CMSC/PHYS 457, Introduction to Quantum Computing
Spring 2023, University of Maryland

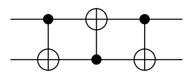
Due: March 2nd, 2023
Date: February 20, 2023

Assignment 2

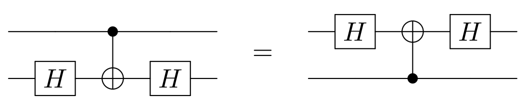
Please submit it electronically to ELMS. This assignment is 6% in your final grade. For the simplicity of the grading, the total number of points for the assignment is 60.

Problem 1. *Circuit identities.*

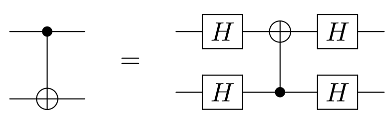
1. *(5 points)* Show that the following circuit swaps two qubits:



2. *(5 points)* Verify the following circuit identity:



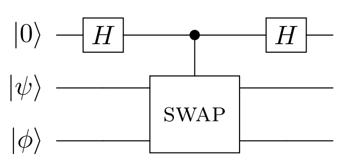
3. *(5 points)* Verify the following circuit identity:



   Give an interpretation of this identity.

Problem 2. *Swap test.*

1. *(5 points)* Let $|\psi\rangle$ and $|\phi\rangle$ be arbitrary single-qubit states (not necessarily computational basis states), and let SWAP denote the 2-qubit gate that swaps its input qubits (i.e., $\text{SWAP}|x\rangle|y\rangle = |y\rangle|x\rangle$ for any $x, y \in \{0, 1\}$). Compute the output of the following quantum circuit:



2. *(5 points)* Suppose the top qubit in the above circuit is measured in the computational basis. What is the probability that the measurement result is 0?

3. *(3 points)* If the result of measuring the top qubit in the computational basis is 0, what is the (normalized) post-measurement state of the remaining two qubits?

4. *(2 points)* How do the results of the previous parts change if $|\psi\rangle$ and $|\phi\rangle$ are n-qubit states, and SWAP denotes the $2n$-qubit gate that swaps the first n qubits with the last n qubits?

Problem 3. *The Hadamard gate and qubit rotations*

1. *(5 points)* Suppose that $(n_x, n_y, n_z) \in \mathcal{R}^3$ is a unit vector and $\theta \in \mathcal{R}$. Show that

$$e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} = \cos(\tfrac{\theta}{2})\, I - i \sin(\tfrac{\theta}{2})\, (n_x X + n_y Y + n_z Z).$$

2. *(5 points)* Find a unit vector $(n_x, n_y, n_z) \in \mathcal{R}^3$ and numbers $\phi, \theta \in \mathcal{R}$ so that

$$H = e^{i\phi} e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)},$$

   where H denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

3. *(5 points)* Write the Hadamard gate as a product of rotations about the x and y axes. In particular, find $\alpha, \beta, \gamma, \phi \in \mathcal{R}$ such that $H = e^{i\phi} R_y(\gamma) R_x(\beta) R_y(\alpha)$.

Problem 4. *Universality of gate sets.* Prove that each of the following gate sets either is or is not universal. You may use the fact that the set $\{\textsc{cnot}, H, T\}$ is universal.

1. *(5 points)* $\{H, T\}$
2. *(5 points)* $\{\textsc{cnot}, T\}$
3. *(5 points)* $\{\textsc{cnot}, H\}$
4. *(Bonus: 10 points)* $\{\textsc{cnot}, H, T^2\}$