CMSC 451 - Algorithm Design
Lecture 18 - NP-Completeness: Clique, Vert. Cov. + Dom. Set

This lecture - 3 easy reductions

Recap: To show L is NP-complete:

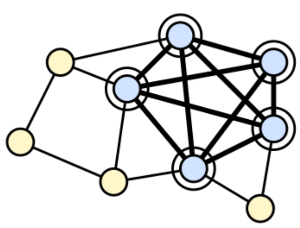
(i) $L \in NP$ - it is verifiable

(ii) L is NP-hard - suffices to show $L' \leq_p L$ for some known NP-hard problem L'

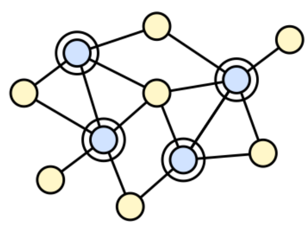
Three problems:

Clique (CLIQUE): Given a graph $G=(V,E)$ + integer k, is there a subset $V' \subseteq V$ of size k s.t. all pairs of vertices in V' are adjacent.

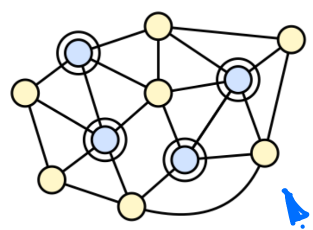
Clique of size 5



Vertex cover size 4



Dominating set size 4



Vertex Cover (VC): Given graph $G=(V,E)$ + integer k, is there a subset $V' \subseteq V$ of size k s.t. every edge of G is incident to some vert. of V'

Dominating Set (DS): Given graph $G=(V,E)$ + integer k, is there a subset $V' \subseteq V$ of size k s.t. every vertex in G is in V' or adjacent to a vertex in V'

Applications:

Clique - Analysis of tightly coupled groups in social networks

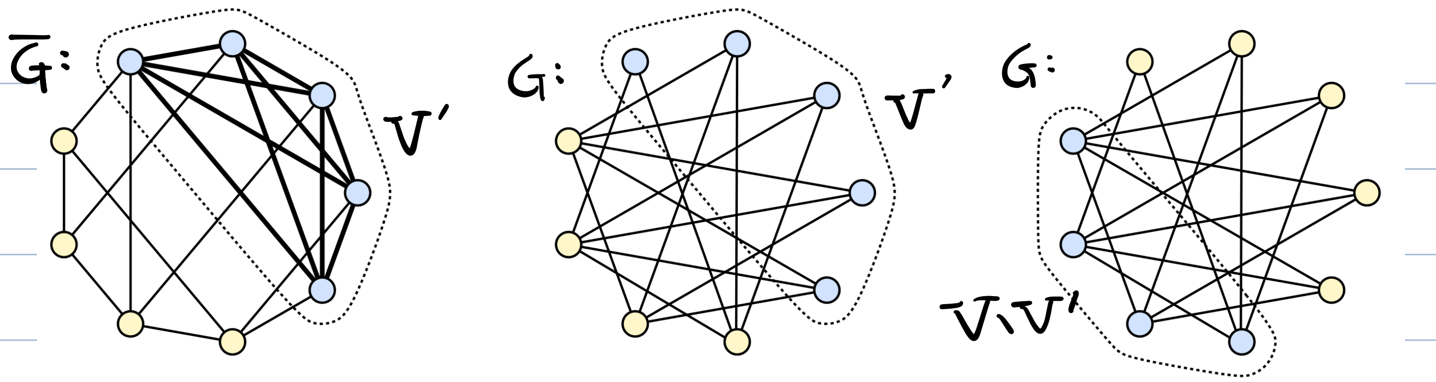
Vertex Cover + **Dominating Set** - Used in applications in facility location (placing service facilities close to clients)

Clique / IS / VC: Closely related.

Recall: $\bar{G}$ = complement graph $(u,v) \in \bar{G}$ iff $(u,v) \notin G$

> **Lemma:** Given a graph $G=(V,E)$ with $n = |V|$. Let $V' \subseteq V$ + $k = |V'|$. Then the following are equivalent:
> (i) V' is a clique of size k in $\bar{G}$
> (ii) V' is an indep. set of size k in G
> (iii) $V \setminus V'$ is a vertex cover of size $n-k$ in G

V' is clique in $\bar{G}$ V' is indep set in G $V \setminus V'$ is VC in G



Proof: (sketch)

(i) $\Leftrightarrow$ (ii): All adjacent in $\bar{G}$

$\Leftrightarrow$ none adjacent in G

(ii) $\Leftrightarrow$ (iii): No edges among V'

$\Leftrightarrow$ all edges touch $V \setminus V'$

Theorem: CLIQUE is NP-complete

Proof: (i) CLIQUE $\in$ NP

Given input $(G=(V,E),k)$

certificate = list of k vertices $\rightarrow V'$

verification:

for each $u,v \in V'$

check that $(u,v) \in E$

if yes for all – accept

else – reject

(ii) IS $\le_p$ CLIQUE

- Given instance (G,k) for IS our translation function outputs $(\bar{G},k)$ for CLIQUE.
- Can complement graph in $O(n^2)$ time
- Correctness:

  G has indep. set of size k

  iff

  $\bar{G}$ has CLIQUE of size k

  (by prior lemma (i) $\Leftrightarrow$ (ii)) $\square$

Theorem: VC is NP-complete

Proof: (i) VC $\in$ NP

Given input $(G=(V,E),k)$

certificate = list of k vertices $\rightarrow V'$

verification:

for each edge $(u,v) \in E$
  check that $u \in V'$ or $v \in V'$
if yes for all – accept
  else – reject

(ii) IS $\leq_P$ VC

- Given instance (G,k) for IS
  our translation function outputs
  $(G, n-k)$ for VC $(n = |V|)$
- Can compute in $O(n)$ time
- Correctness:
  G has indep. set of size k
  iff
  G has vertex cover of size $n-k$
  (by prior lemma (ii) $\Leftrightarrow$ (iii)) $\square$

Theorem: DS is NP-complete

- This is trickier!
  - Looks similar to VC (if V' is a vertex cover then it's a dom. set, right?)
  - But, translation must be faithful (if V' is a dom. set it may not be VC)

- Goal: Poly-time function $f: (G,k) \longrightarrow (G',k')$ s.t.

G has VC of size k

iff

G' has DS of size k'

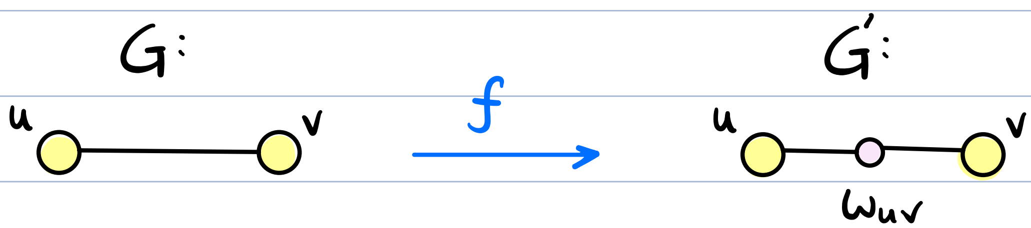
- What are corresponding elements?

VC: Every edge is incident to V'

DS: Every vertex is adjacent (or in) V'

$\Rightarrow$ Need to make edges behave like vertices

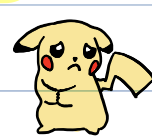
Idea 1: Add a vertex within each edge:



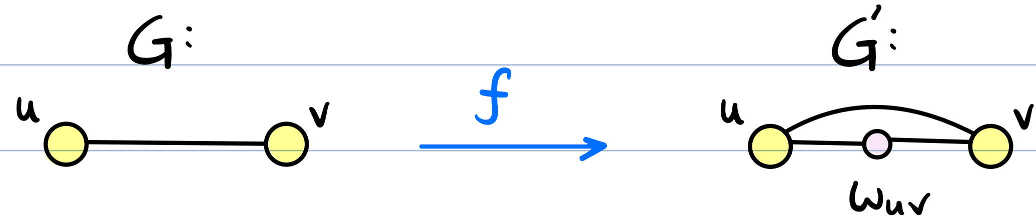
Putting u or v in VC covers (u,v)

Putting u or v in DS dominates w_{uv}

$\rightarrow$ But if u in DS, who dominates v?



Idea 2: Add a vertex within each edge + keep the old edge.



Putting u or v in VC covers (u,v)

Putting u/v in DS dominates w_{uv} + v/u

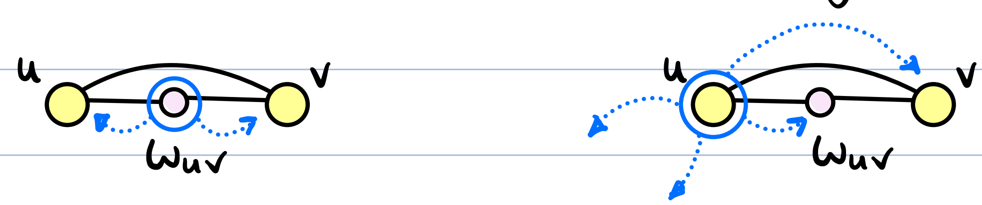
This is almost perfect. Two issues:

① Don't want DS to use the mid-edge vertices. How to enforce this?

② Who dominates isolated vertices? (no incident edges)

Fixes:

① Not a problem. No advantage in using w_{uv}, since u or v dominates at least as many vertices.



② Isolated vertices are trivial to handle:

VC – They never help

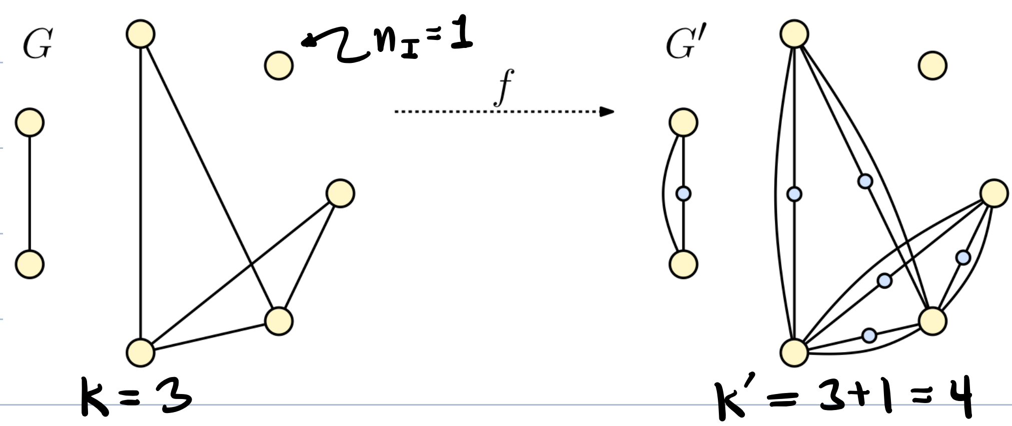
DS – They must be included

Let n_I denote no. of isolated vertices in G. Set $k' \leftarrow k + n_I$

Final reduction function f:

- Given (G,k) for VC, $G = (V, E)$
- for each edge $(u,v) \in E$
  - create vertex w_{uv}
  - add new edges (u, w_{uv}) + (v, w_{uv})

- Call resulting graph G'
- let n_I = no. isolated vertices in G
- set $k' \leftarrow k + n_I$
- return (G', k')



Lemma: G has a vertex cover of size k iff G' has a dom. set of size k'

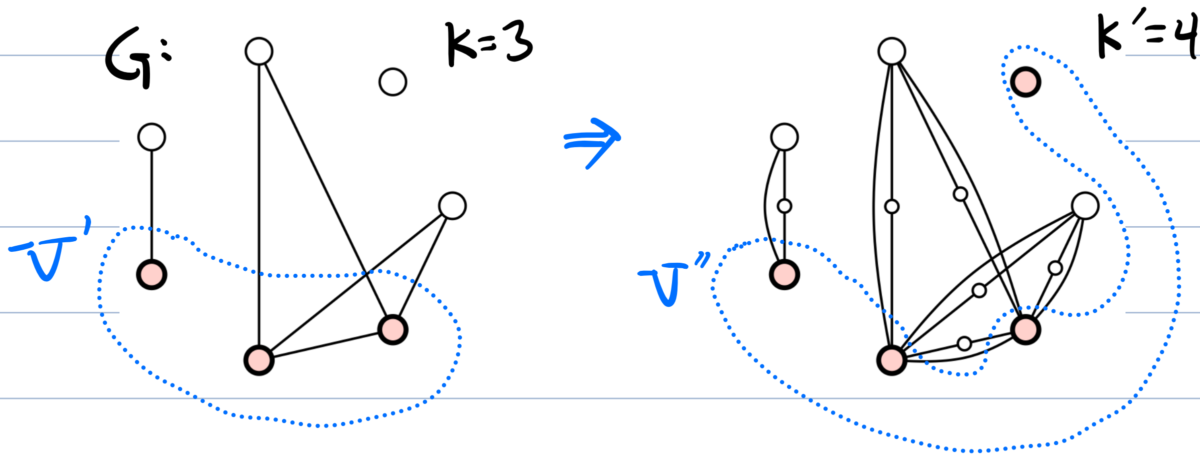
Proof:

($\Rightarrow$) Let V' be a VC in G of size k

- Let:

$$V'' = V' \cup \{\text{all isolated vertices}\}$$

- Clearly, $|V''| = k + n_I = k'$



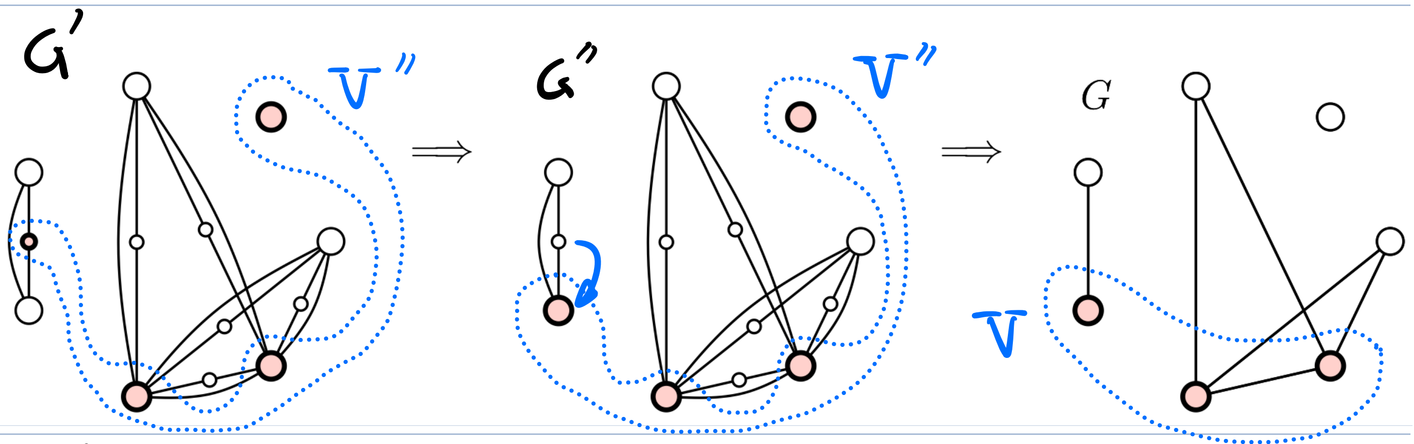
- Since V' is VC, $\forall (u,v) \in E$, either $u \in V'$ or $v \in V'$

- In either case we dominate u, v, w_{uv}

$\Rightarrow$ V'' is dom. set of size k' in G' ✓

(⇐) Let V'' be a D.S. in G' of size k'

- If V'' contains any mid-edge vertices w_{uv}, replace with either u or v.
- Still a dom. set (as observed earlier)



$k' = 4$

- Let $V \leftarrow V''$ with isolated vertices removed
  (Isolated vertices must be in dom. set)

- $|V| = k' - n_I = k$

- Since V'' dominated all vertices
  $\Rightarrow V''$ dominates all mid-edge vertices
  $\Rightarrow V$ covers all edges

$\Rightarrow V$ is a vertex cover in G of size k ✓

□

Theorem: DS is NP-complete

Proof: (i) DS $\in$ NP

Given input $(G=(V,E),k)$

certificate = list of k vertices $\rightarrow V'$

verification:

for each $v \in V$

check that v or some neigh. in V'

if yes for all - accept

else - reject

(ii) $VC \leq_p DS$

Follows from previous lemma. $\square$

Summary: Three new NP-Complete problems

- CLIQUE
- Vertex cover (VC)
- Dominating Set (DS)