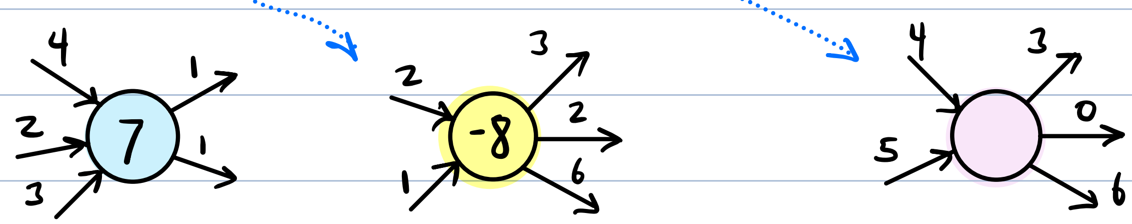
# CMSC 451 - Algorithm Design
## Lecture 14 - Network Flow - Extensions + Applications

### Circulation with Demands:

**Transportion:**
- Network (where commodities flow)
  - Edge capacities
- Supply nodes - generate flow
- Demand nodes - consume flow

**Decision problem:**
- Can we manage all the flow from suppliers to consumers?

**Supply + Demand:** Given digraph $G=(V,E)$
- Each node $v \in V$ has demand $d(v)$:
  - demand: $d(v) > 0$ ($d_v$ net incoming)
  - supply: $d(v) < 0$ ($-d_v$ net outgoing)
  - balanced: $d(v) = 0$ (incoming = outgoing)



$f^{in} - f^{out} = 7$ $\quad$ $f^{in} - f^{out} = -8$ $\quad$ $f^{in} - f^{out} = 0$

**Circulation:** Given a directed graph $G=(V,E)$ with edge capacities $c(u,v) > 0$ and vertex demands $d(v)$, a circulation is a function $f: E \to \mathbb{R}$ such that:

**Capacity constraint:** $0 \le f(u,v) \le c(u,v)$, $\forall (u,v) \in E$

**Supply/Demand constraint:**

$$f^{in}(v) - f^{out}(v) = d(v), \quad \forall v \in V$$

Recall:

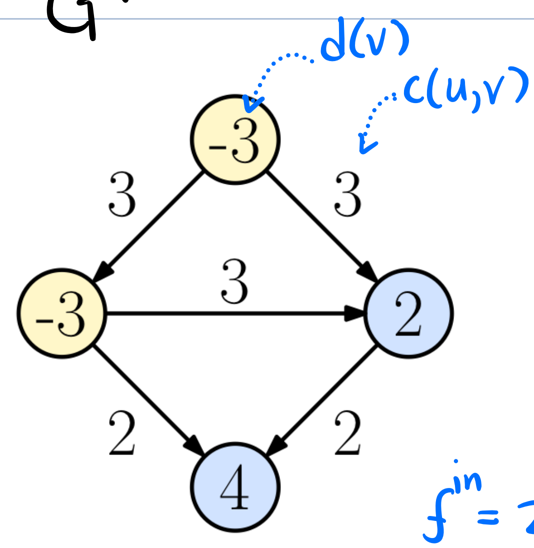
$$f^{in}(v) = \sum_{(u,v)\in E} f(u,v)$$

$$f^{out}(v) = \sum_{(v,w)\in E} f(v,w)$$

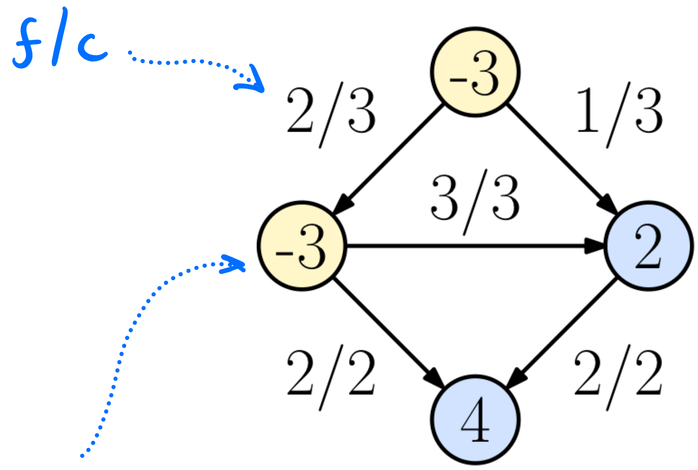
-5 supply node

4 demand node

G:




A circulation:




$f^{in} = 2$

$f^{out} = 3 + 2 = 5$

$f^{in} - f^{out} = 2 - 5 = -3 = d(v)$

Define: $S = \{v \mid d(v) < 0\}$ Supply nodes

$T = \{v \mid d(v) > 0\}$ Demand nodes

Clearly: $\sum_{v \in T} d(v) = -\sum_{v \in S} d(v)$

demand = supply

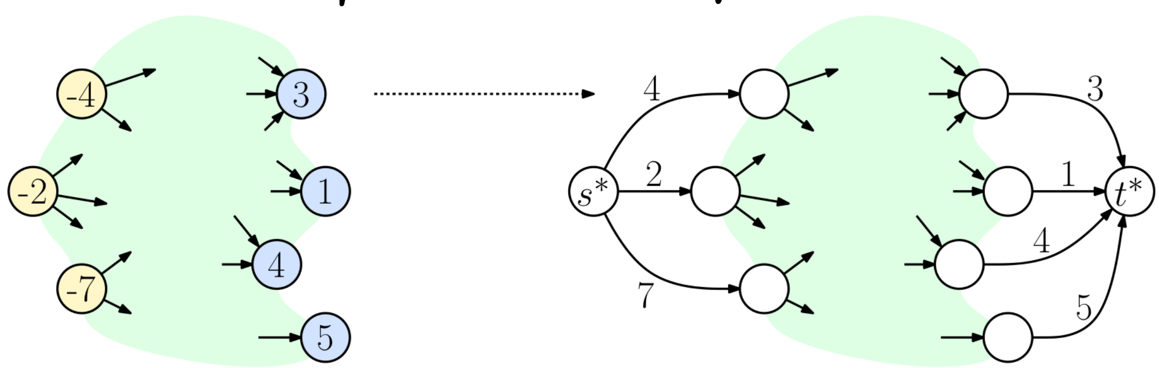
(or no feasible circulation exists)

Define: Total demand: $\sum_{v \in T} d(v)$ denoted $D(G)$

## Solving Circulation by Network Flow

- Treat supply nodes as sources
- Treat demand nodes as sinks
- Add super-source + super-sink to connect

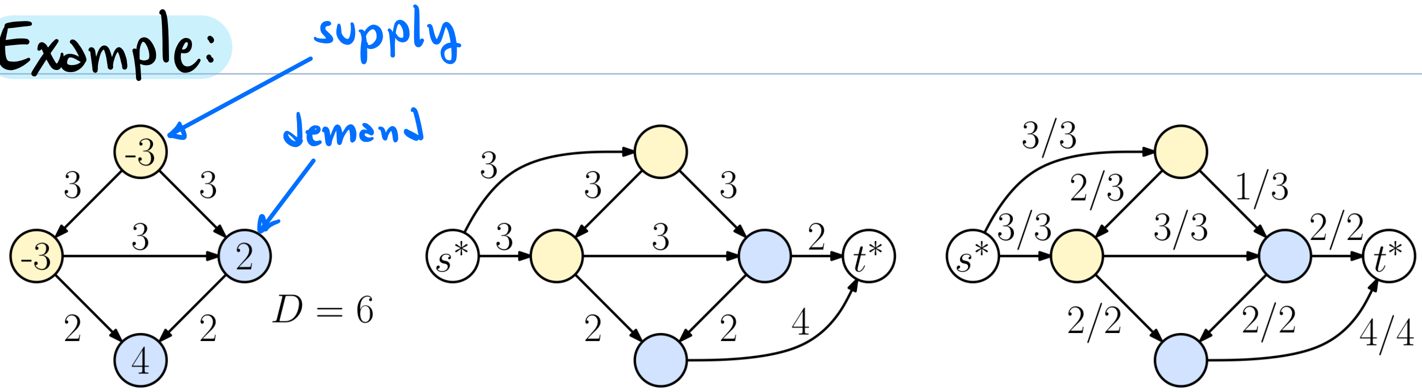
$D = 13$




- Create network $G' = (V', E')$, $V' \leftarrow V$, $E' \leftarrow E$
- Add $s^*$ + $t^*$ to $V'$
- For $v \in V$: Add $(s^*, v)$ capacity $= -d(v)$ if $v \in S$

  Add $(v, t^*)$ capacity $= d(v)$ if $v \in T$

**Example:**




**Claim:** $G$ has a feasible circulation iff $G'$ has a flow of value $D(G)$

Recall: $= \sum_{v \in T} d(v)$

$= -\sum_{v \in S} d(v)$

**Proof:**

($\Rightarrow$) Let $f$ be a circulation in $G$.

Create flow $f'$ by:

- saturate all edges out of $s^*$
- " " " into $t^*$
- all other edges same as $f$

This satisfies –

- capacity constraints
- supply-demand constraints

Exercise

($\Leftarrow$) Let $f'$ be flow of value $D$ in $G'$

Obs: All edges out of $s^*$ + into $t^*$ must be saturated

$\Rightarrow$ All supply/demands satisfied

Exercise

# Circulations with Upper + Lower Capacity Bounds

- Sometimes, we want to apply both upper + lower bounds on flow amounts
- Lower bound ⇒ guarantees minmum utilization

- For each edge $(u,v) \in E$, let:

  $c(u,v)$ = upper bound on flow

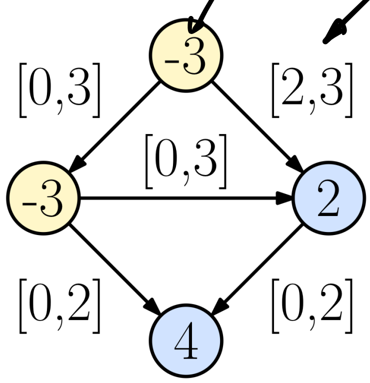
  $\ell(u,v)$ = lower bound on flow
- New conditions: A flow is a circulation if

  - $\ell(u,v) \leq f(u,v) \leq c(u,v), \quad \forall (u,v) \in E$
  - $f^{in}(v) - f^{out}(v) = d(v), \quad \forall v \in V$

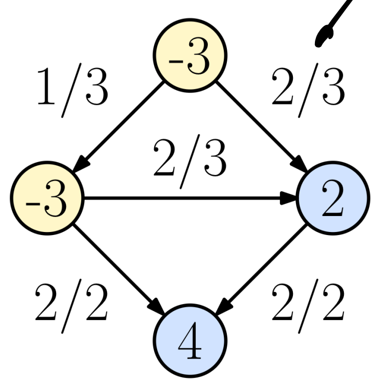
Example:

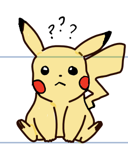
$d(u)$ $[\ell(u,v), c(u,v)]$ $f(u,v)/c(u,v)$

G:




f:

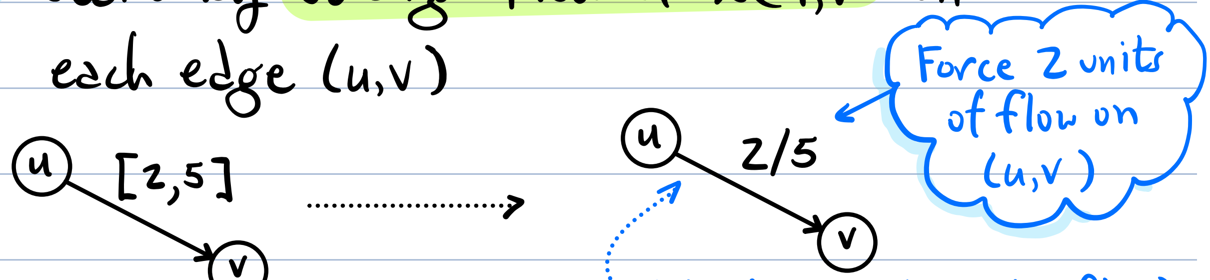


But how can we enforce these lower bounds?

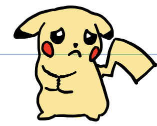
- Can do this by "mucking" with demands!

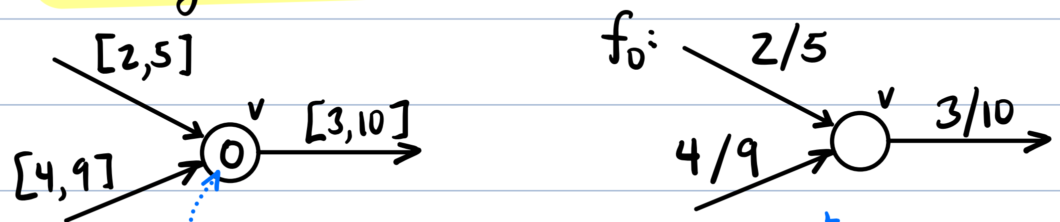
**Step 1:** Start by forcing flow of $\ell(u,v)$ on each edge $(u,v)$




Call this $f_0$ (pseudo-flow)

But this forced flow won't generally satisfy vertex demands:



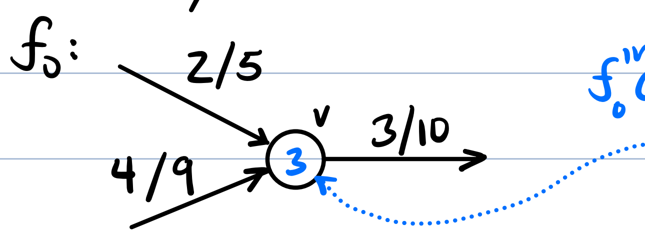



$d(v) = 0$

$$f_0^{in}(v) - f_0^{out}(v) = (2+4)-3$$
$$= 3 \neq d(v) \;☹$$

**Step 2:** Determine excess flow through each vertex

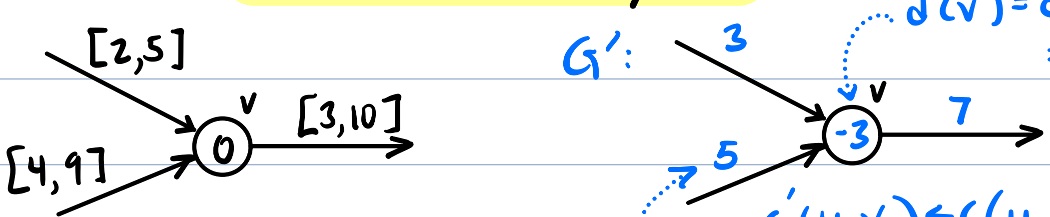
$$x(v) = f_0^{in}(v) - f_0^{out}(v)$$




$$f_0^{in}(v) - f_0^{out}(v) = (2+4)-3$$
$$= 3 = x(v)$$

**Step 3:** Eliminate $f_0$ by reducing upper capacities + adjusting demands:

$$c'(u,v) \leftarrow c(u,v) - f_0(u,v) = c(u,v) - \ell(u,v)$$
$$d'(v) \leftarrow d(v) - x(v)$$




$$d'(v) = d(v) - x(v)$$
$$= 0 - 3$$

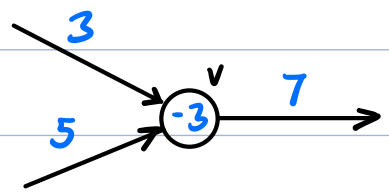
$$c'(u,v) \leftarrow c(u,v) - \ell(u,v)$$
$$= 9 - 4 = 5$$

Observe: Lower constraints gone - just upper remain

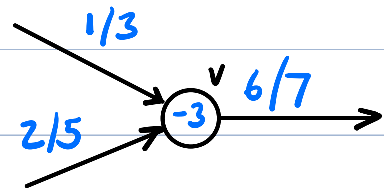
**Step 4:** Apply standard circulation algorithm to this modified network $G'$

$\rightarrow$ Let $f_1$ be result

$G'$:



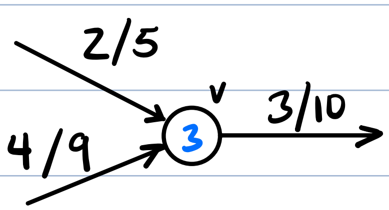

$f_1$:



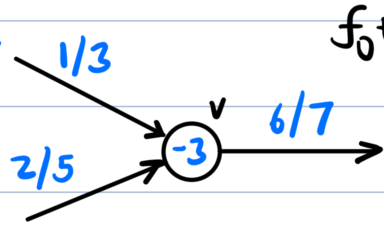

**Step 5:** Add back the (eliminated) forced lower flow:

return $f_0 + f_1$

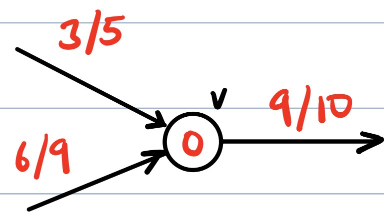
$f_0$:




$f_1$:




$f_0 + f_1$:




**Summary:** Given $G$ with $[\ell, u]$ capacity constraints

- Create pseudo-flow $f_0(u,v) \leftarrow \ell(u,v)$
- Create network $G'$ (same vertices + edges)

  $c'(u,v) \leftarrow c(u,v) - \ell(u,v)$

  $d'(u,v) \leftarrow d(v) - x(v)$, $x(v) = f_0^{in}(v) - f_0^{out}(v)$
- Compute (standard) circulation in $G' \rightarrow f_1$
- If $G'$ has no circulation — report infeasible

  else return $f \leftarrow f_0 + f_1$

## Correctness:

> **Lemma:** $G$ (with both lower + upper constraints) has a valid circulation iff $G'$ (just upper constraints) has a valid circulation.

See Kleinberg + Tardos for proof

- Intuitively: $f_0$ handles lower bounds
  $d'$ handles imbalance in demands
  $c'$ limits additional flow above $\ell(u,v)$ so it doesn't exceed $c(u,v)$
  $f_0 + f_1$ combines the two flows

## Application - Survey Design

**Setup:**

- A company sells $k$ products
- Survey $n$ customers - "How satisfied..."
- Rules:
  1. Only ask about products customer purchased
  2. Customer $i$: Ask about $\geq c_i^-$ products
     and $\leq c_i^+$ products
  3. Product $j$: Survey $\geq p_j^-$ customers
     and $\leq p_j^+$ customers

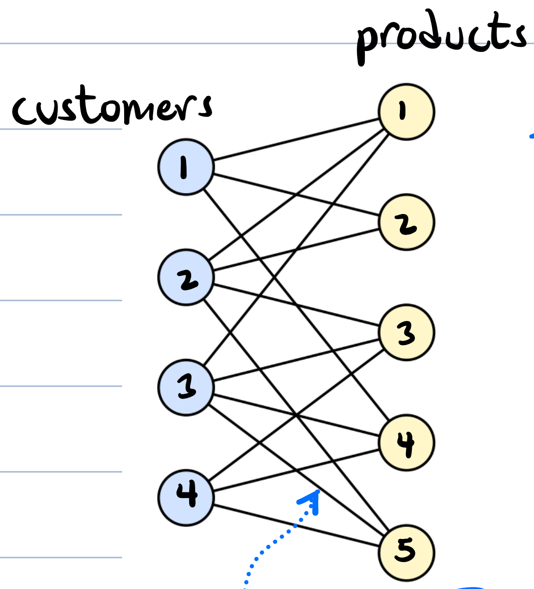
Q: Given $C = \{[c_1^-, c_1^+], \dots, [c_n^-, c_n^+]\}$

$P = \{[p_1^-, p_1^+], \dots, [p_k^-, p_k^+]\}$

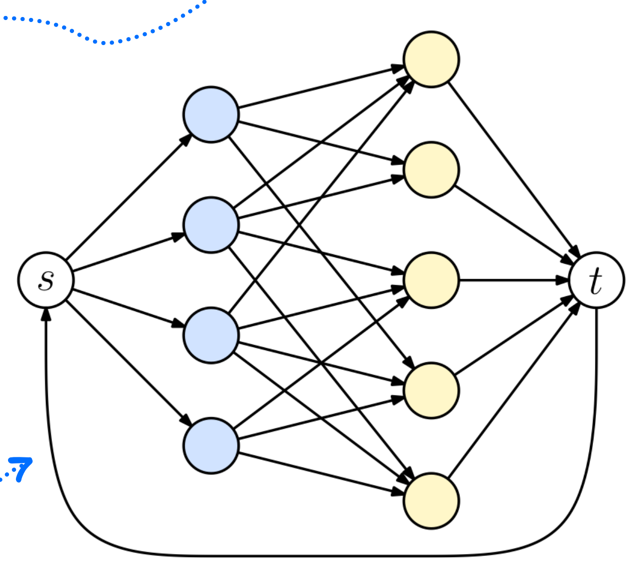
+ purchase information

does there exist a survey satisfying all rules?

Purchase information – Modeled as bipartite graph




customer 3 purchased product 5




- Direct edges from cust. to prod.
- Add vertex s + edges to customers
- Add vertex t + edges from products
- Add edge (t, s)

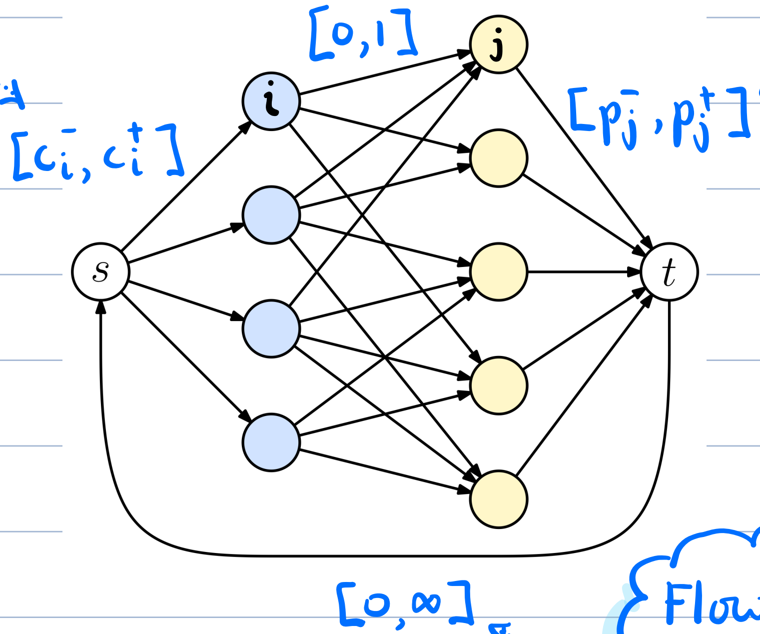
All demands = 0
(flow conservation)

Capacities:

- Customer edges $(s, i)$: $[c_i^-, c_j^+]$
- Product edges $(j, t)$: $[p_j^-, p_j^+]$
- Customer/product edges: $[0, 1]$
- $(t, s)$: $[0, \infty]$

## Explanation:

Customer $i$ surveyed on product $j$




Num. surveys to cust. $i$ in $[c_i^-, c_i^+]$

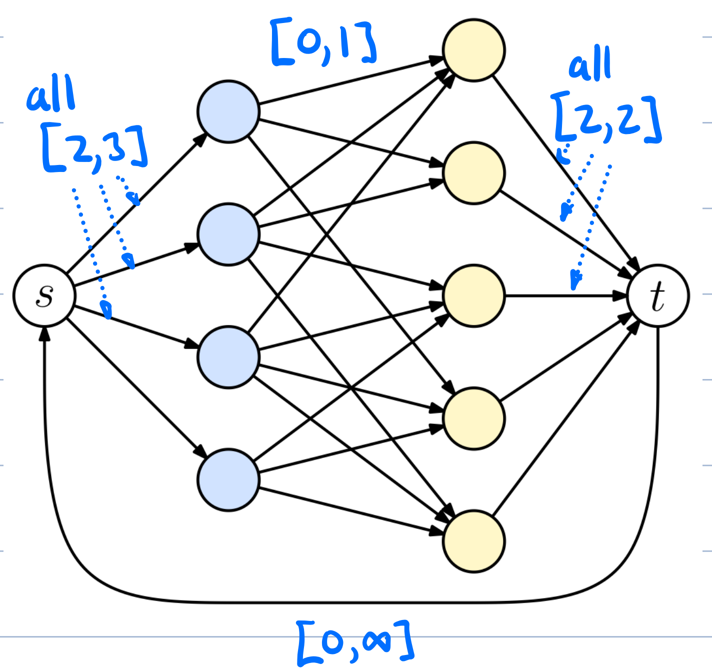
Num. of surveys on prod. $j$ in $[p_j^-, p_j^+]$

Flow on this edge = total num. surveys

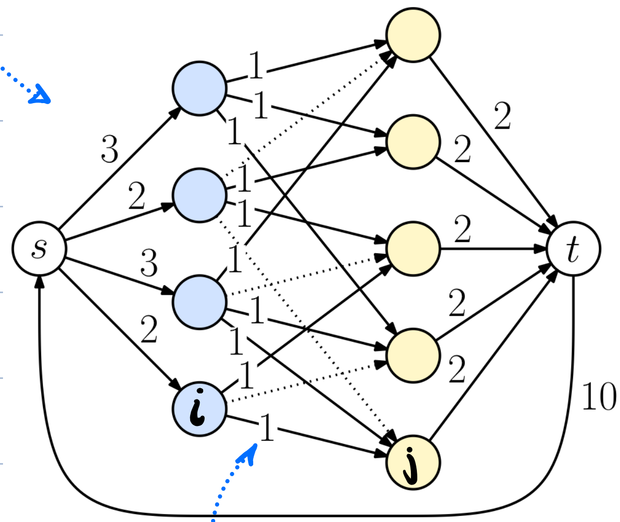
## Example:

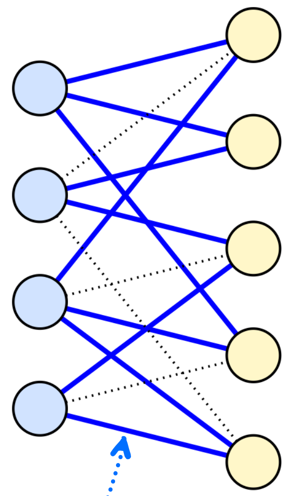
- For simplicity, set $[c_i^-, c_i^+] = [2,3]$, for all $i$

  $[p_j^-, p_j^+] = [2,2]$, for all $j$



Compute Circulation - $f$ (integer valued)



if $f(i,j)=1$ survey customer $i$ about product $j$

Is this a faithful implementation of survey rules?

**Claim:** There exists a valid circulation in G iff there is a valid survey design

**Proof:**

($\Rightarrow$) Suppose G has a valid circulation $f$.

- Survey: Ask cust. $i$ about prod $j$ if $f(i,j)=1$
- We'll show this survey is valid:

① Only survey customers about purchases

$\Leftarrow$ Create edge $(i,j)$ if $i$ purchases $j$

② Num. of surveys for cust $i$ in $[c_i^-, c_i^+]$

$\Leftarrow$ Since $f(s,i) \in [c_i^-, c_i^+]$

+ flow conservation $(d(i) = 0)$

③ Num. of surveys for prod. $j$ in $[p_j^-, p_j^-]$

$\Leftarrow$ Since $f(j,t) \in [p_j^-, p_j^+]$

+ flow conservation $(d(j) = 0)$

$\Rightarrow$ This is a valid survey ✓

($\Leftarrow$) Suppose there is a valid survey.

Construct flow in $G$:

- $f(i,j) = 1$ if cust. $i$ surveyed on prod $j$
- $f(s,i) =$ total surveys for cust $i$
- $f(j,t) =$ total surveys for prod $j$
- $f(t,s) =$ total num. of surveys

We'll show this is a valid circulation:

① Cust. $i$ surveyed about prod. $j$ only if purchased $\Rightarrow$ edge $(i,j) \in G$

② Cust. $i$ surveyed about $[c_i^-, c_i^+]$ products $\Rightarrow f(s,i)$ satisfies capacities

③ Prod. $j$ involved in $[p_j^-, p_j^+]$ surveys $\Rightarrow f(j,t)$ satisfies capacities

Finally flows-in = flows-out so all demands (zero) are met $\Rightarrow$ valid circulation ✓

Ugh! 

- These proofs are lengthy + rather tedious.
- Important to understand structure
  - Similer to NP-complete reductions

Summary:

- Circulations – vertex demand/supply
  - Reduction to max flow
- Circulations with lower + upper edge capacities
  - Reduction standard circulations
- Application - Survey Design
  - Reduction to circulations w. lower/upper capacities