Homework 5: NP-Completeness and Approximations (Preliminary)

Problem 1. Given an undirected graph $G = (V, E)$, we say that a subset $V' \subseteq V$ is a *strong independent set* if for any two vertices $u, v \in V'$ these vertices are separated by a distance of at least three (that is, they are not adjacent, and further, they share no neighbors in common). The *Strong Independent Set problem* (SIS) is, given an undirected graph $G = (V, E)$ and an integer k, does G have a strong independent set of size k. Prove that SIS is NP-complete. (Hint: Remember that this involves both showing that SIS $\in$ NP and SIS is NP-hard. Reduction from the standard Independent Set problem (IS).)

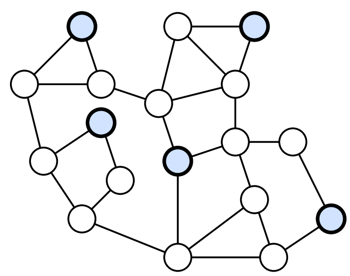

Figure 1: Strong Independent Set (SIS).

Problem 2. In this problem, we will explore three problems that are closely related to the Directed Hamiltonian Cycle problem (DHC) discussed in class. In each case, present a reduction from a known NP-complete problem to the given decision problem. (You do not need show that the problem is in NP.)

Briefly (in a sentence or so) explain why your reduction runs in polynomial time, and give a careful proof of the correctness of your reduction.

(a) *Directed Hamiltonian Path* (DHP): Given a directed graph $G = (V, E)$, does there exist a path that visits all the vertices exactly once? (See Fig. 2(a).) (Hint: Reduction from DHC. Create a start and end vertex for the path.) While it is possible to do this by modifying the 3SAT $\leq_P$ DHC reduction from class, for full credit, reduce directly from DHC.

(b) *Hamiltonian Path* (HP): Given an *undirected* graph $G = (V, E)$, does there exists a path that visits all the vertices exactly once? (See Fig. 2(b).) (Hint: Reduction from DHP. One approach involves replacing each vertex of the graph with a small cluster of vertices.)

(c) *Degree-3 Spanning Tree* (D3ST): Given an *undirected* graph $G = (V, E)$, does there exists a spanning tree such that the degree of each vertex in the tree (that is, the number of edges of E' incident on any vertex) is at most three (see Fig. 2(c)). Recall that a *spanning tree* is, a subset $E' \subseteq E$ that induces a connected, acyclic subgraph containing all the vertices of G. (Hint: Reduction from HP. This can be done by a local modification to each vertex in the graph.)

Problem 3. (TBD)

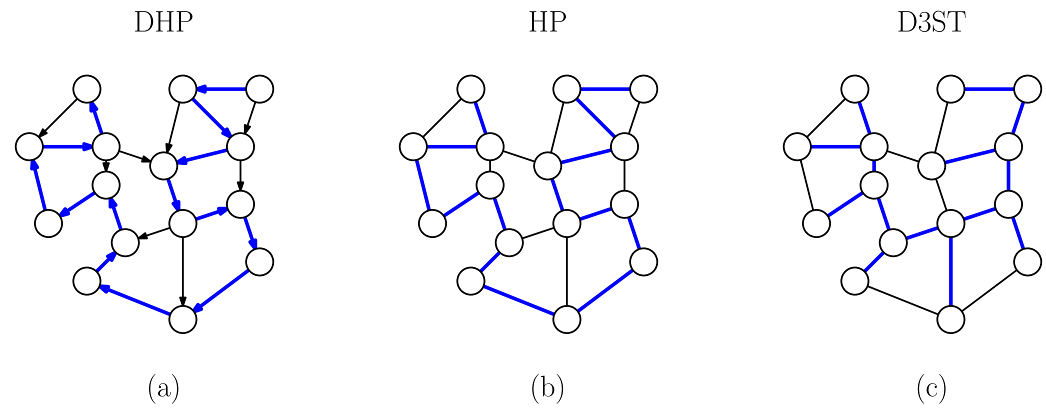

Figure 2: NP-Complete problems related to Hamiltonian cycle.

Problem 4. (TBD)

Problem 5. (TBD)

Challenge Problem. (TBD)