Problem Set #2

Quantum Error Correction
Instructor: Daniel Gottesman

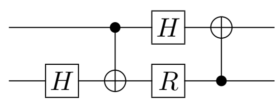
Due Tuesday, Mar. 5, 2024, 5 PM

Note: This problem set has problems on 2 pages.

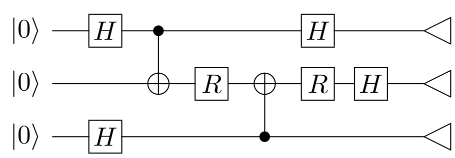
Problem #1. Analyzing Clifford group circuits (25 points)

In the following diagrams, $R = R_{\pi/4}$ is the matrix $\mathrm{diag}(1, i)$ and H is the Hadamard transform.

a) (10 points) For the following Clifford group circuit, compute the overall action on Paulis and use that to write down the 4×4 unitary matrix performed by the circuit:



b) (15 points) For the following Clifford group circuit, use Clifford simulation techniques to compute the full probability distribution of the 8 possible classical outputs after measuring all qubits in the computational basis:



Problem #2. Twirling (20 points)

Let $S(\rho)$ be a quantum operation (a completely positive trace-preserving map) taking n qubits to n qubits. **Hint:** (For both parts) Any $2^n \times 2^n$ matrix can be expanded in the basis of Pauli operators.

a) (10 points) Consider the following quantum operation: Choose a uniformly random $P \in \mathcal{P}_n/\{\pm I, \pm iI\}$ (i.e., a Pauli ignoring global phase). Apply $P^\dagger$, then S, then P (for the same P). Show that, averaging over P, the resulting quantum operation is a Pauli channel. A Pauli channel is any channel $\mathcal{S}$ such that $\mathcal{S}(rho) = (1 - \sum_P p_P)\rho + \sum_P p_P P\rho P^\dagger$. (Where the sums are over non-trivial Paulis without phases, $\mathcal{P}_n/\{\pm 1, \pm i\} \setminus \{I\}$.)

b) (10 points) Now instead of choosing a random Pauli, choose a random Clifford and do the same thing, i.e., uniformly random $C \in \mathcal{C}_n/\{e^{i\phi}I\}$, apply $C^\dagger$, then S, then C. Show that, averaging over C, the resulting quantum channel is a depolarizing channel. A depolarizing channel is a channel $\mathcal{S}$ such that $\mathcal{S}(rho) = [1 - (2^n - 1)p]\rho + \sum_P pP\rho P^\dagger$. (Where the sums are over non-trivial Paulis without phases, $\mathcal{P}_n/\{\pm 1, \pm i\} \setminus \{I\}$.) That is, an n-qubit depolarizing channel is a Pauli channel where all probabilities p_P are the same.

Problem #3. Logical operations for qudit code (15 points)

Consider the following stabilizer code for qutrits (qudits with dimension $p = 3$):

$$\begin{matrix} X & X & Z & Z \\ Z & Z & X & X \end{matrix}$$

a) (5 points) Show that these generators define a qudit stabilizer code. What are its parameters as a QECC?

b) (5 points) Find a generating set for the logical Pauli group. (I.e., coset representatives for $\overline{X}_i$ and $\overline{Z}_i$).

c) (5 points) For your choice of logical Pauli operators, write down the codeword for logical $|\overline{0}\rangle$ states (for all logical qubits) expanded in the standard basis for the physical qubits.