CMSC 451 - Algorithm Design
Lecture 17 - NP-Completeness: 3SAT + Independent Set

Recap:

P: Set of languages (decision probs.) solvable in poly-time

NP: Set of languages verifiable in poly-time

$\leq_p$: $L_1 \leq_p L_2$ means there is a poly-time function f s.t.

$\forall x \quad x \in L_1$ iff $f(x) \in L_2$

NP-hard: L is NP-hard if $\forall L' \in NP, \; L' \leq_p L$

NP-complete: L is NP-complete if

(1) $L \in NP$

(2) L is NP-hard

If any NP-complete problem is solvable in poly-time, they all are ($P = NP$)

If any problem in NP is not solvable in poly-time, then no NP-complete problem is solvable in poly-time ($P \neq NP$)

Boolean Satisfiability + Cook's Theorem:

- Prove problems NP-complete through reductions
- Need one initial NP-complete problem to start

Boolean Satisfiability (SAT)

Boolean formula:

variables: x, y, z or $x_1, x_2, \ldots$

→ Each can be assigned true or false

operations:

- $\wedge$ – "and" $x \wedge y$ - true if both true
- $\vee$ – "or" $x \vee y$ - true if either/both true
- $\neg$ – "not" $\neg x$ - reverses true ↔ false

often written: $\bar{x} \equiv \neg x$

Examples:

$$F_1(x,y,z) = (x \wedge (y \vee \bar{z})) \wedge ((\bar{y} \wedge \bar{z}) \vee \bar{x})$$

$x = T$
$y = z = F$

$$F_2(x,y) = (\bar{z} \vee x) \wedge (z \vee y) \wedge (\bar{x} \wedge \bar{y})$$

Def: A formula is satisfiable if it is possible to assign values to variables so it evaluates to true

Not satisfiable

Cook's Theorem: SAT is NP-Complete

The proof is long + technical, but here is a high-level intuitive overview

SAT ∈ NP: Given a formula $F(x_1, \ldots, x_n)$

certificate is assignment to variables

$x_1 \leftarrow T$, $x_2 \leftarrow F$, $x_3 \leftarrow T$, ...

To verify, plug these into formula

evaluate + check that result = true

SAT is NP-hard:

- Given any language $L \in NP$,
  want to show $L \leq_p SAT$ ($x \in L$ iff $f(x) \in SAT$)
- For any instance x for L we know
  - Can verify that $x \in L$
  - Involves a certificate + verification program
  - Encode certificate as a bit string

    e.g. $C = 0101101\ldots$
  - Express bits as variables

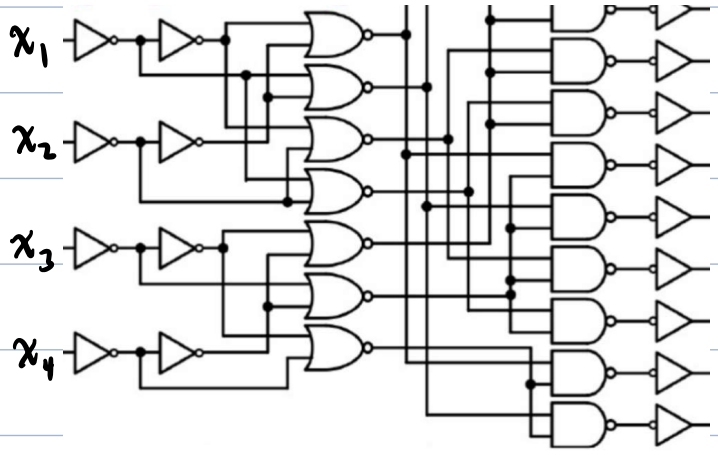
    x_1 = 1st bit of C (0 = F, 1 = T)

    x_2 = 2nd bit of C

    ⋮

- Verification is a poly-time program.

- Program can be expressed as a circuit of polynomial size



- Circuit can be expressed as boolean formula $f(x)$ of polynomial size.

- This formula faithfully simulates the verification program:

$$x \in L \text{ iff } f(x) \in SAT$$

$\therefore \forall L \in NP \quad L \leq_p SAT \Rightarrow$ SAT is NP-hard!

□

Cook proved a stronger result -

Satisfiability is NP-complete, even if we restrict the format of the formulas to 3CNF.

Notation:

literal: A variable or its complement, x_i or $\bar{x}_i$

3-conjunctive normal form (3CNF):

A formula that is the conjunction ("$\wedge$") of clauses, each of which is the disjunction ("$\vee$") of 3 literals.

Example:

$$(x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_4) \wedge \dots$$

joined by "ands"

clause = "or" of 3 literals

Def: 3SAT - Satisfiability of formulas expressed in 3CNF form.

Theorem (Cook). 3SAT is NP-complete

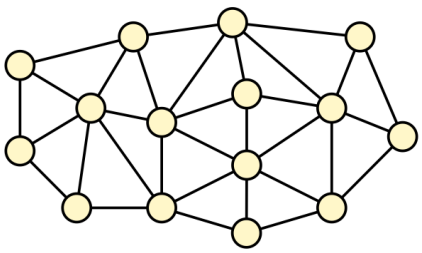
Reductions are simpler

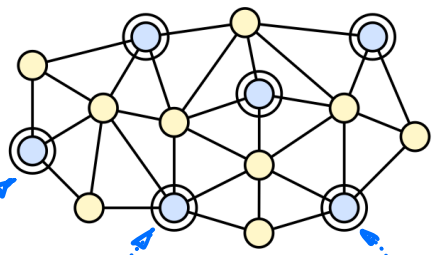
Given that 3SAT is NP-Complete, let's try another.

Independent Set (IS): Given a graph $G=(V,E)$ + integer k, does G contain a subset of vertices $V' \subseteq V$ of size k so that no two are adjacent?

Example:

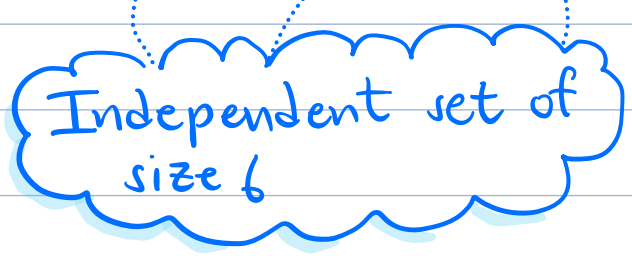
G:





$(G,6) \in IS$

$(G,7) \notin IS$



Claim: IS is NP-Complete

Need to show: (1) IS $\in$ NP (verifiable)

(2) IS is NP-hard

(1) IS $\in$ NP:

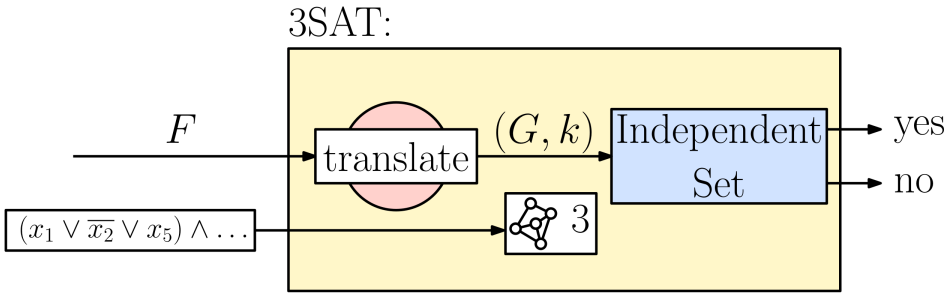
Given instance (G,k), the certificate consists of a set of k vertices of G. We consider each pair u,v from this set & check that they are not adjacent. If so, we accept the certificate & otherwise we reject it.

(2) IS is NP-hard:

We'll show that: $3SAT \leq_p IS$

There exists poly-time function f that maps a 3CNF formula F to a pair (G,k) s.t.

$F \in 3SAT$ iff $(G,k) \in IS$



How? These problems seem unrelated!

Let's see what they share in common:

Selection:

3SAT: Which variables are true
≡ which literals are true

IS: Which vertices are in indep. set

literals map to vertices?

Requirements:

3SAT: Logical "and" of clauses, each is "or" of 3 literals
⇒ at least one true literal per clause

IS: V' must contain k vertices

set k = number of clauses

Restrictions:

3SAT: x_i + $\bar{x}_i$ cannot both be true

IS: If $(u,v) \in E$, u + v cannot both be in indep. set

Create edge between x_i — $\bar{x}_i$

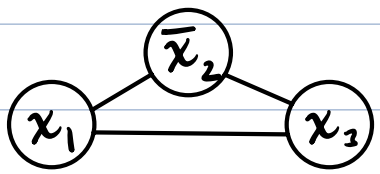
Plan:

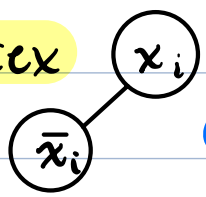
- For each clause, create a vertex for each literal. (Clusters of 3 vertices)

$(x_1 \vee \bar{x}_2 \vee x_3) \longrightarrow$

- Connect vertices with each cluster (Forces us to select only one as the "satisfying" literal)

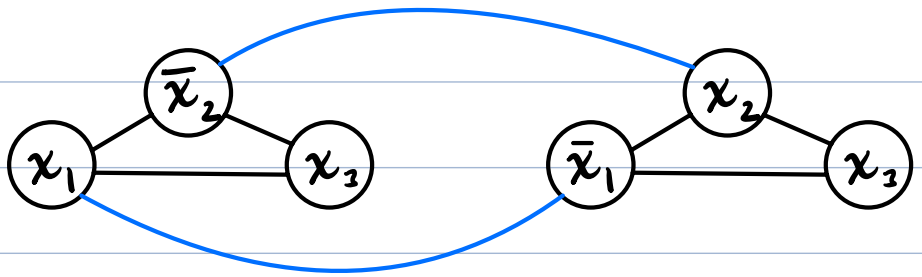
$(x_1 \vee \bar{x}_2 \vee x_3) \longrightarrow$



- Connect each vertex 

with its complements (conflict links)
(Forces us to choose one or other to be true)

$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3)$



Reduction: Given formula F in 3CNF:

$k \leftarrow$ number of clauses

for each clause $(x_a \vee x_b \vee x_c)$:

- create vertices x_a x_b x_c
- create edges $(x_a, x_b), (x_b, x_c), (x_c, x_a)$

for each variable x_i:

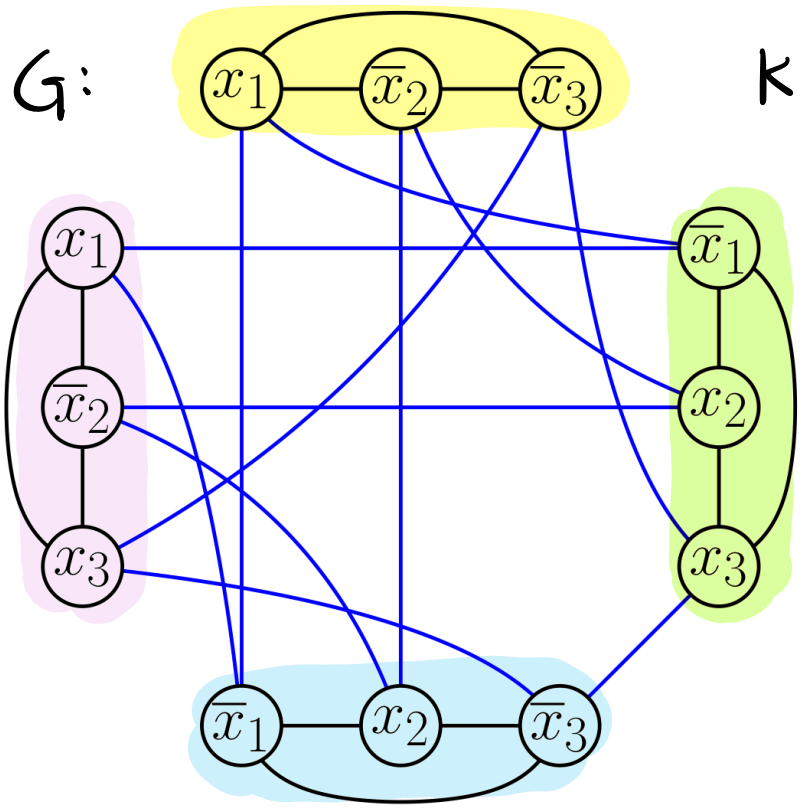
- create edges between all copies of x_i and $\bar{x}_i$

return (G, k)

Example: $F =$

$$(x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3)$$

G:



$k = 4$

This can computed in poly-time in size of formula.

(Exercise)

Why does it work? Some intuition...

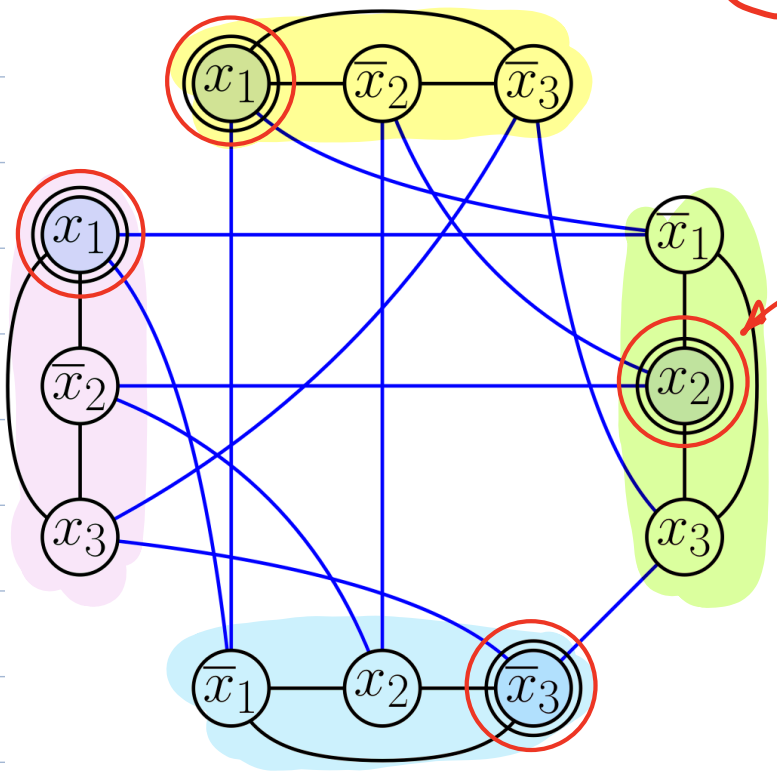
$\Rightarrow$ If F is satisfiable, there must be at least one true literal per clause $\longrightarrow$ add it to indep. set

$\Leftarrow$ If G has indep. set of size k, must be one vertex from each cluster. Set this literal true. This assignment satisfies F.

$$(x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3)$$

T T T T

$x_1 = T$
$x_2 = T$
$x_3 = F$



There may be choices on which vertex to use

Circled vertices form an indep. set size $k = 4$

Formal proof:

Claim: F is satisfiable iff G has ind. set size k

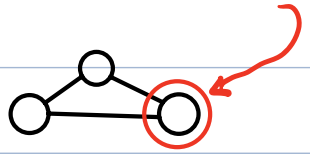
($\Rightarrow$) If F is satisfiable then (because F in CNF form) each clause of F must have at least one true literal.

- Select one true literal from each clause and add to V'
- There are k clauses $\Rightarrow |V'| = k$
- Both x_i and $\bar{x}_i$ cannot be true, so no edge between vertices of V'

$\Rightarrow$ V' is ind. set of size k ✓

($\Leftarrow$) If G has ind. set V' of size k

- Because clusters are connected, can take at most one from each cluster.



- Because k = no. of clauses = no. of clusters must take exactly one from each cluster.

- Because x_i — $\bar{x}_i$ are adjacent, both cannot be in V'.
  - if $x_i \in V' \rightarrow$ set $x_i \leftarrow T$
  - $\bar{x}_i \in V' \rightarrow$ set $x_i \leftarrow F$
  - if neither $\rightarrow$ set $x_i \leftarrow$ either T or F
- This assignment satisfies every clause of F

$\Rightarrow$ F has a satifying assignment ✓

□

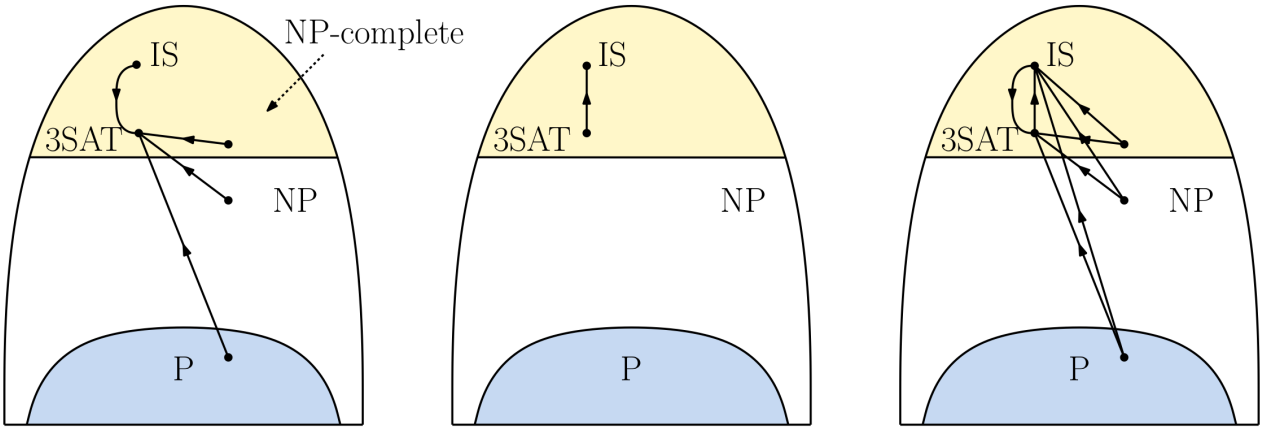
Done! Showed IS is NP-complete.

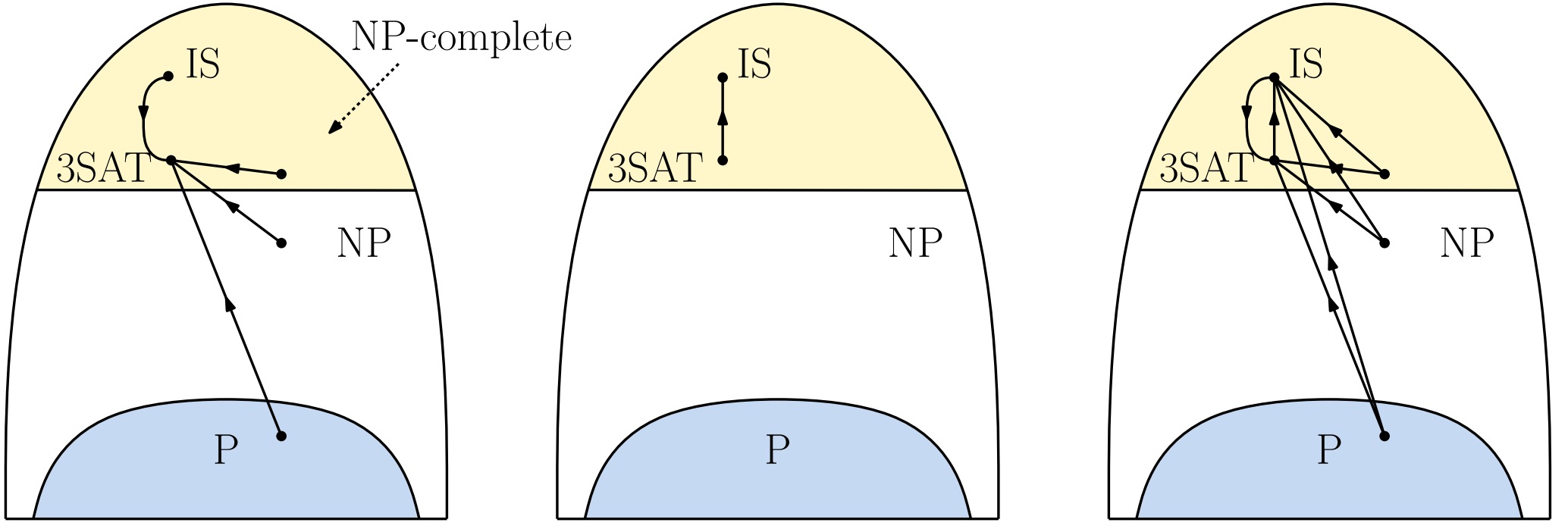
Notes:

- Hardest part is the reduction
- Reduce known NP-hard to our problem
- Reduction merely translates common elements.
- It is agnostic (doesn't care) which variables are true, which vertices are in ind. set or even if an ind. set exists!

Summary:

- 3SAT is NP-complete
- IS is NP-complete ($3SAT \leq_p IS$)

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