_0, F, δ)
where
Σ is the alphabet
Q is set of states
q_0 is starting state
F is set of final states
δ is transition relation

- Base case: ∅

<∅> = (∅, {S0, S1}, S0, {S1}, ∅)

Reduction: Concatenation

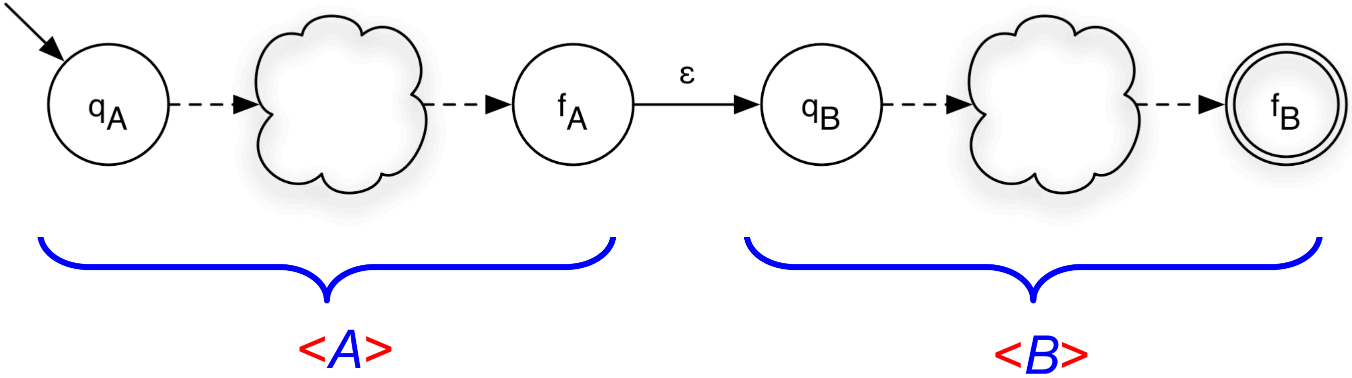
- Induction: *AB*



- <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- <*B*> = $(\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$

Reduction: Concatenation

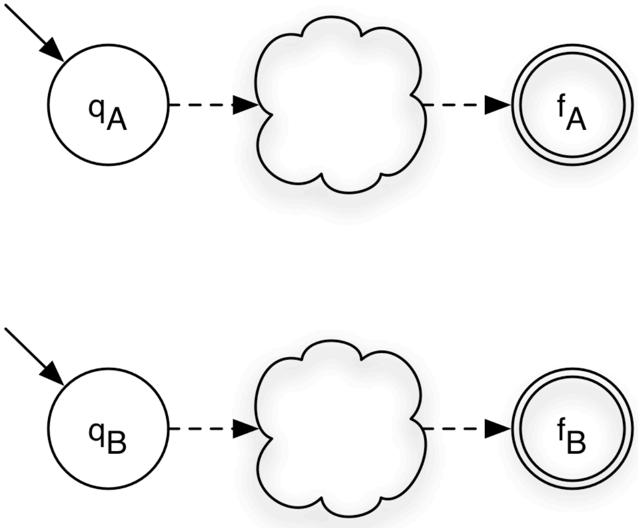
- Induction: *AB*



- $\langle A \rangle = (\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- $\langle B \rangle = (\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$
- $\langle AB \rangle = (\Sigma_A \cup \Sigma_B, Q_A \cup Q_B, q_A, \{f_B\}, \delta_A \cup \delta_B \cup \{(f_A,\varepsilon,q_B)\})$

Reduction: Union

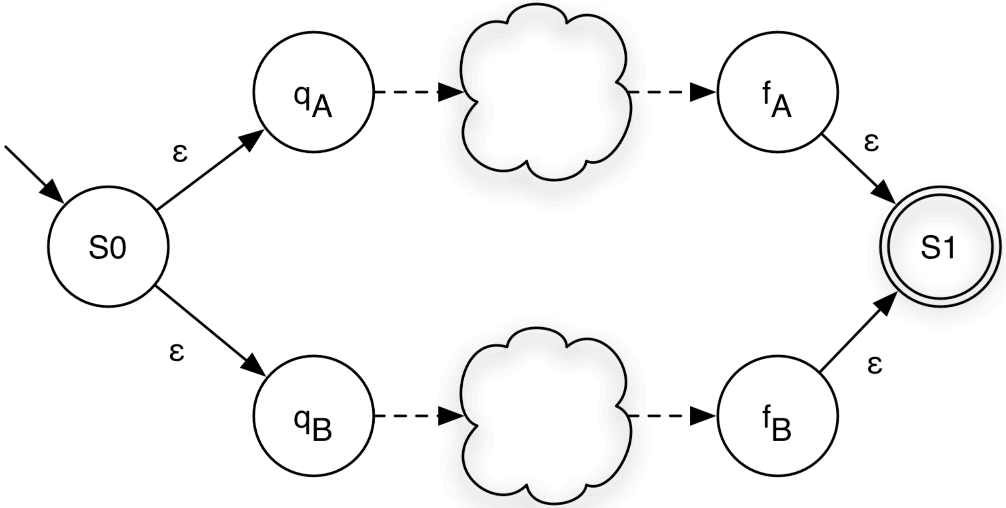
- Induction: *A*|*B*



  - <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
  - <*B*> = $(\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$

Reduction: Union

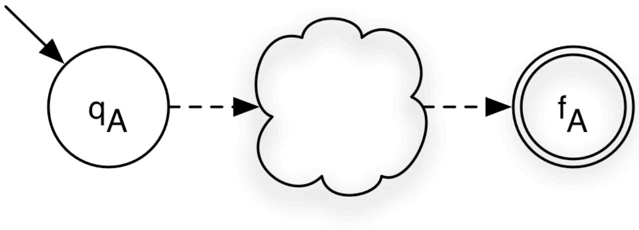
- Induction: *A*|*B*



  - <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
  - <*B*> = $(\Sigma_B, Q_B, q_B, \{f_B\}, \delta_B)$
  - <*A*|*B*> = $(\Sigma_A \cup \Sigma_B, Q_A \cup Q_B \cup \{S0,S1\}, S0, \{S1\},$
    $\delta_A \cup \delta_B \cup \{(S0,\varepsilon,q_A), (S0,\varepsilon,q_B), (f_A,\varepsilon,S1), (f_B,\varepsilon,S1)\})$

Reduction: Closure

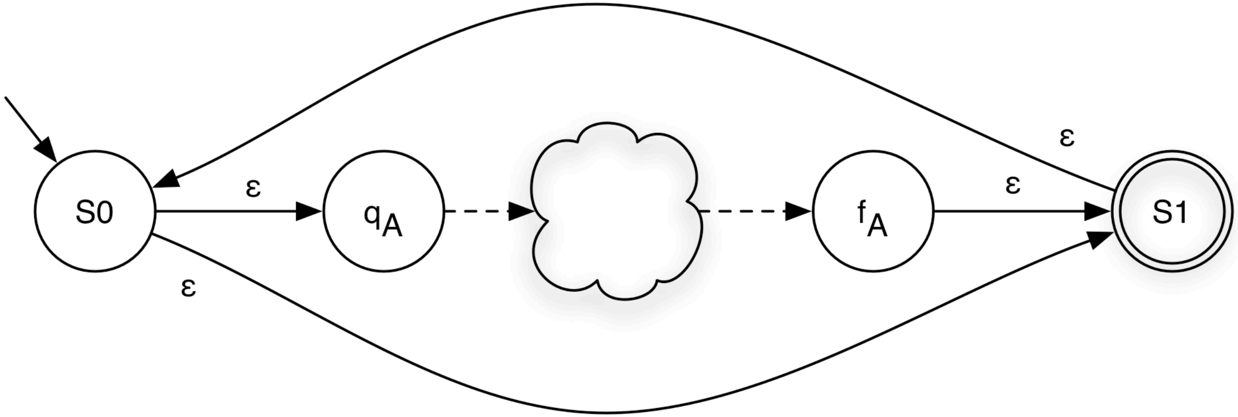
- Induction: *A**



- <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$

Reduction: Closure

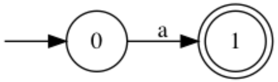
- Induction: *A**



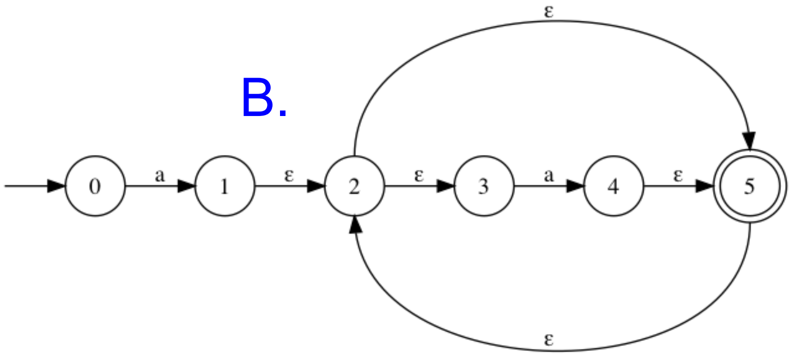
- <*A*> = $(\Sigma_A, Q_A, q_A, \{f_A\}, \delta_A)$
- <*A**> = $(\Sigma_A, Q_A \cup \{S0,S1\}, S0, \{S1\},$
  $\delta_A \cup \{(f_A,\varepsilon,S1), (S0,\varepsilon,q_A), (S0,\varepsilon,S1), (S1,\varepsilon,S0)\})$

Quiz 2: Which NFA matches **a*** ?

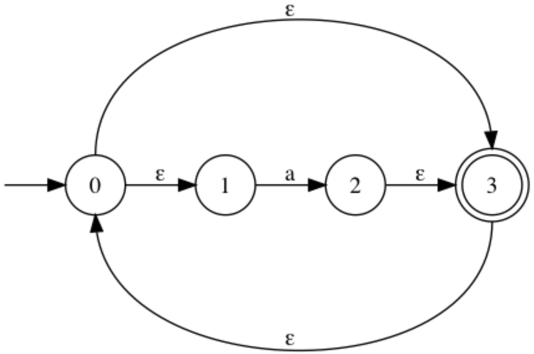
A.



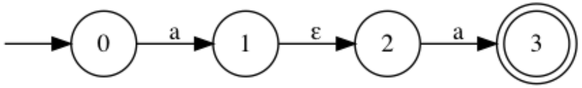
B.



C.



D.



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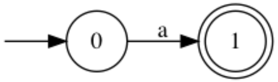

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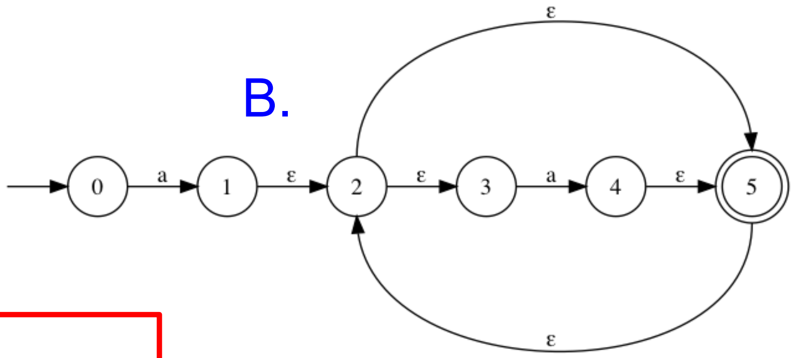

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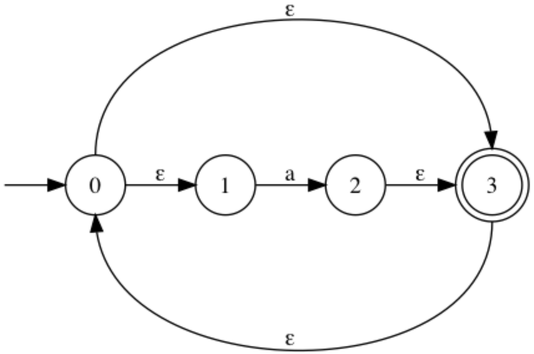

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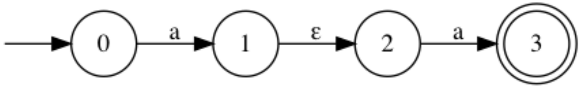
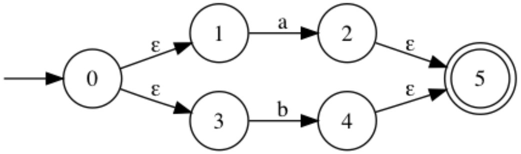
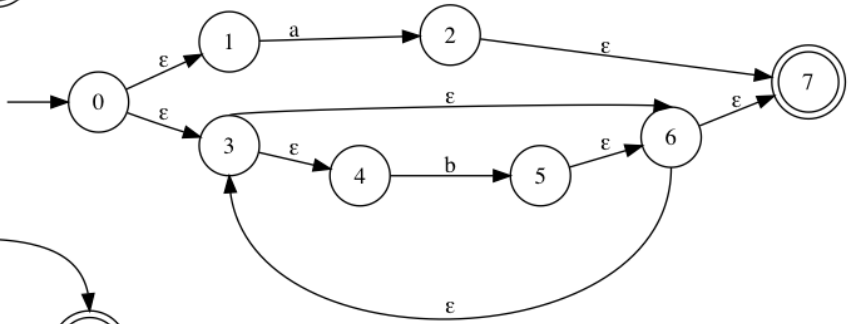


Quiz 3: Which NFA matches **a|b*** ?

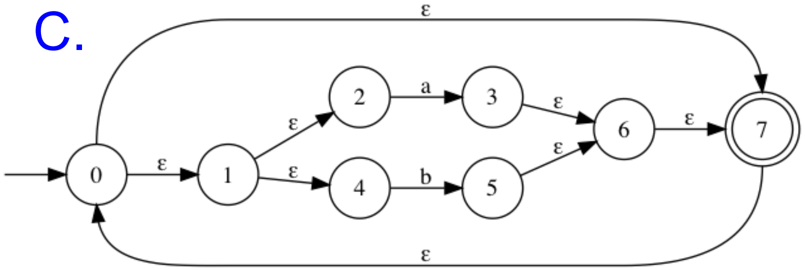
A.



B.



C.



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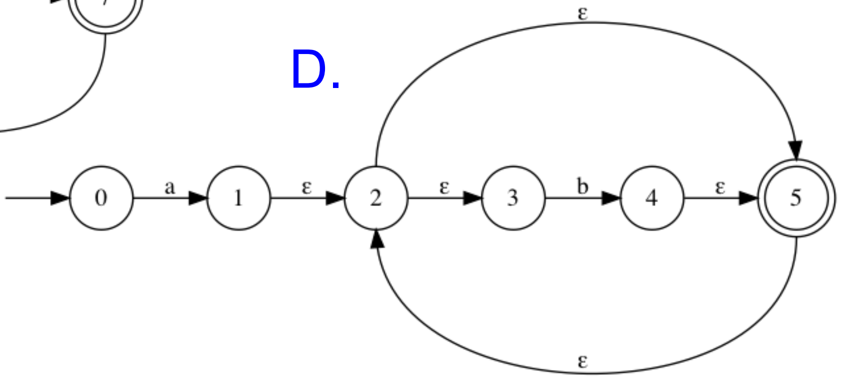

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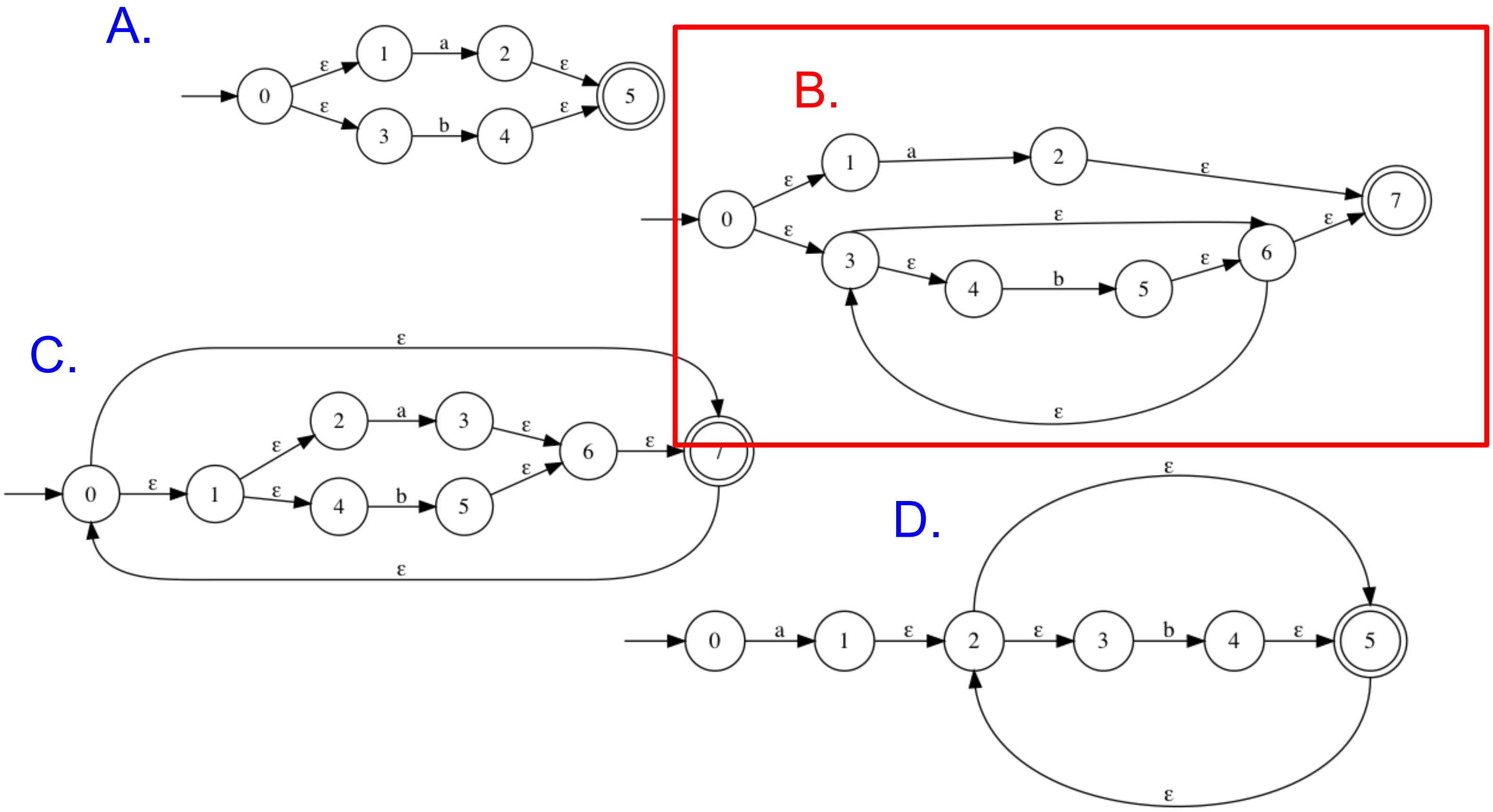
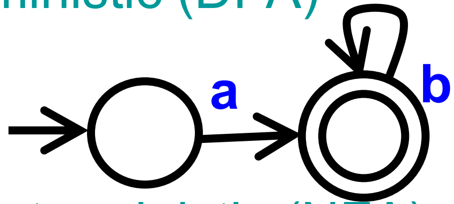
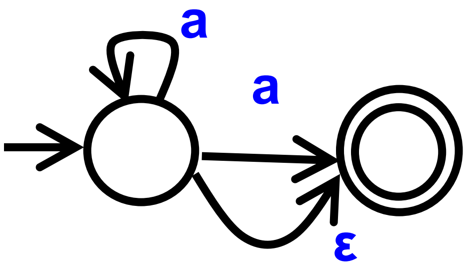


Recap

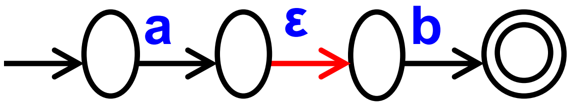
- Finite automata
  - Alphabet, states…
  - (Σ, Q, q_0, F, δ)
- Types
  - Deterministic (DFA)



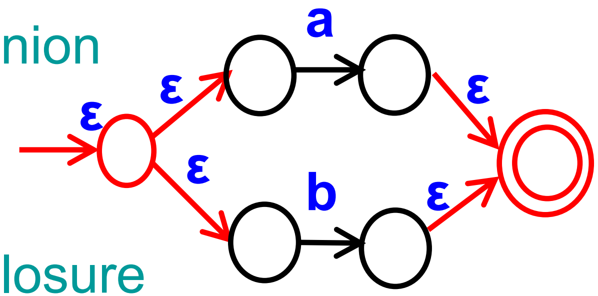
  - Non-deterministic (NFA)



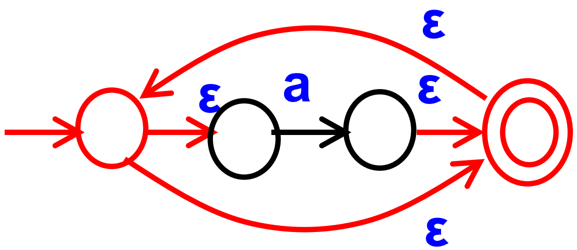
- Reducing RE to NFA
  - Concatenation



  - Union



  - Closure



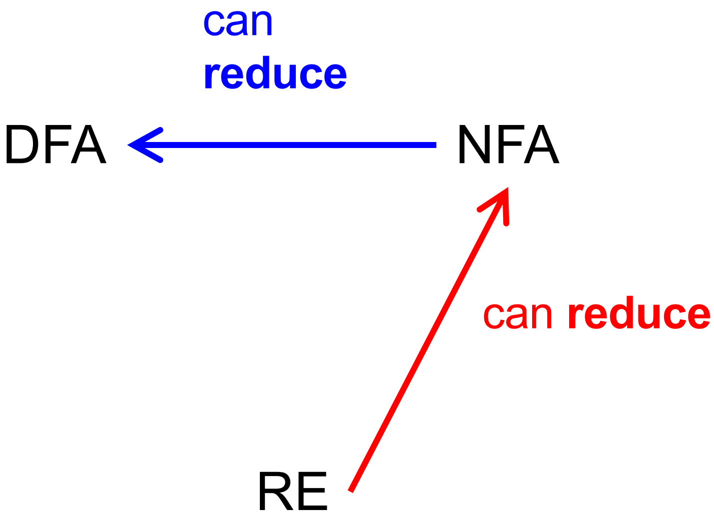
Reduction Complexity

- Given a regular expression *A* of size n...

  Size = # of symbols + # of operations

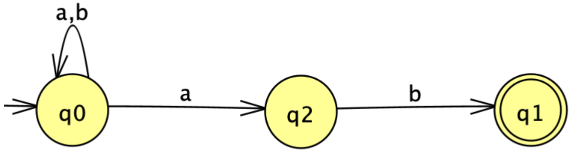
- How many states does <*A*> have?
  - Two added for each **|**, two added for each *****
  - O(n)
  - That’s pretty good!

Reducing NFA to DFA

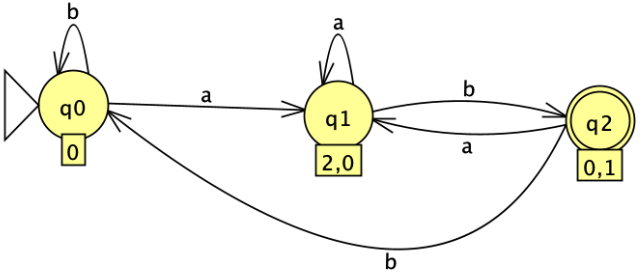


Why NFA → DFA

- DFA is generally more efficient than NFA



NFA



DFA

Language: (a|b)*ab

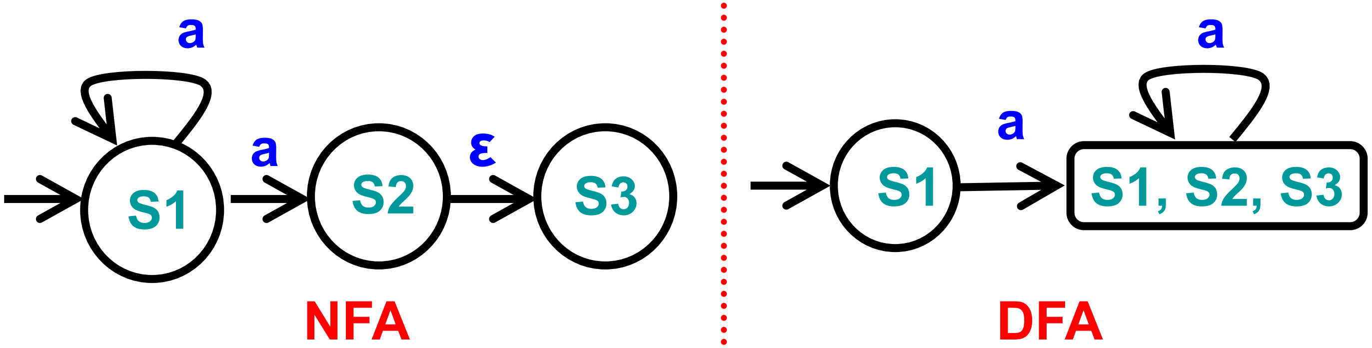
Why NFA → DFA

- DFA has the same expressive power as NFAs.
  - Let language $L \subseteq \Sigma^*$, and suppose L is accepted by NFA $N = (\Sigma, Q, q_0, F, \delta)$. There exists a DFA $D= (\Sigma, Q', q'_0, F', \delta')$ that also accepts L. $(L(N) = L(D))$
- NFAs are more flexible and easier to build. But DFAs have no less power than NFAs.

NFA ↔ DFA

Reducing NFA to DFA

- NFA may be reduced to DFA
  - By explicitly tracking the set of NFA states
- Intuition
  - Build DFA where
    - Each DFA state represents a set of NFA “current states”
- Example



Algorithm for Reducing NFA to DFA

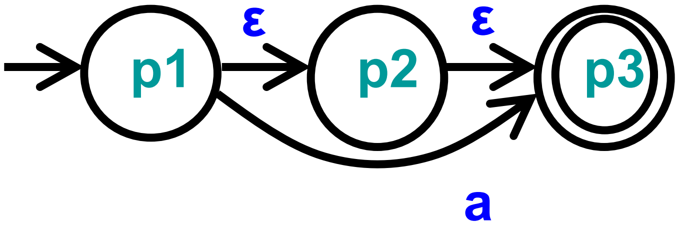
- Reduction applied using the subset algorithm
  - DFA state is a subset of set of all NFA states
- Algorithm
  - Input
    - NFA $(\Sigma, Q, q_0, F_n, \delta)$
  - Output
    - DFA $(\Sigma, R, r_0, F_d, \delta)$
  - Using two subroutines
    - ε-closure(δ, p) (and ε-closure(δ, Q))
    - move(δ, p, σ) (and move(δ, Q, σ))
      - (where p is an NFA state)

ε-transitions and ε-closure

- We say $p \xrightarrow{\varepsilon} q$
  - If it is possible to go from state p to state q by taking only ε-transitions in δ
  - If $\exists\ p, p_1, p_2, \ldots p_n, q \in Q$ such that
    - $\{p,\varepsilon,p_1\} \in \delta, \{p_1,\varepsilon,p_2\} \in \delta, \ldots, \{p_n,\varepsilon,q\} \in \delta$
- ε-closure(δ, p)
  - Set of states reachable from p using ε-transitions alone
    - Set of states q such that $p \xrightarrow{\varepsilon} q$ according to δ
    - $\varepsilon\text{-closure}(\delta, p) = \{q \mid p \xrightarrow{\varepsilon} q \text{ in } \delta\}$
    - $\varepsilon\text{-closure}(\delta, Q) = \{ q \mid p \in Q, p \xrightarrow{\varepsilon} q \text{ in } \delta \}$
  - Notes
    - ε-closure(δ, p) always includes p
    - We write ε-closure(p) or ε-closure(Q) when δ is clear from context

ε-closure: Example 1

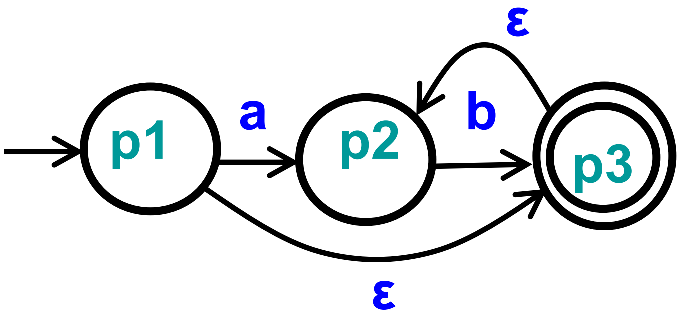
- Following NFA contains
  - p1 $\xrightarrow{\varepsilon}$ p2
  - p2 $\xrightarrow{\varepsilon}$ p3
  - p1 $\xrightarrow{\varepsilon}$ p3
    - Since p1 $\xrightarrow{\varepsilon}$ p2 and p2 $\xrightarrow{\varepsilon}$ p3



- ε-closures
  - ε-closure(p1) = { p1, p2, p3 }
  - ε-closure(p2) = { p2, p3 }
  - ε-closure(p3) = { p3 }
  - ε-closure({ p1, p2 }) = { p1, p2, p3 } ∪ { p2, p3 }

ε-closure: Example 2

- Following NFA contains
  - p1 $\xrightarrow{\varepsilon}$ p3
  - p3 $\xrightarrow{\varepsilon}$ p2
  - p1 $\xrightarrow{\varepsilon}$ p2
    - Since p1 $\xrightarrow{\varepsilon}$ p3 and p3 $\xrightarrow{\varepsilon}$ p2



- ε-closures
  - ε-closure(p1) = { p1, p2, p3 }
  - ε-closure(p2) = { p2 }
  - ε-closure(p3) = { p2, p3 }
  - ε-closure({ p2,p3 }) = { p2 } ∪ { p2, p3 }

ε-closure Algorithm: Approach

- Input: NFA $(\Sigma, Q, q_0, F_n, \delta)$, State Set R
- Output: State Set R'
- Algorithm

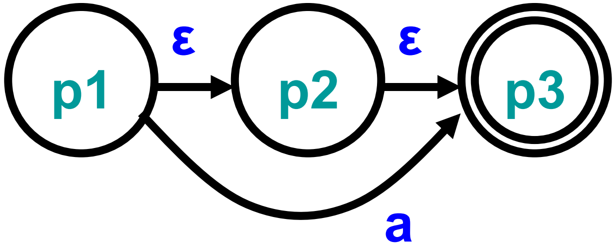
```
Let R' = R                                   // start states
Repeat
    Let R = R'                               // continue from previous
    Let R' = R ∪ {q | p ∈ R, (p, ε, q) ∈ δ}  // new ε-reachable states
Until R = R'                                 // stop when no new states
```

This algorithm computes a fixed point

ε-closure Algorithm Example

- Calculate ε-closure(δ,{p1})

R	R’
{p1}	{p1}
{p1}	{p1, p2}
{p1, p2}	{p1, p2, p3}
{p1, p2, p3}	{p1, p2, p3}



```
Let R’ = R
Repeat
   Let R= R’
   Let R’ = $R \cup \{q \mid p \in R, (p, \varepsilon, q) \in \delta\}$
Until R = R’
```

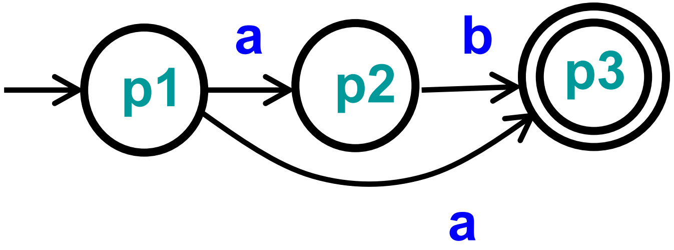
Calculating move(p,σ)

- move(δ,p,σ)
  - Set of states reachable from p using exactly one transition on symbol σ
    - Set of states q such that {p, σ, q} ∈ δ
    - move(δ,p,σ) = { q | {p, σ, q} ∈ δ }
    - move(δ,Q,σ) = { q | p ∈ Q, {p, σ, q} ∈ δ }
      - i.e., can "lift" move() to a *set* of states Q

  - Notes:
    - move(δ,p,σ) is Ø if no transition (p,σ,q) ∈ δ, for any q
    - We write move(p,σ) or move(R,σ) when δ clear from context

move(p,σ) : Example 1

- Following NFA
  - Σ = { a, b }

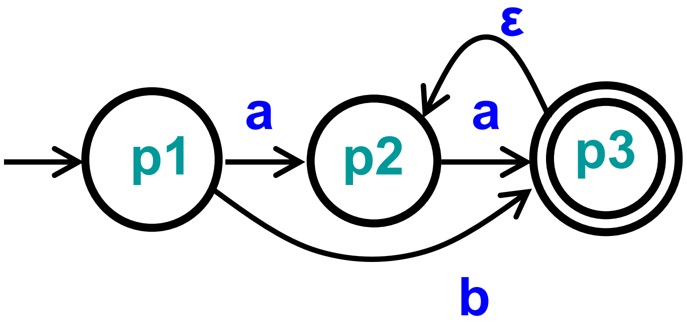


- Move
  - move(p1, a) = { p2, p3 }
  - move(p1, b) = Ø
  - move(p2, a) = Ø
  - move(p2, b) = { p3 }
  - move(p3, a) = Ø
  - move(p3, b) = Ø

move({p1,p2},b) = { p3 }

move(p,σ) : Example 2

- Following NFA
  - Σ = { a, b }



- Move
  - move(p1, a) = { p2 }
  - move(p1, b) = { p3 }
  - move(p2, a) = { p3 }
  - move(p2, b) = Ø
  - move(p3, a) = Ø
  - move(p3, b) = Ø

move({p1,p2},a) = {p2,p3}

NFA → DFA Reduction Algorithm (“subset”)

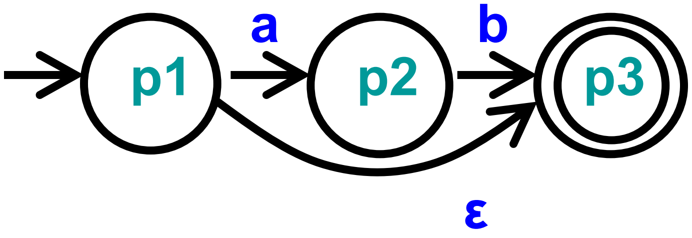
- Input NFA (Σ, Q, q_0, F_n, δ), Output DFA (Σ, R, r_0, F_d, δ')
- Algorithm

```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R      // DFA start state
While $\exists$ an unmarked state $r \in R$                          // process DFA state r
    Mark r                                                           // each state visited once
    For each $\sigma \in \Sigma$                                     // for each symbol $\sigma$
        Let E = move($\delta$,r,$\sigma$)                            // states reached via $\sigma$
        Let e = $\varepsilon$-closure($\delta$,E)                    // states reached via $\varepsilon$
        If $e \notin R$                                              // if state e is new
            Let $R = R \cup \{e\}$                                   // add e to R (unmarked)
        Let $\delta' = \delta' \cup \{r, \sigma, e\}$                // add transition r→e on $\sigma$
Let $F_d = \{r \mid \exists s \in r \text{ with } s \in F_n\}$       // final if include state in $F_n$
```

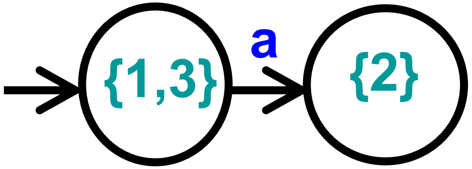
NFA → DFA Example

- Start = ε-closure(δ,p1) = { {p1,p3} }
- R = { {p1,p3} }
- r ∈ R = {p1,p3}
- move(δ,{p1,p3},a) = {p2}
  - e = ε-closure(δ,{p2}) = {p2}
  - R = R ∪ {{p2}} = { {p1,p3}, {p2} }
  - δ’ = δ’ ∪ {{p1,p3}, a, {p2}}
- move(δ,{p1,p3},b) = Ø

NFA



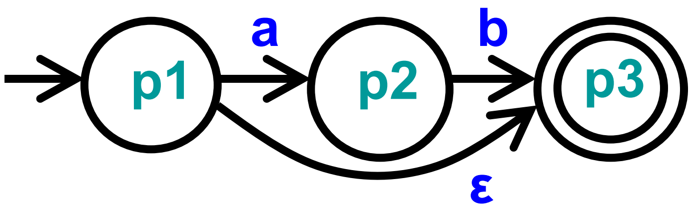
DFA



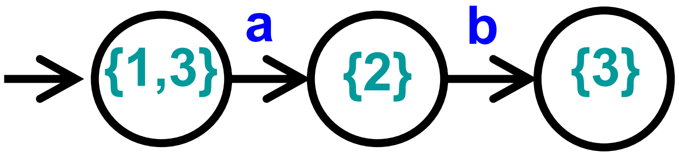
NFA → DFA Example (cont.)

- $R = \{ \{p1,p3\}, \{p2\} \}$
- $r \in R = \{p2\}$
- $\text{move}(\delta,\{p2\},a) = \emptyset$
- $\text{move}(\delta,\{p2\},b) = \{p3\}$
  - $e = \varepsilon\text{-closure}(\delta,\{p3\}) = \{p3\}$
  - $R = R \cup \{\{p3\}\} = \{ \{p1,p3\}, \{p2\}, \{p3\} \}$
  - $\delta' = \delta' \cup \{\{p2\}, b, \{p3\}\}$

NFA



DFA



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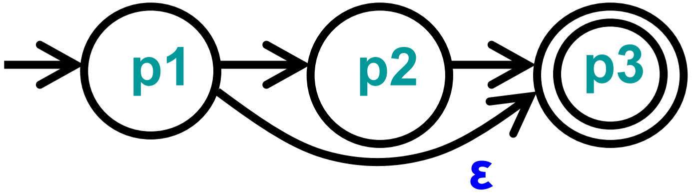
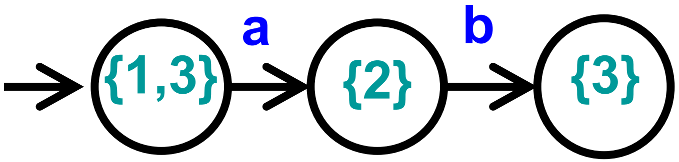
NFA → DFA Example (cont.)

- R = { {p1,p3}, {p2}, {p3} }
- r ∈ R = {p3}
- Move({p3},a) = Ø
- Move({p3},b) = Ø
- Mark {p3}, exit loop
- F_d = {{p1,p3}, {p3}}
  - Since p3 ∈ F_n
- Done!

NFA

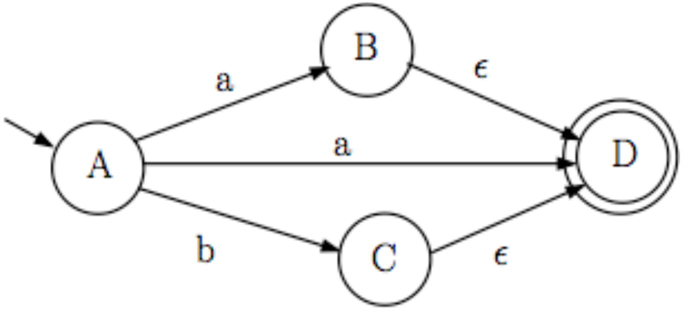


DFA

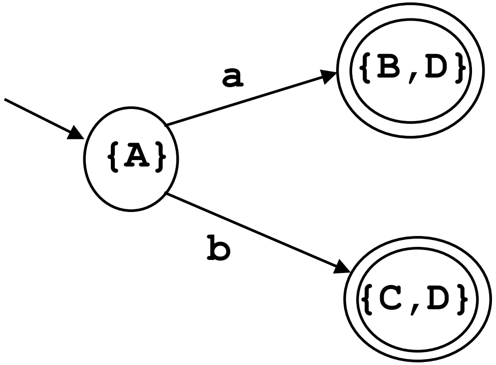


NFA → DFA Example 2

- NFA

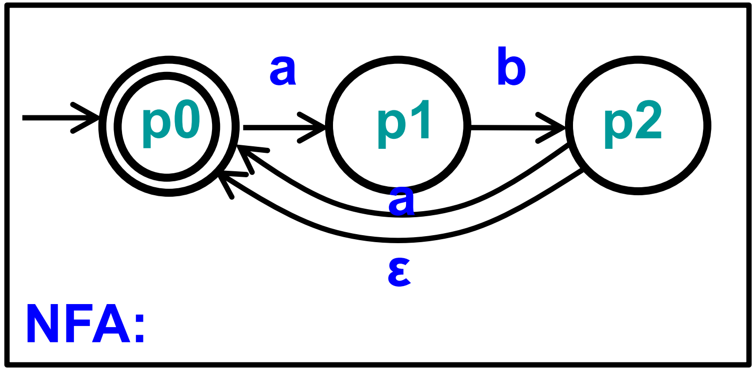


- DFA

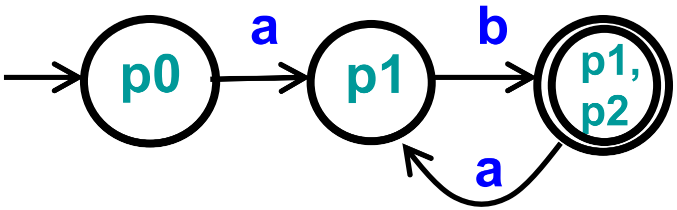


`R = { {A}, {B,D}, {C,D} }`

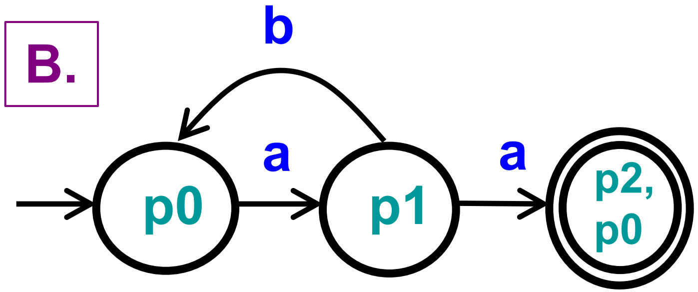
Quiz 4: Which DFA is equiv to this NFA?



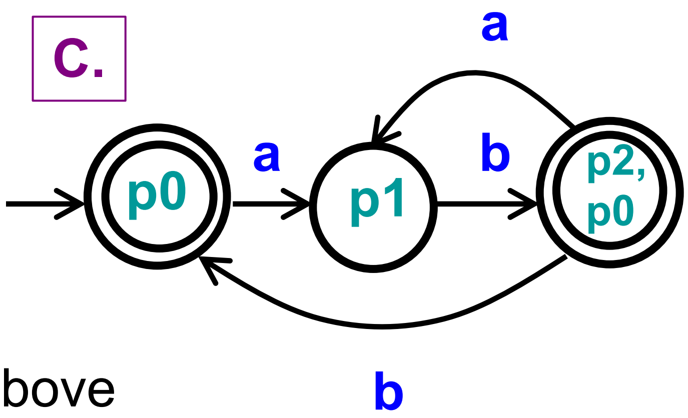
A.



B.

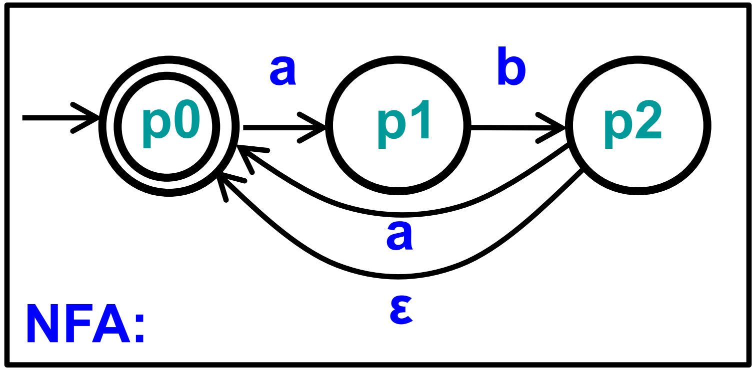


C.

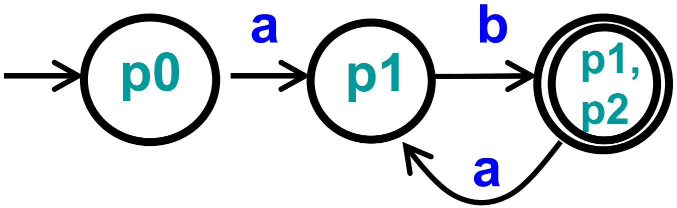


D. None of the above

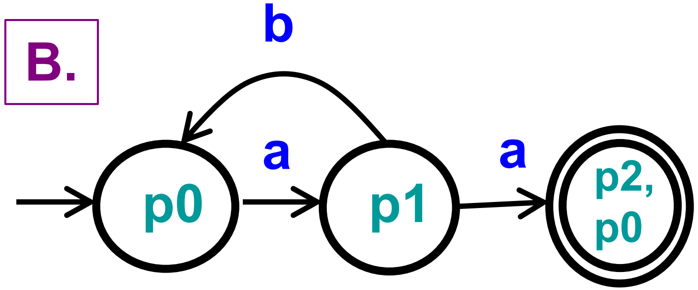
Quiz 4: Which DFA is equiv to this NFA?



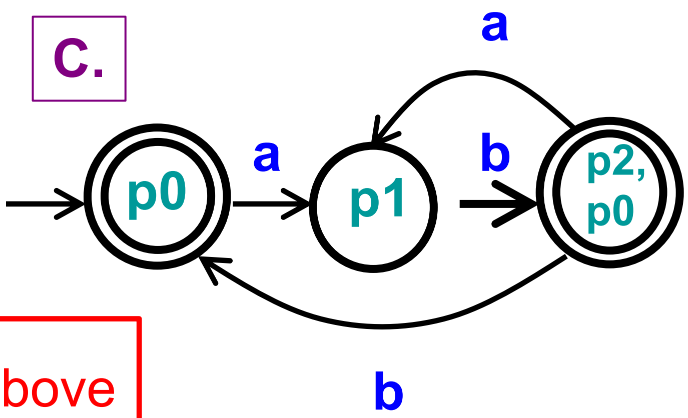
A.



B.

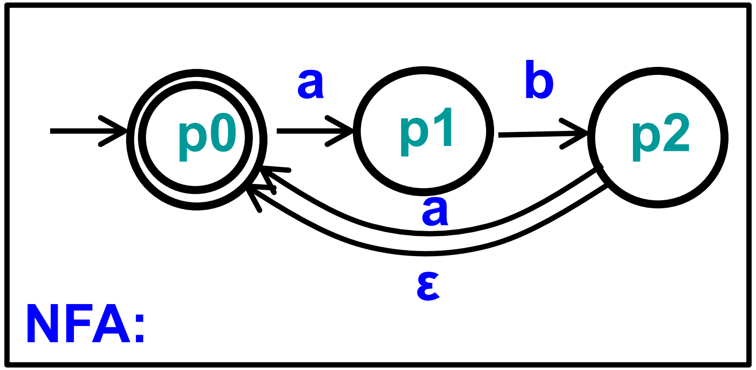


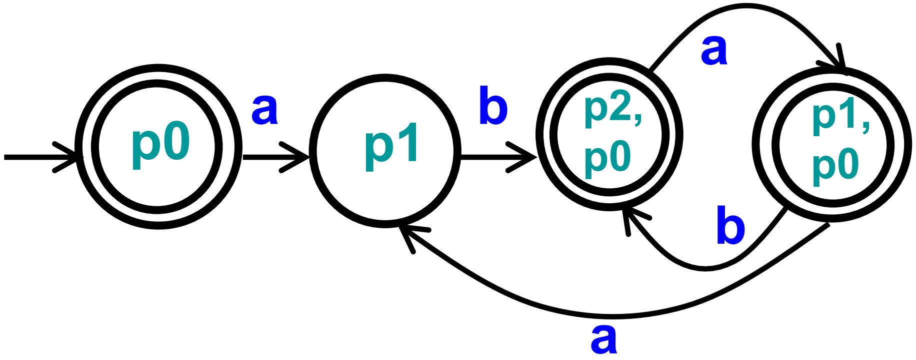
C.



D. None of the above

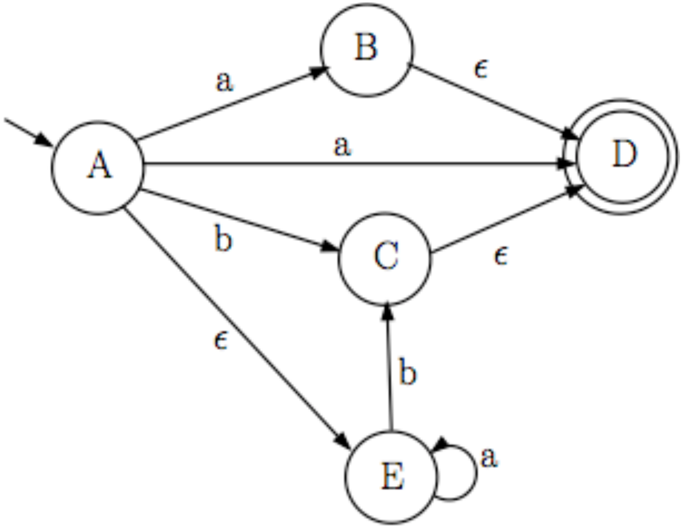
Actual Answer



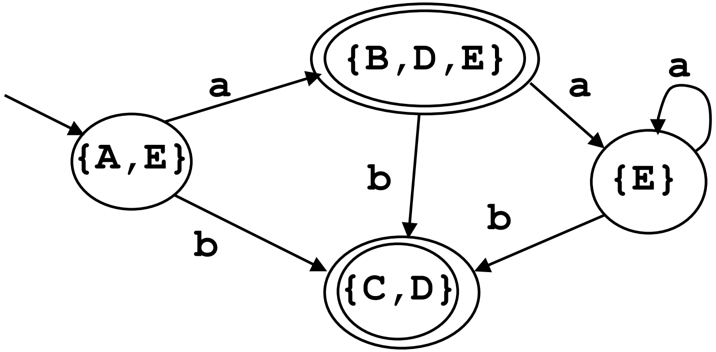


NFA → DFA Example 3

- NFA



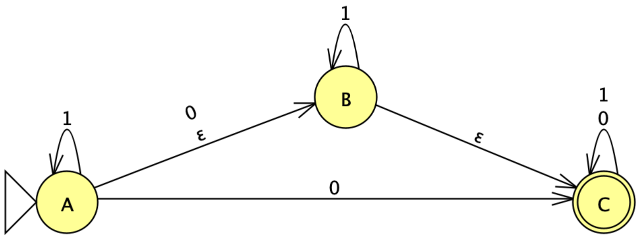
- DFA



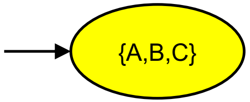
R = { {A,E}, {B,D,E}, {C,D}, {E} }

Detailed NFA → DFA Example

NFA



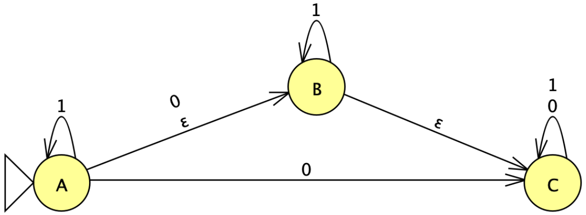
DFA

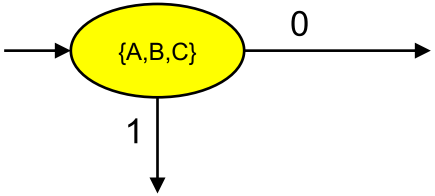


New Start State

```
Let $r_0 = \varepsilon$-closure$(\delta, q_0)$, add it to R
While $\exists$ an unmarked state $r \in R$
    Mark r
    For each $\sigma \in \Sigma$
    Let $E = move(\delta, r, \sigma)$
        Let $e = \varepsilon$-closure$(\delta, E)$
        If $e \notin R$
            Let $R = R \cup \{e\}$
        Let $\delta' = \delta' \cup \{r, \sigma, e\}$
Let $F_d = \{r \mid \exists s \in r \text{ with } s \in F_n\}$
```

Detailed NFA → DFA Example

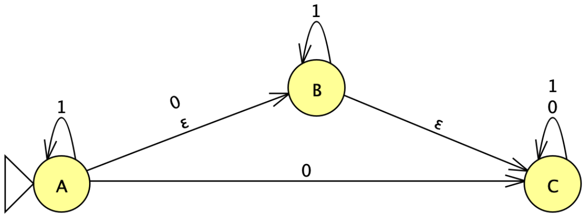


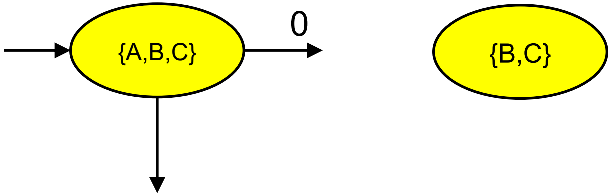


```
Let $r_0$ = ε-closure(δ,$q_0$), add it to R
While ∃ an unmarked state r ∈ R
    Mark r
    For each σ ∈ Σ            //0
    Let E = move(δ,r,σ)
        Let e = ε-closure(δ,E)
        If e ∉ R
            Let R = R ∪ {e}
        Let δ' = δ' ∪ {r, σ, e}
Let $F_d$ = {r | ∃ s ∈ r with s ∈ $F_n$}
```

	0	1
{A,B,C}		

Detailed NFA → DFA Example

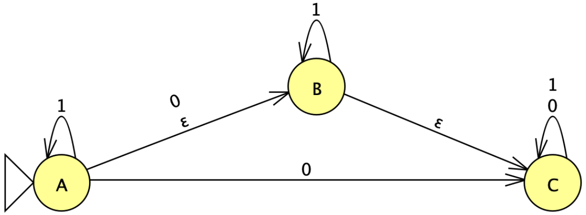


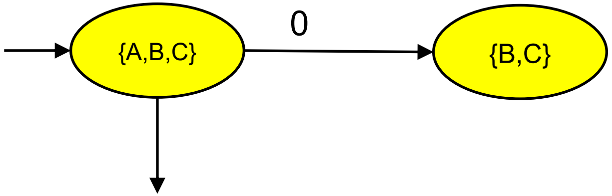


```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
While $\exists$ an unmarked state r $\in$ R
    Mark r
    For each $\sigma \in \Sigma$
    Let E = move($\delta$,r,$\sigma$)
        Let e = $\varepsilon$-closure($\delta$,E)
        If e $\notin$ R
            Let R = R $\cup$ {e}
        Let $\delta'$ = $\delta'$ $\cup$ {r, $\sigma$, e}
Let $F_d$ = {r | $\exists$ s $\in$ r with s $\in$ $F_n$}
```

	0	1
{A,B,C}	{B,C}	

Detailed NFA → DFA Example

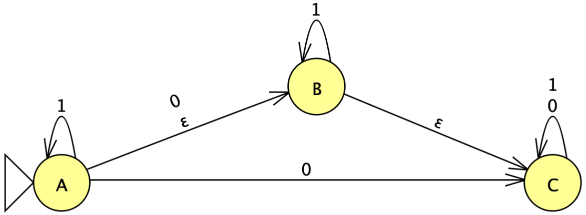


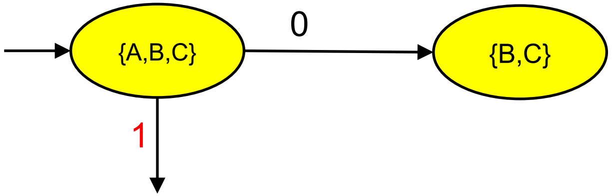


	0	1
{A,B,C}	{B,C}	
{B,C}		

```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
While $\exists$ an unmarked state r $\in$ R
    Mark r
    For each $\sigma \in \Sigma$
    Let E = move($\delta$,r,$\sigma$)
        Let e = $\varepsilon$-closure($\delta$,E)
        If e $\notin$ R
            Let R = R $\cup$ {e}
→       Let $\delta'$ = $\delta'$ $\cup$ {r, $\sigma$, e}
Let $F_d$ = {r | $\exists$ s $\in$ r with s $\in$ $F_n$}
```

Detailed NFA → DFA Example

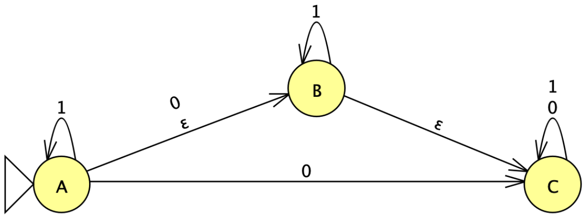


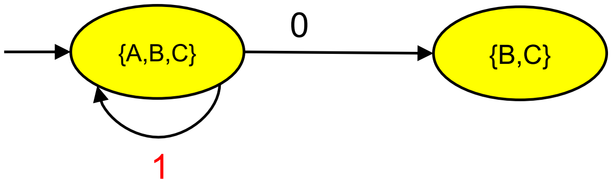


	0	1
{A,B,C}	{B,C}	
{B,C}		

```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
While $\exists$ an unmarked state r $\in$ R
        Mark r
  →     For each $\sigma \in \Sigma$            //1
        Let E = move($\delta$,r,$\sigma$)
            Let e = $\varepsilon$-closure($\delta$,E)
            If e $\notin$ R
              Let R = R $\cup$ {e}
            Let $\delta'$ = $\delta'$ $\cup$ {r, $\sigma$, e}
Let $F_d$ = {r | $\exists$ s $\in$ r with s $\in$ $F_n$}
```

Detailed NFA → DFA Example

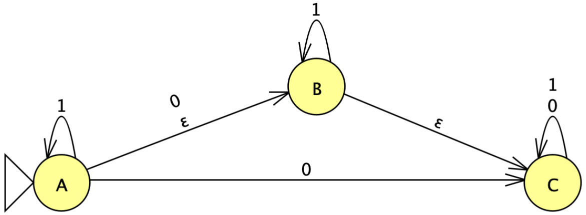


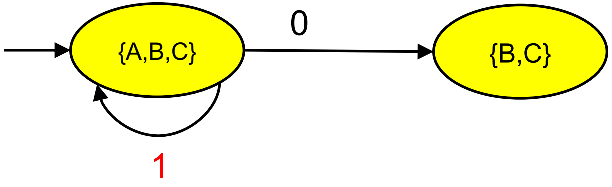


	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}		

```
Let r_0 = ε-closure(δ,q_0), add it to R
While ∃ an unmarked state r ∈ R
    Mark r
    For each σ ∈ Σ                //1
    Let E = move(δ,r,σ)
→       Let e = ε-closure(δ,E)
        If e ∉ R
            Let R = R ∪ {e}
        Let δ' = δ' ∪ {r, σ, e}
Let F_d = {r | ∃ s ∈ r with s ∈ F_n}
```

Detailed NFA → DFA Example

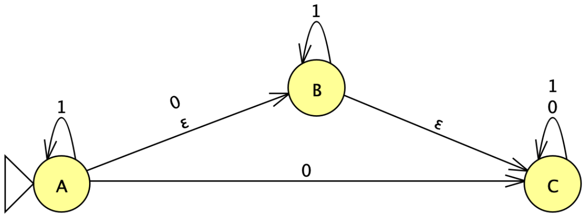


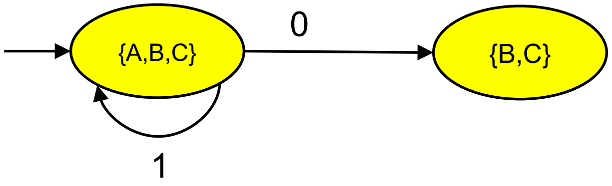


	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}		

```
Let $r_0$ = ε-closure(δ,$q_0$), add it to R
While ∃ an unmarked state r ∈ R
    Mark r
    For each σ ∈ Σ                //1
    Let E = move(δ,r,σ)
        Let e = ε-closure(δ,E)
        If e ∉ R
            Let R = R ∪ {e}
        Let δ’ = δ’ ∪ {r, σ, e}
Let $F_d$ = {r | ∃ s ∈ r with s ∈ $F_n$}
```

Detailed NFA → DFA Example

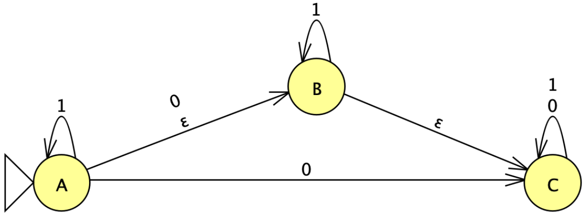


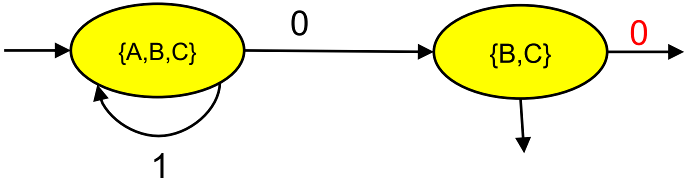


	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}		

```
Let $r_0$ = ε-closure(δ,$q_0$), add it to R
While ∃ an unmarked state r ∈ R
    Mark r
    For each σ ∈ Σ              //1
    Let E = move(δ,r,σ)
        Let e = ε-closure(δ,E)
        If e ∉ R
            Let R = R ∪ {e}
        Let δ' = δ' ∪ {r, σ, e}
Let $F_d$ = {r | ∃ s ∈ r with s ∈ $F_n$}
```

Detailed NFA → DFA Example

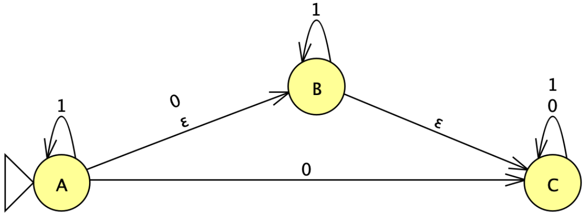


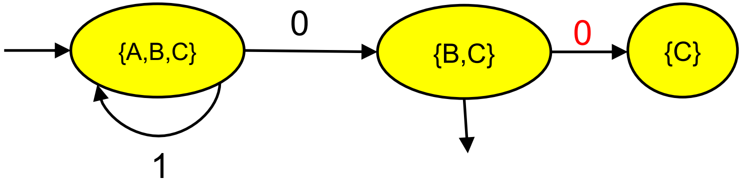


```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
While $\exists$ an unmarked state r $\in$ R
    Mark r
    For each $\sigma \in \Sigma$            //0
    Let E = move($\delta$,r,$\sigma$)
        Let e = $\varepsilon$-closure($\delta$,E)
        If e $\notin$ R
            Let R = R $\cup$ {e}
        Let $\delta'$ = $\delta'$ $\cup$ {r, $\sigma$, e}
Let $F_d$ = {r | $\exists$ s $\in$ r with s $\in$ $F_n$}
```

	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}		

Detailed NFA → DFA Example





```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
While $\exists$ an unmarked state r $\in$ R
    Mark r
    For each $\sigma \in \Sigma$                //0
    Let E = move($\delta$,r,$\sigma$)
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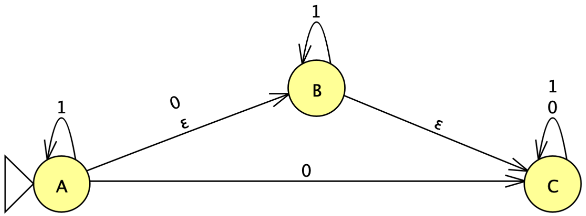
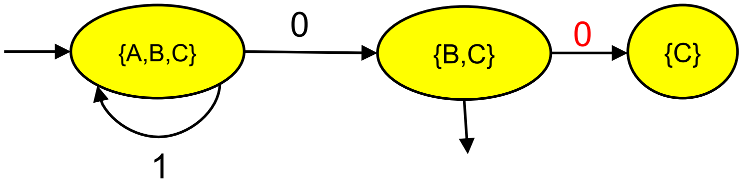
        Let $\delta'$ = $\delta' \cup$ {r, $\sigma$, e}
Let $F_d$ = {r | $\exists$ s $\in$ r with s $\in$ $F_n$}
```

	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	

Detailed NFA → DFA Example





	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	
{C}		

```
Let $r_0$ = ε-closure(δ,$q_0$), add it to R
While ∃ an unmarked state r ∈ R
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        Let e = ε-closure(δ,E)
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            Let R = R ∪ {e}
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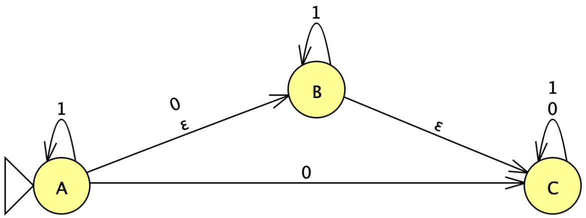
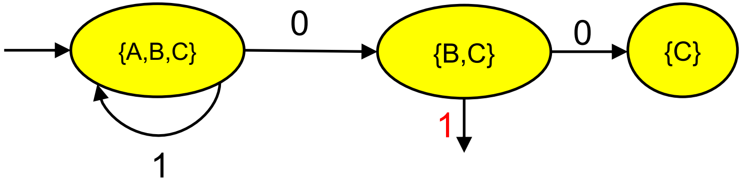
Let $F_d$ = {r | ∃ s ∈ r with s ∈ $F_n$}
```

Detailed NFA → DFA Example





	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	?
{C}		

```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
While $\exists$ an unmarked state r $\in$ R
    Mark r
    For each $\sigma \in \Sigma$            //1
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```

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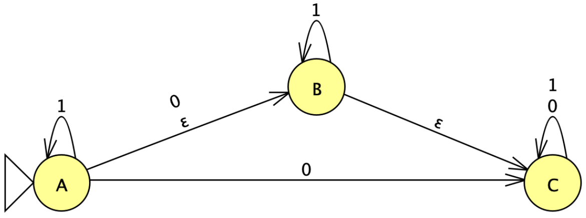

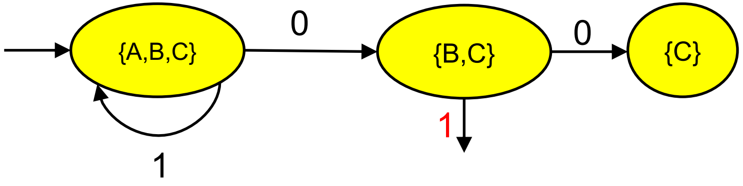

	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}		

```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
While $\exists$ an unmarked state r $\in$ R
    Mark r
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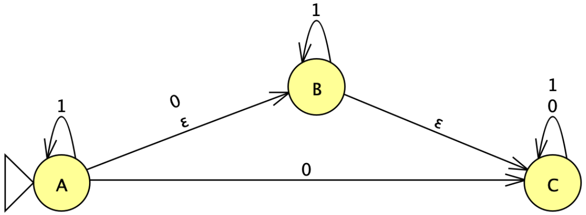
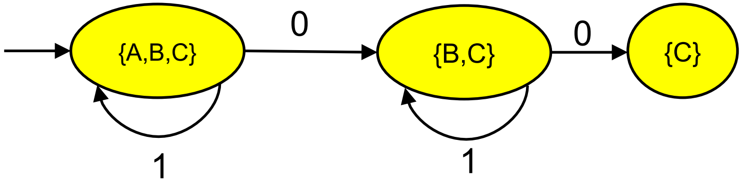
Let $F_d$ = {r | $\exists$ s $\in$ r with s $\in F_n$}
```

Detailed NFA → DFA Example





	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}		

```
Let $r_0$ = ε-closure(δ,$q_0$), add it to R
While ∃ an unmarked state r ∈ R
    Mark r
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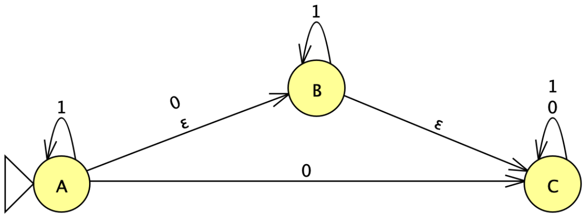
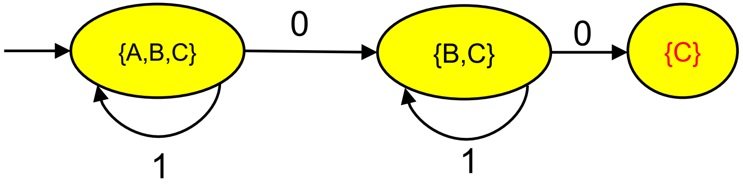
Detailed NFA → DFA Example





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While $\exists$ an unmarked state r $\in$ R
    Mark r
    For each $\sigma \in \Sigma$            //1
    Let E = move($\delta$,r,$\sigma$)
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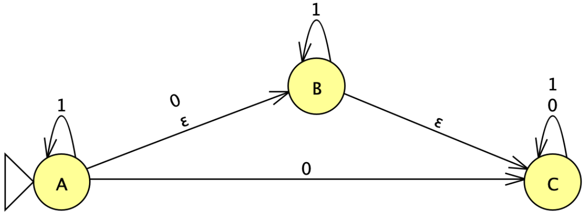
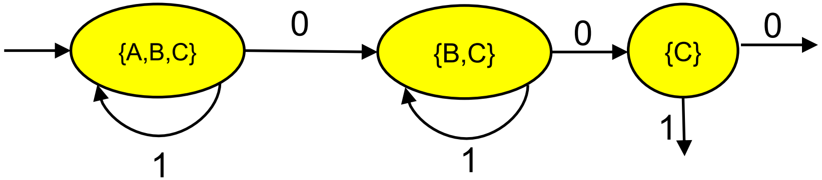
        Let $\delta' = \delta' \cup$ {r, $\sigma$, e}
Let $F_d$ = {r | $\exists$ s $\in$ r with s $\in F_n$}
```

	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}		

Detailed NFA → DFA Example

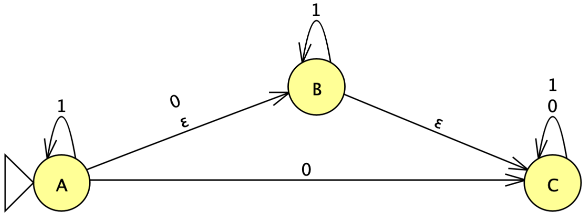


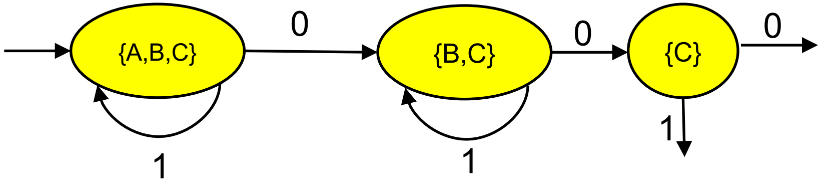


```
Let $r_0$ = ε-closure(δ,$q_0$), add it to R
While ∃ an unmarked state r ∈ R
      Mark r
→     For each σ ∈ Σ
      Let E = move(δ,r,σ)
          Let e = ε-closure(δ,E)
          If e ∉ R
              Let R = R ∪ {e}
          Let δ' = δ' ∪ {r, σ, e}
Let $F_d$ = {r | ∃ s ∈ r with s ∈ $F_n$}
```

	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}		

Detailed NFA → DFA Example





```
Let $r_0$ = ε-closure(δ,$q_0$), add it to R
While ∃ an unmarked state r ∈ R
    Mark r
    For each σ ∈ Σ                //0
    Let E = move(δ,r,σ)
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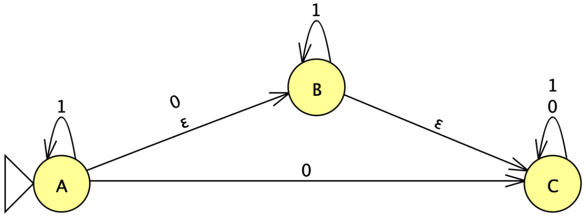
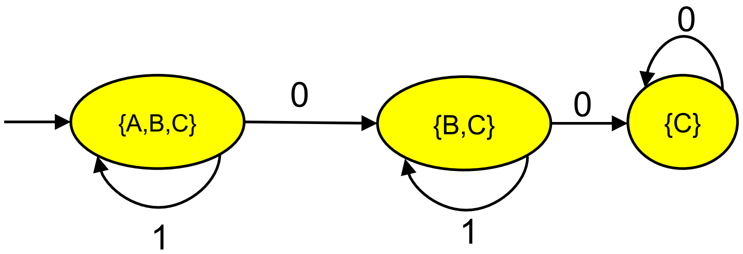
        Let δ' = δ' ∪ {r, σ, e}
Let $F_d$ = {r | ∃ s ∈ r with s ∈ $F_n$}
```

	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}	{C}	

Detailed NFA → DFA Example

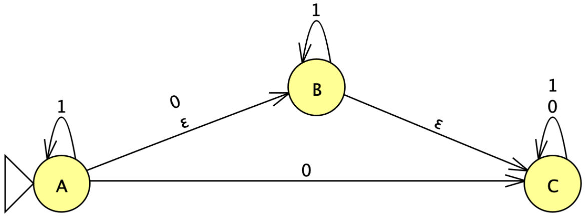


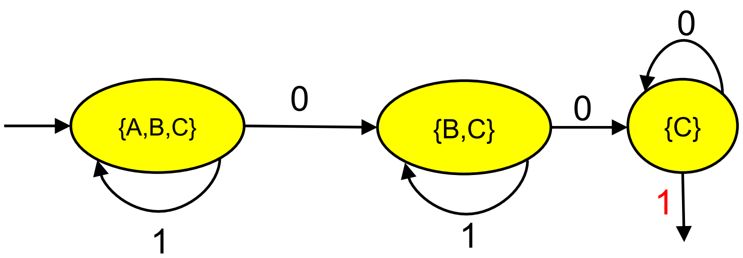


	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}	{C}	

```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
While $\exists$ an unmarked state r $\in$ R
    Mark r
    For each $\sigma \in \Sigma$                //0
    Let E = move($\delta$,r,$\sigma$)
        Let e = $\varepsilon$-closure($\delta$,E)
        If e $\notin$ R
          Let R = R $\cup$ {e}
        Let $\delta'$ = $\delta' \cup$ {r, $\sigma$, e}
Let $F_d$ = {r | $\exists$ s $\in$ r with s $\in$ $F_n$}
```

Detailed NFA → DFA Example

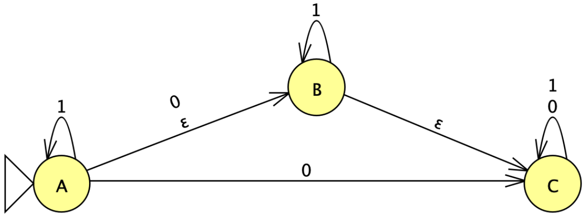


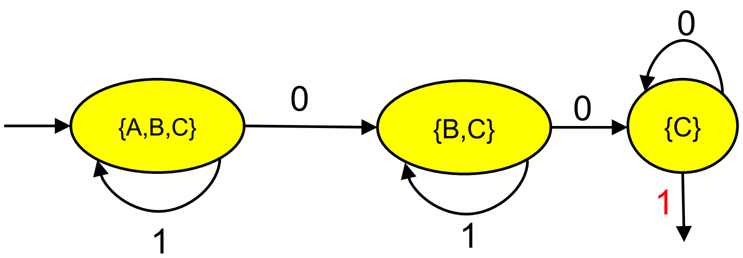


```
Let $r_0$ = ε-closure(δ,$q_0$), add it to R
While ∃ an unmarked state r ∈ R
    Mark r
→   For each σ ∈ Σ            //1
    Let E = move(δ,r,σ)
        Let e = ε-closure(δ,E)
        If e ∉ R
          Let R = R ∪ {e}
        Let δ' = δ' ∪ {r, σ, e}
Let $F_d$ = {r | ∃ s ∈ r with s ∈ $F_n$}
```

	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}	{C}	

Detailed NFA → DFA Example





	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}	{C}	{C}

```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
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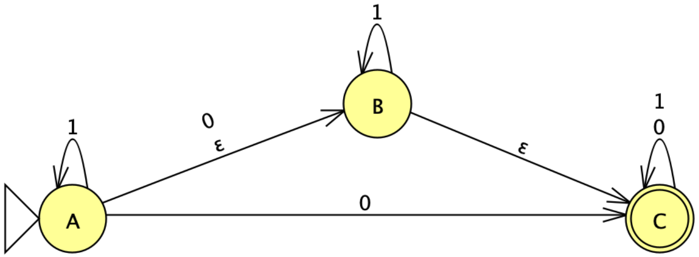
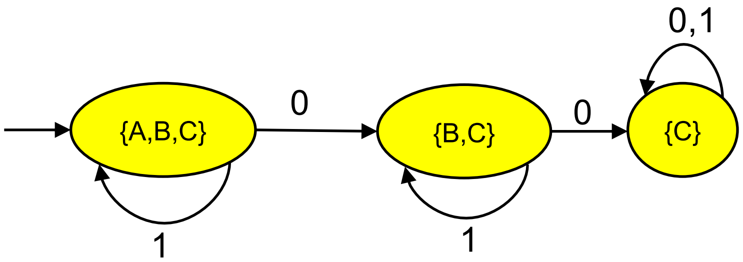
Detailed NFA → DFA Example

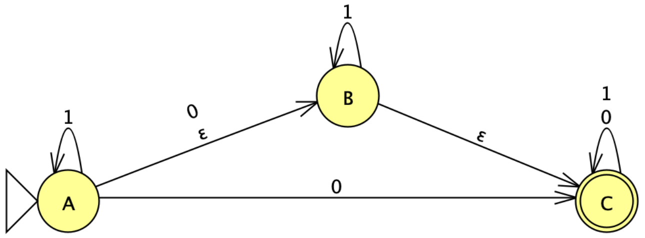


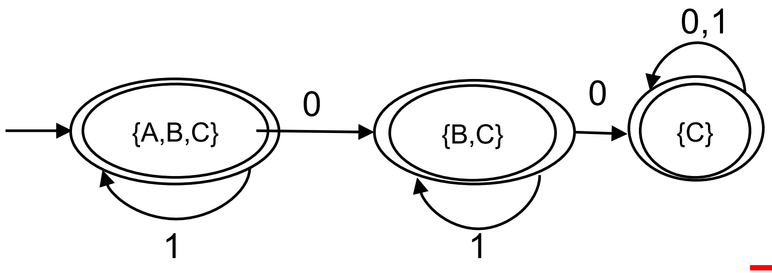


```
Let $r_0$ = ε-closure(δ,$q_0$), add it to R
While ∃ an unmarked state r ∈ R
    Mark r
    For each σ ∈ Σ            //1
    Let E = move(δ,r,σ)
        Let e = ε-closure(δ,E)
        If e ∉ R
          Let R = R ∪ {e}
        Let δ’ = δ’ ∪ {r, σ, e}
Let $F_d$ = {r | ∃ s ∈ r with s ∈ $F_n$}
```

	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}	{C}	{C}

Detailed NFA → DFA Example





	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}	{C}	{C}

```
Let $r_0$ = $\varepsilon$-closure($\delta$,$q_0$), add it to R
While $\exists$ an unmarked state r $\in$ R
    Mark r
    For each $\sigma \in \Sigma$          //1
    Let E = move($\delta$,r,$\sigma$)
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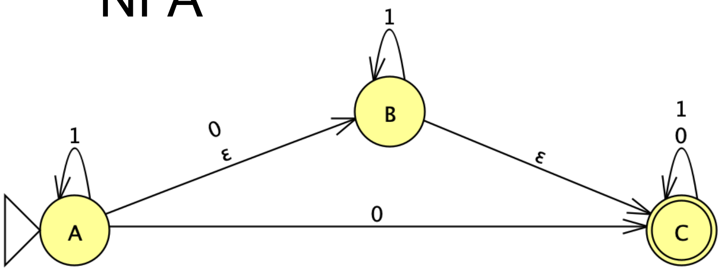
        Let $\delta'$ = $\delta' \cup$ {r, $\sigma$, e}
Let $F_d$ = {r | $\exists$ s $\in$ r with s $\in F_n$}
```

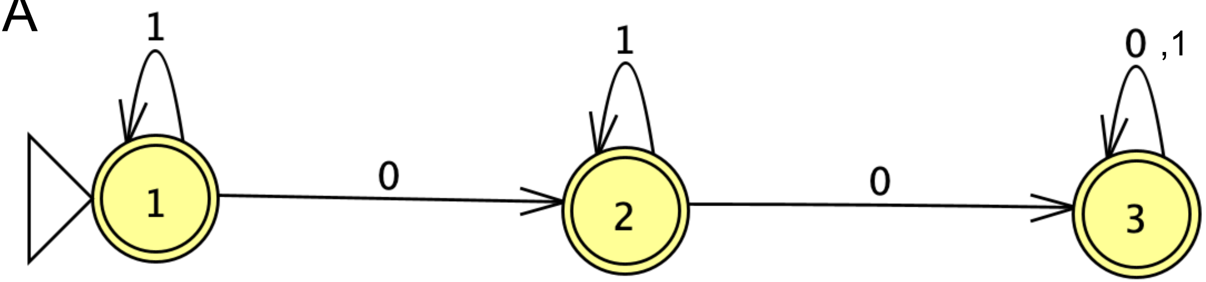
Detailed NFA → DFA Example: Completed

NFA

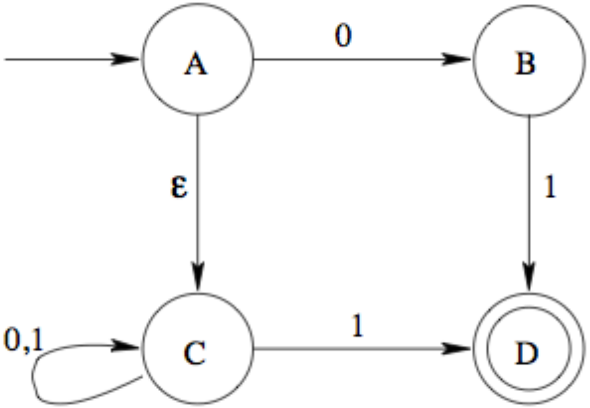


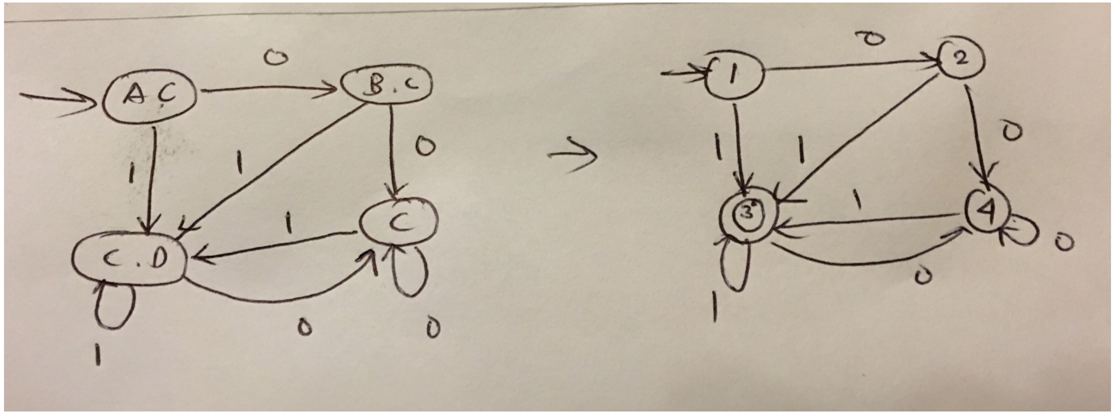
	0	1
{A,B,C}	{B,C}	{A,B,C}
{B,C}	{C}	{B,C}
{C}	{C}	{C}

DFA



NFA → DFA Example

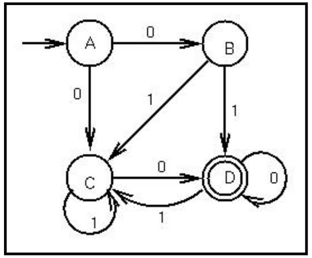


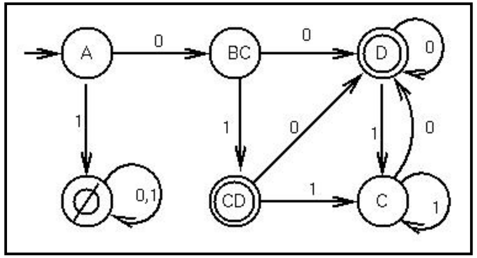


Analyzing the Reduction

- Can reduce any NFA to a DFA using subset alg.
- How many states in the DFA?
  - Each DFA state is a subset of the set of NFA states
  - Given NFA with n states, DFA may have 2^n states
    - Since a set with n items may have 2^n subsets
  - Corollary
    - Reducing a NFA with n states may be $O(2^n)$

NFA





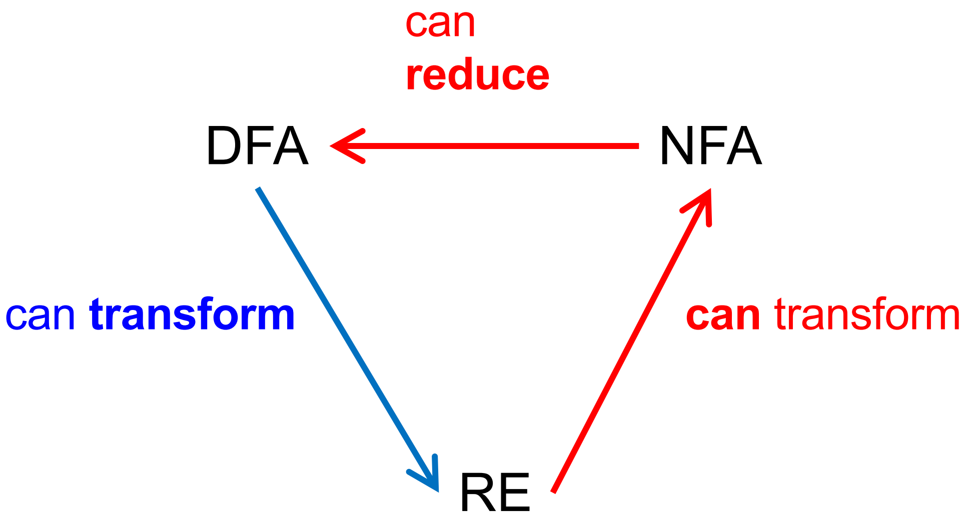
DFA

Recap: Matching a Regexp *R*

- Given *R*, construct NFA. Takes time O(*R*)
- Convert NFA to DFA. Takes time $O(2^{|R|})$
  - But usually not the worst case in practice
- Use DFA to accept/reject string s
  - Assume we can compute $\delta(q,\sigma)$ in constant time
  - Then time to process s is O(|s|)
    - Can't get much faster!

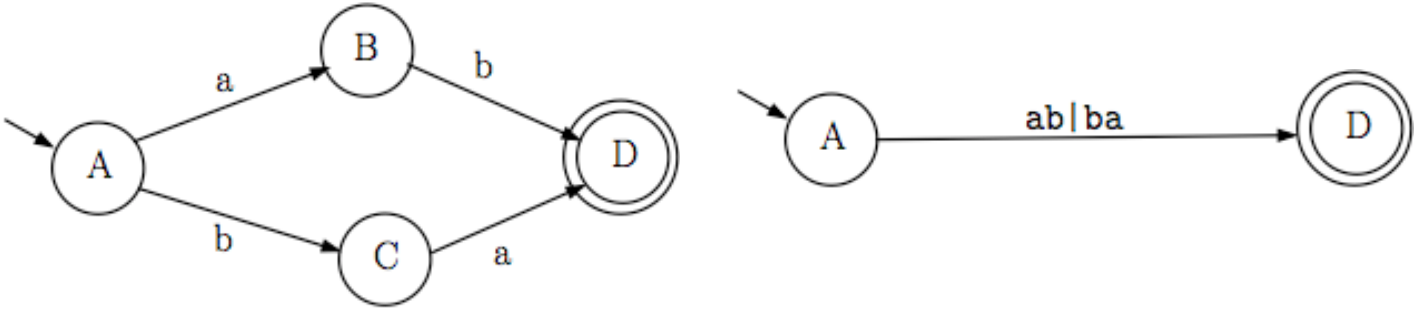
- Constructing the DFA is a one-time cost
  - But then processing strings is fast

Closing the Loop: Reducing DFA to RE



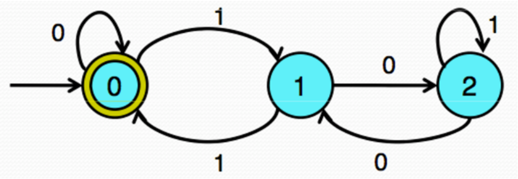
Reducing DFAs to REs

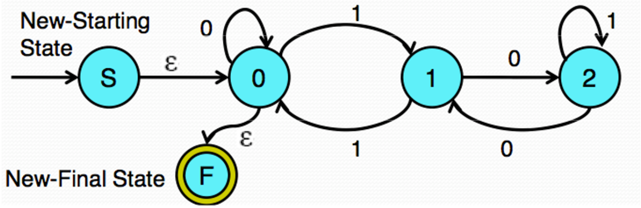
- General idea
  - Remove states one by one, labeling transitions with regular expressions
  - When two states are left (start and final), the transition label is the regular expression for the DFA

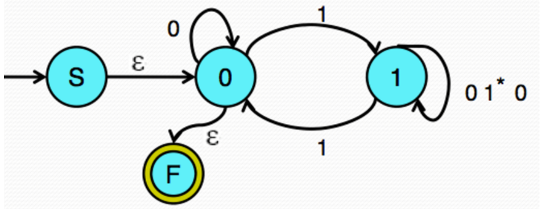


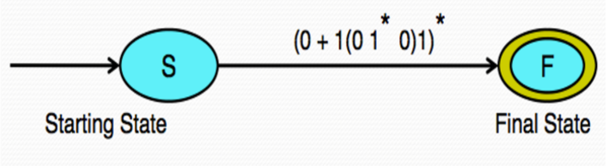
DFA to RE example

Language over Σ= {0,1} such that every string is a multiple of 3 in binary





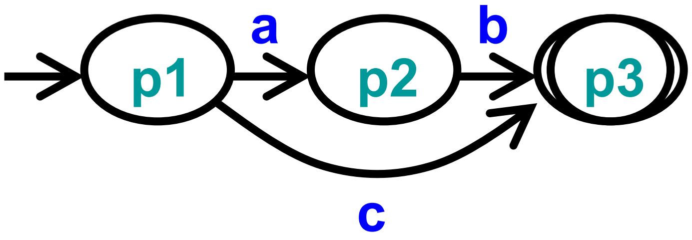


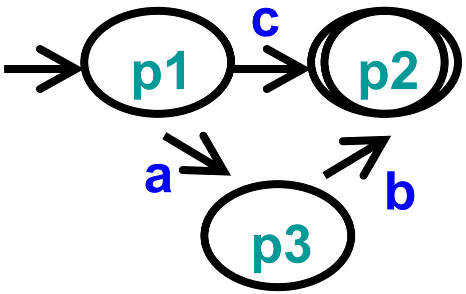


$$(0 + 1(0\,1^{*}\;0)1)^{*}$$

Minimizing DFAs

- Every regular language is recognizable by a unique minimum-state DFA
  - Ignoring the particular names of states
- In other words
  - For every DFA, there is a unique DFA with minimum number of states that accepts the same language





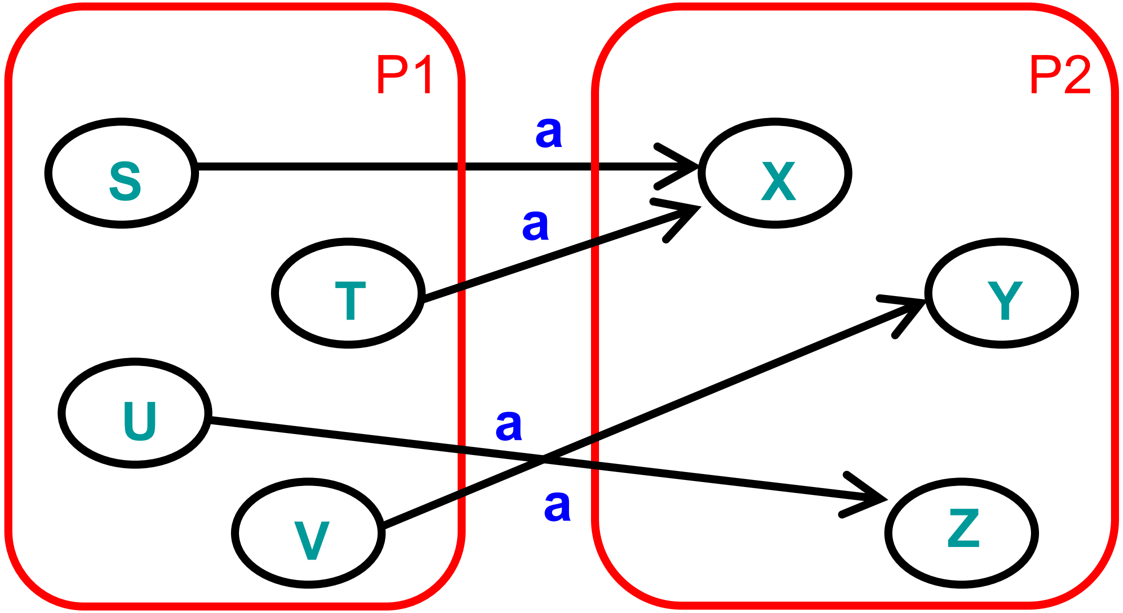
J. Hopcroft, “An n log n algorithm for minimizing states in a finite automaton,” 1971

Minimizing DFA: Hopcroft Reduction

- Intuition
  - Look to distinguish states from each other
    - End up in different accept / non-accept state with identical input
- Algorithm
  - Construct initial partition
    - Accepting & non-accepting states
  - Iteratively split partitions (until partitions remain fixed)
    - Split a partition if **members in partition have transitions to different partitions for same input**
      - Two states x, y belong in same partition if and only if for all symbols in Σ they transition to the same partition
  - Update transitions & remove dead states

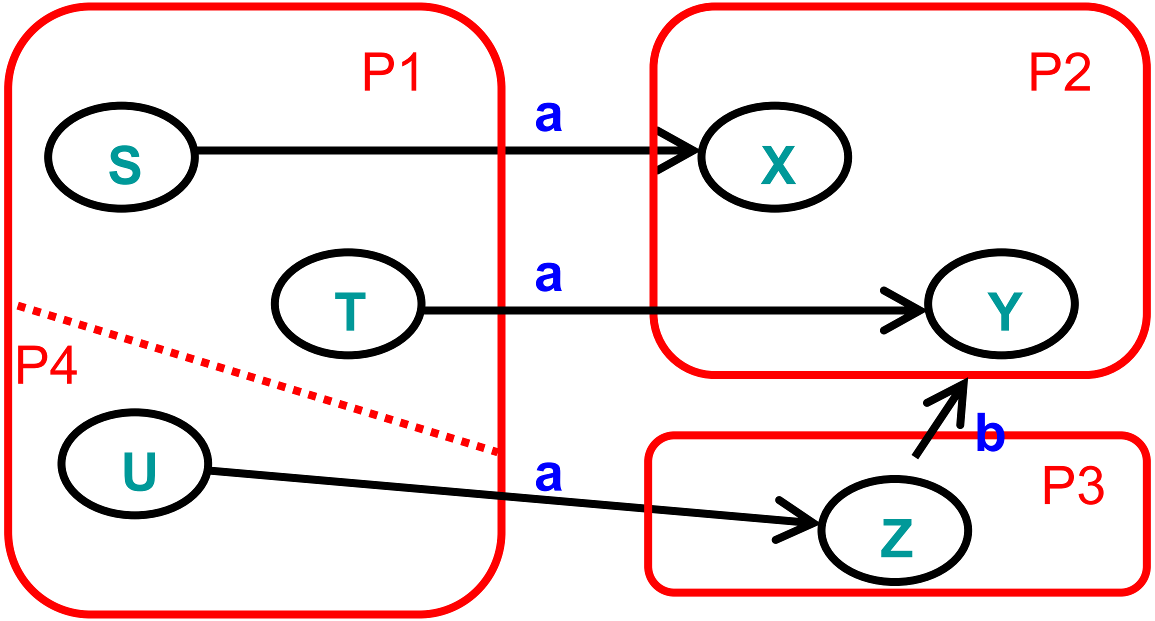
Splitting Partitions

- No need to split partition {S,T,U,V}
  - All transitions on a lead to identical partition P2
  - Even though transitions on a lead to different states



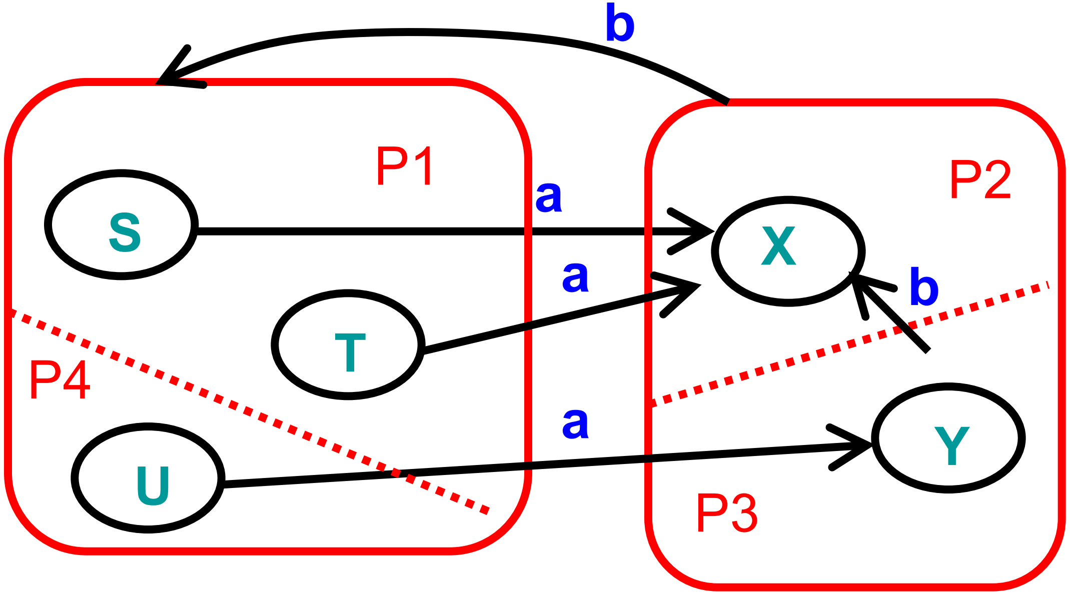
Splitting Partitions (cont.)

- Need to split partition {S,T,U} into {S,T}, {U}
  - Transitions on a from S,T lead to partition P2
  - Transition on a from U lead to partition P3



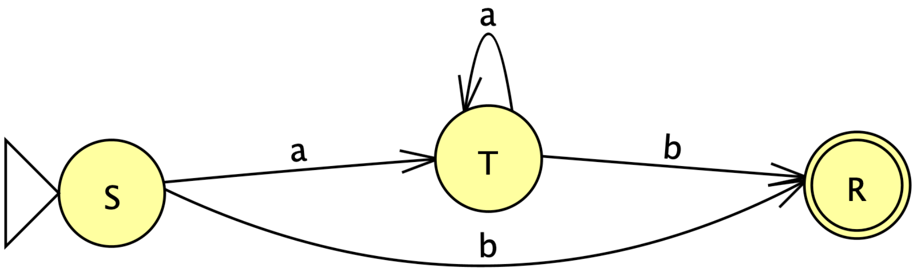
Resplitting Partitions

- Need to reexamine partitions after splits
  - Initially no need to split partition {S,T,U}
  - After splitting partition {X,Y} into {X}, {Y} we need to split partition {S,T,U} into {S,T}, {U}



Minimizing DFA: Example 1

- DFA

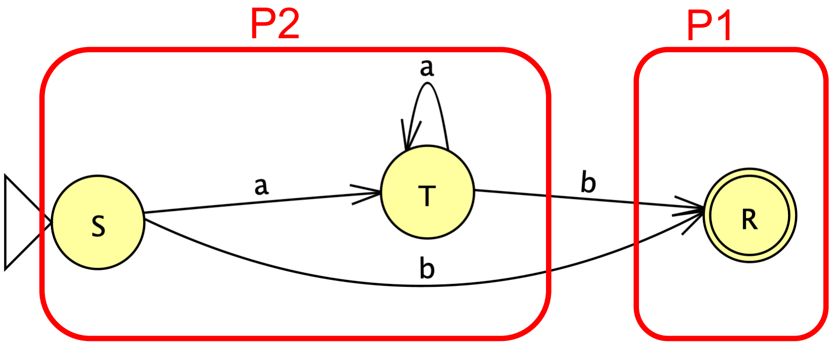


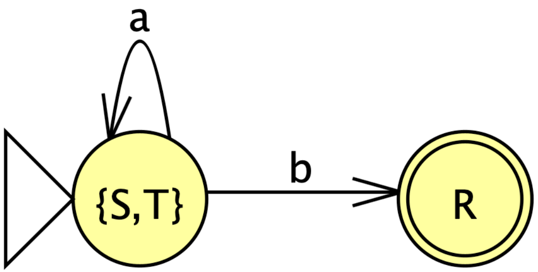
- Initial partitions

- Split partition

Minimizing DFA: Example 1

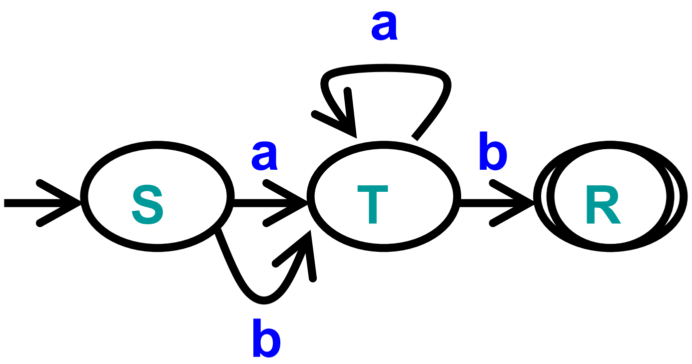
- DFA





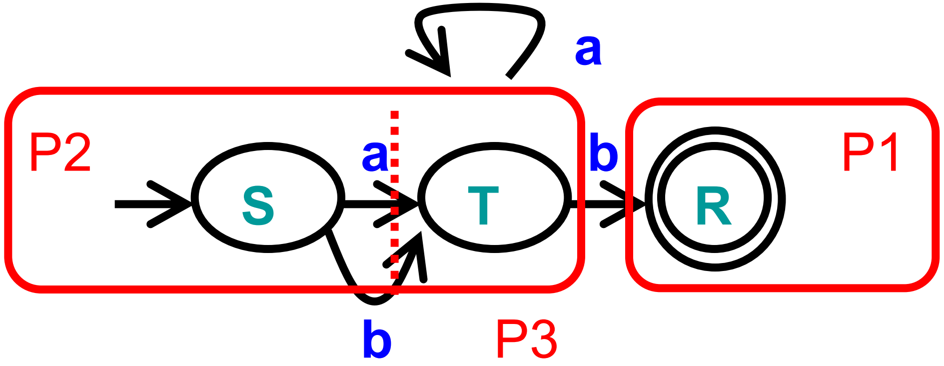
- Initial partitions
  - Accept { R } = P1
  - Reject { S, T } = P2
- Split partition? → Not required, minimization done
  - move(S,a) = T ∊ P2
    - move(S,b) = R ∊ P1
  - move(T,a) = T ∊ P2
    - move (T,b) = R ∊ P1

Minimizing DFA: Example 2



Minimizing DFA: Example 2

- DFA



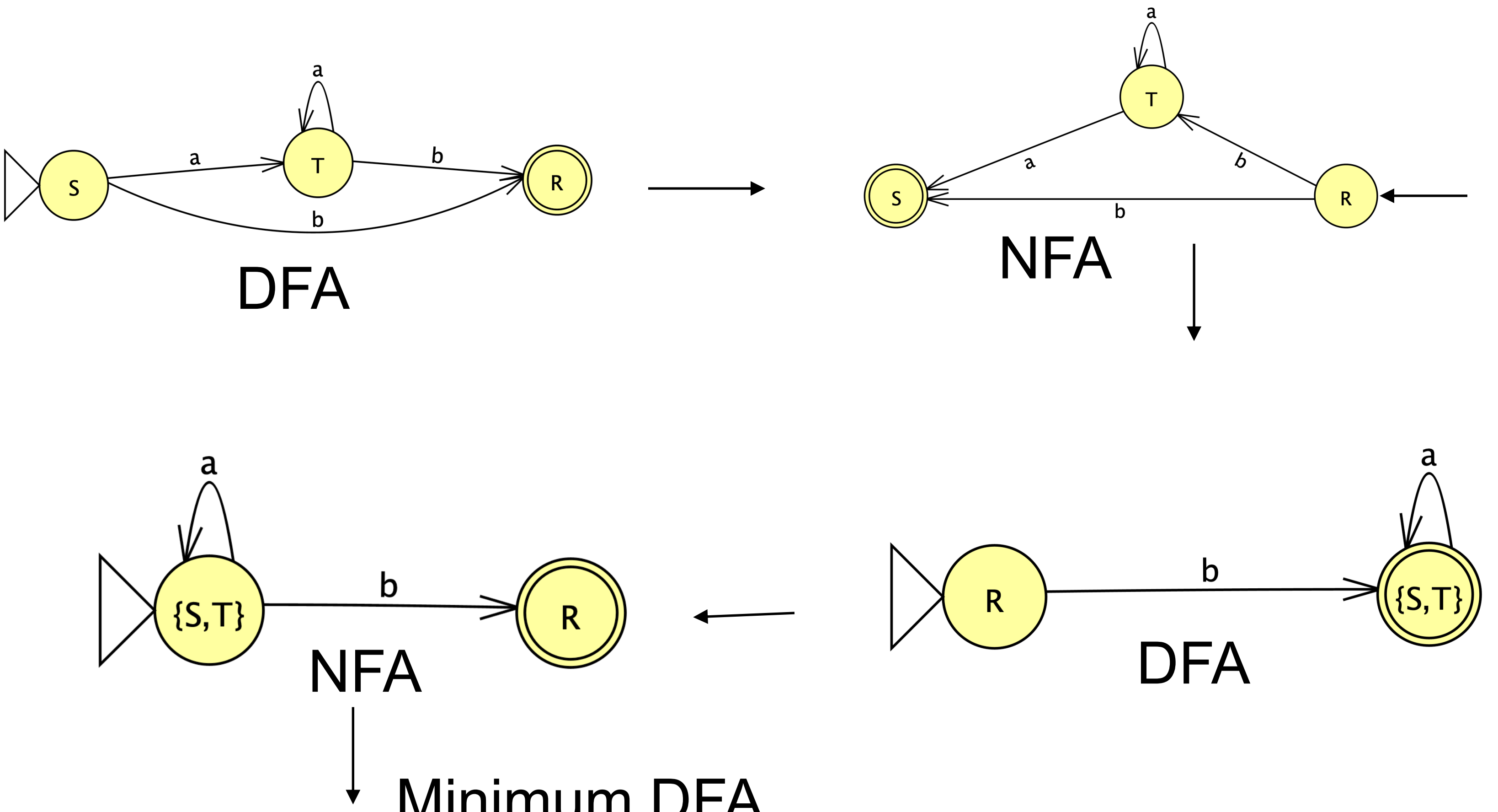
DFA already minimal

- Initial partitions
  - Accept { R }= P1
  - Reject { S, T } = P2
- Split partition? → Yes, different partitions for B
  - move(S,a) = T ∊ P2
    - move(S,b) = T ∊ P2
  - move(T,a) = T ∊ P2
    - move (T,b) = R ∊ P1

Brzozowski's Algorithm: DFA Minimization

1. Given a DFA, reverse all the edges, make the initial state an accept state, and the accept states initial, to get an NFA
2. NFA-> DFA
3. For the new DFA, reverse the edges (and initial-accept swap) get an NFA
4. NFA -> DFA

Brzozowski's algorithm



DFA

NFA

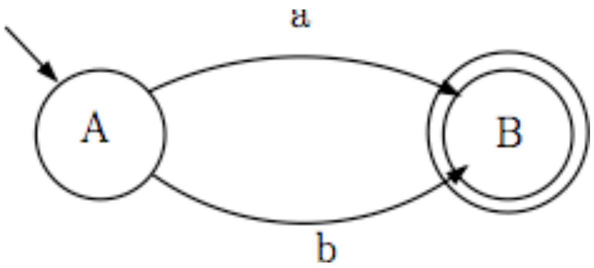
DFA

NFA

Minimum DFA

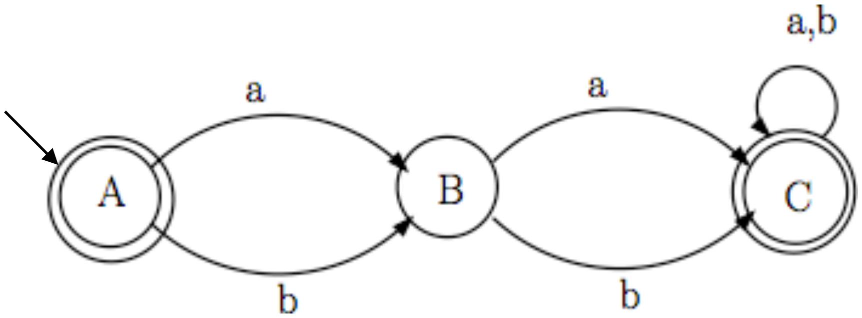
Complement of DFA

- Given a DFA accepting language L
  - How can we create a DFA accepting its complement?
  - Example DFA
    - Σ = {a,b}



Complement of DFA

- Algorithm
  - Add explicit transitions to a dead state
  - Change every accepting state to a non-accepting state & every non-accepting state to an accepting state
- Note this only works with DFAs
  - Why not with NFAs?



Summary of Regular Expression Theory

- Finite automata
  - DFA, NFA
- Equivalence of RE, NFA, DFA
  - RE → NFA
    - Concatenation, union, closure
  - NFA → DFA
    - ε-closure & subset algorithm
- DFA
  - Minimization, complementation