Problem Set #4

CMSC 858L
Instructor: Daniel Gottesman

Due on Gradescope Apr. 6, 2023, noon

The late deadline to turn the problem set in without penalty is Apr. 9, 2023, noon.

Problem #1. Adversary Bounds (40 points)

a) (10 point) When f is a non-constant symmetric total function, let $f_k = f(x)$ for some x of weight k (all give the same value),

$$\Gamma(f) = \min\{|2k - N + 1| \text{ s.t. } f_k \neq f_{k+1}\} \tag{1}$$

$$\Upsilon(f) = \max\{\sqrt{(k+1)(N-k)} \text{ s.t. } f_k \neq f_{k+1}\}. \tag{2}$$

Recall that in class we saw results using the polynomial method to lower bound the query complexity of f as $\Omega(\sqrt{N(N - \Gamma(f))})$. Show, using results about the adversary method, that the query complexity of f is at least $\Omega(\Upsilon(f))$.

b) (10 points) Prove that the two bounds from part a give the same value (up to constant factors).

c) (10 points) Let

$$g_a(X_0, \ldots, X_{N-1}) = X_{a\sqrt{N}} \text{ OR } X_{a\sqrt{N}+1} \text{ OR } \ldots \text{ OR } X_{(a+1)\sqrt{N}-1} \tag{3}$$

(that is, the OR of a block of $\sqrt{N}$ variables), and let

$$f(X_0 \ldots, X_{N-1}) = \oplus_{a=0}^{\sqrt{N}-1} g_a(X_0, \ldots, X_{N-1}) \tag{4}$$

(that is, the XOR or parity of the different blocks). Using the adversary method, find a lower bound on the query complexity.

d) (10 points) Find an upper bound on the query complexity of the function from part c.

Note: You should be able to match the upper and lower bounds in parts c and d, up to polylog factors.

Problem #2. LOCAL HAMILTONIAN variants (60 points)

For this problem, all languages refer to instances of the form (H, E, Δ), with H a 5-local Hermitian operator, Δ polynomially small, and all coefficients of appropriately bounded accuracy. You may use any results we discussed in class, even if we did not prove them.

Remember that a problem is QMA-complete if it is QMA-hard and in QMA. Be sure the address both parts in your answers.

a) (15 points) Prove that the following problem is QMA-complete: Given (H, E, Δ), is the *largest* eigenvalue of H greater than E? You are promised that the largest eigenvalue of H is either greater than E or less than $E - \Delta$.

b) (15 points) Prove that the following problem is QMA-complete: Given (H, E, Δ), is the *second-smallest* eigenvalue of H less than E? You are promised that the second-smallest eigenvalue of H is either less than E or greater than $E + \Delta$.

c) (15 points) Prove that the following problem is in QMA: Given (H, E, Δ), does there exist an eigenvalue of H between $E - \Delta$ and $E + \Delta$? You are promised that there are no eigenvalues in the ranges $E - 2\Delta$ to $E - \Delta$ or $E + \Delta$ to $E + 2\Delta$.

   Note: Changed from showing it is QMA-complete to just showing that it is QMA.

d) (15 points) Prove that the following problem is QMA-*hard*: Given (H, E, Δ), does the ground state $|\psi\rangle$ of H have the property that $\langle\psi|Z_1|\psi\rangle \leq E$? Here Z_1 is the Z operator on the first qubit of $|\psi\rangle$. You are promised that all ground states of H satisfy either $\langle\psi|Z_1|\psi\rangle \leq E$ or $\langle\psi|Z_1|\psi\rangle \geq E + \Delta$.

   What is the difficulty in showing that this problem is in QMA?

   Note: Changed from unique ground state.