# Problem Set #3

CMSC 858L
Instructor: Daniel Gottesman

Due on Gradescope Mar. 16, 2023, noon

The late deadline to turn the problem set in without penalty is Mar. 19, 2023, noon.

**Problem #1. Phase vs. Bit Flip Oracles (25 pts.)**

In class, we have seen two different ways to convert a classical oracle $O(x)$ into a quantum one. (In this problem, restrict attention to oracles with a one-bit output.) The "standard" answer is the *bit flip oracle* $O_b|x\rangle|c\rangle = |x\rangle|c \oplus O(x)\rangle$. The *phase oracle* is $O_p|x\rangle = (-1)^{O(x)}|x\rangle$ is simpler since it doesn't require an ancilla. We saw in class that you can implement $O_p$ given one use of $O_b$ by using the ancilla $|-\rangle = |0\rangle - |1\rangle$. In this problem, we will invesitgate the opposite direction.

a) (10 pts.) Show that with any number of queries to $O_p$, it is not possible to distinguish $O(x)$ from its complement $\overline{O}$, for which $\overline{O}(x) = 1 \oplus O(x)$.

b) (15 pts.) Assume we know that $O(00\ldots0) = 1$. Find a way to implement $O_b$ using one query to $O_p$.

**Problem #2. Query Complexity Problems (75 pts.)**

A marked element for an oracle $O$ is an input $x$ such that $O(x) = 1$.

a) (25 pts.) Consider the following problem: Given an oracle $O$ with $N$ possible inputs, return 1 if there are either 0 or exactly 2 marked elements; otherwise return 0.

   Find an algorithm to solve this problem using $O(\sqrt{N})$ queries. Then use the method of polynomials to prove that the query complexity is $\Omega(\sqrt{N})$.

b) (25 pts.) Consider the following problem: Given an oracle $O$ with $N$ possible inputs ($N$ a multiple of 4), return 1 if there are at most $N/4$ marked elements and 0 otherwise. Note that the threshold $N/4$ is sharp: If there are $N/4$ marked elements, the algorithm should return 1 with probability at least 2/3, and if there are $N/4+1$ marked elements, the algorithm should return 0 with probability at least 2/3.

   Find the quantum query complexity (upper and lower bounds, up to a constant factor) of this problem.

c) (25 pts.) Consider the following problem: Given an oracle $O(x, y)$ which takes two inputs, each with $N$ possible values. There exist $x_0$ and $y_0$ such that $O(x, y) = 1$ iff exactly one of $x = x_0, y = y_0$, otherwise $O(x, y) = 0$. That is, $O(x_0, y) = 1$ if $y \neq y_0$, $O(x, y_0) = 1$ if $x \neq x_0$, $O(x, y) = 0$ if $x \neq x_0, y \neq y_0$, and $O(x_0, y_0) = 0$. The algorithm should return $(x_0, y_0)$.

   Find an algorithm to solve this problem using $O(\sqrt{N})$ queries. Then prove that the query complexity is $\Omega(\sqrt{N})$.

   **Hint:**There are probably many ways to prove a lower bound, but my solution involves a reduction.