CMSC/PHYS 457, Introduction to Quantum Computing
Spring 2022, University of Maryland

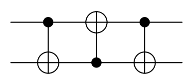
Due: February 25th, 2022
Date: February 13, 2022

Assignment 2

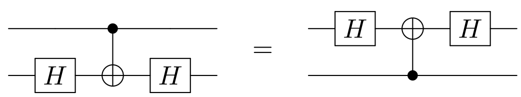
Please submit it electronically to ELMS. This assignment is 7% in your final grade. For the simplicity of the grading, the total number of points for the assignment is 70.

Problem 1. *Circuit identities.*

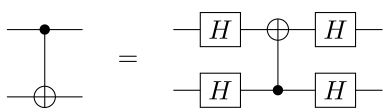
1. *(5 points)* Show that the following circuit swaps two qubits:



2. *(5 points)* Verify the following circuit identity:



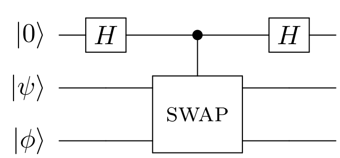
3. *(5 points)* Verify the following circuit identity:



   Give an interpretation of this identity.

Problem 2. *Swap test.*

1. *(5 points)* Let $|\psi\rangle$ and $|\phi\rangle$ be arbitrary single-qubit states (not necessarily computational basis states), and let SWAP denote the 2-qubit gate that swaps its input qubits (i.e., $\text{SWAP}|x\rangle|y\rangle = |y\rangle|x\rangle$ for any $x, y \in \{0, 1\}$). Compute the output of the following quantum circuit:

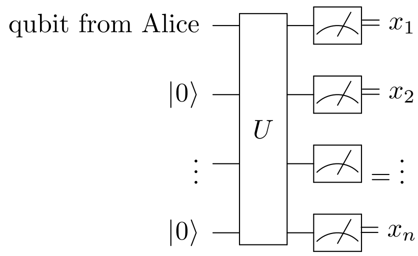


2. *(5 points)* Suppose the top qubit in the above circuit is measured in the computational basis. What is the probability that the measurement result is 0?

3. *(3 points)* If the result of measuring the top qubit in the computational basis is 0, what is the (normalized) post-measurement state of the remaining two qubits?

4. *(2 points)* How do the results of the previous parts change if $|\psi\rangle$ and $|\phi\rangle$ are n-qubit states, and SWAP denotes the $2n$-qubit gate that swaps the first n qubits with the last n qubits?

Problem 3. *A qubit cannot be used to communicate a trit perfectly* Suppose that Alice wants to convey a trit of information (an element of $\{0,1,2\}$) to Bob and all she is allowed to do is prepare one qubit and send it to Bob. Bob is allowed to prepare $n-1$ additional qubits, each in state $|0\rangle$, and apply an n-qubit unitary U operation to the entire n qubit system followed by a measurement in the computational basis.



The outcome will be an element of $\{0,1\}^n$. It is conceivable that such a scheme could exist where Bob can determine the trit from these n bits (e.g., by a function $f(x_1,\cdots,x_n)\in\{0,1,2\}$). We shall prove that this is impossible.

The framework is that Alice starts with a trit $j\in\{0,1,2\}$ (unknown to Bob) and, based on j, prepares a one-qubit state, $\alpha_j|0\rangle+\beta_j|1\rangle, j\in\{0,1,2\}$. and sends it to Bob.

Then Bob applies some n-qubit unitary U to $(\alpha_j|0\rangle+\beta_j|1\rangle)|00\cdots0\rangle$ and measures each qubit in the computational basis, obtaining some $x\in\{0,1\}^n$ as outcome. Finally, Bob applies some function $f:\{0,1\}^n\to\{0,1,2\}$ to x to obtain a trit. The scheme *works* if and only if, starting with any $j\in\{0,1,2\}$, the resulting x will satisfy $f(x)=j$ with probability 1.

1. *(5 points)* Note that each row of the matrix U is a 2^n-dimensional vector. For $j\in\{0,1,2\}$, define the space V_j to be the span of all rows of U that are indexed by an element of the set $f^{-1}(j)\subseteq\{0,1\}^n$. Prove that V_0,V_1, and V_2 are mutually orthogonal spaces.
2. *(5 points)* Explain why, for a scheme to work, $(\alpha_j|0\rangle+\beta_j|1\rangle)|00\cdots0\rangle\in V_j$ must hold for all $j\in\{0,1,2\}$.
3. *(5 points)* Prove that it is impossible for $(\alpha_j|0\rangle+\beta_j|1\rangle)|00\cdots0\rangle\in V_j$ to hold for all $j\in\{0,1,2\}$.

Problem 4. *The Hadamard gate and qubit rotations*

1. *(5 points)* Suppose that $(n_x,n_y,n_z)\in\mathcal{R}^3$ is a unit vector and $\theta\in\mathcal{R}$. Show that
$$e^{-i\frac{\theta}{2}(n_xX+n_yY+n_zZ)}=\cos(\tfrac{\theta}{2})\,I-i\sin(\tfrac{\theta}{2})\,(n_xX+n_yY+n_zZ).$$
2. *(5 points)* Find a unit vector $(n_x,n_y,n_z)\in\mathcal{R}^3$ and numbers $\phi,\theta\in\mathcal{R}$ so that
$$H=e^{i\phi}e^{-i\frac{\theta}{2}(n_xX+n_yY+n_zZ)},$$
where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?
3. *(5 points)* Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha,\beta,\gamma,\phi\in\mathcal{R}$ such that $H=e^{i\phi}R_y(\gamma)R_x(\beta)R_y(\alpha)$.

Problem 5. *Universality of gate sets.* Prove that each of the following gate sets either is or is not universal. You may use the fact that the set $\{\textsc{cnot},H,T\}$ is universal.

1. *(3 points)* $\{H, T\}$
2. *(3 points)* $\{\textsc{cnot}, T\}$
3. *(4 points)* $\{\textsc{cnot}, H\}$
4. *(Bonus: 10 points)* $\{\textsc{cnot}, H, T^2\}$