# Homework 4: NP-Completeness

Due: May 3, 11:59pm

**Problem 1 (20 points)**

1. (5 points) State the definition of an NP-complete language.
2. (5 points) Explain why proving $A \leq_m^p B$ is useful when we already know that $A$ is NP-complete.
3. (10 points) Suppose you want to prove that CLIQUE is NP-complete. Which of the following would be the correct direction of reduction to prove NP-hardness? Circle one and give a one- or two-sentence justification.

$$\text{CLIQUE} \leq_m^p \text{3-SAT} \qquad \text{or} \qquad \text{3-SAT} \leq_m^p \text{CLIQUE}.$$

**Problem 2 (20 points)** Define

$$\text{DOUBLE-SAT} = \{\phi \mid \phi \text{ has at least two distinct satisfying assignments}\}.$$

Show that DOUBLE-SAT is NP-hard.

**Problem 3 (30 points)** Define

$$\text{SET-COVER} = \{(U, \mathcal{S}, k) \mid \text{there exist } S_1, \ldots, S_k \in \mathcal{S} \text{ whose union is } U\}.$$

Your task is to prove SET-COVER is NP-complete.

1. (10 points) First, show that SET-COVER $\in$ NP.
2. (10 points) To prove NP-hardness, we reduce from vertex cover. Let $(G, k)$ be an instance of vertex cover, where $G = (V, E)$. Your task is to perform the reduction and construct a corresponding instance $(U, \mathcal{S}, k)$ of SET-COVER. Explicitly describe:
   - the universe $U$, and
   - the collection of sets $\mathcal{S}$.
3. (5 points) Prove that if $G$ has a vertex cover of size at most $k$, then the SET-COVER instance you constructed has a set cover of size at most $k$.
4. (5 points) Prove the converse: if the SET-COVER instance you constructed has a set cover of size at most $k$, then $G$ has a vertex cover of size at most $k$.

**Problem 4 (20 points)** For the fixed constant 100, define

$$\text{CLIQUE}_{100} = \{G \mid G \text{ has a clique of size at least } 100\}.$$

Show that $\text{CLIQUE}_{100} \in \text{P}$.

**Problem 5 (10 points)** Draw a diagram of all NP-complete problems covered in class, indicating the direction of the reductions.