NP-Completeness of Graph Problems

Lecture 18

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Outline

- We start to work on graph problems.
- We study three problems: Clique, Vertex Cover, and Independent Set.
- We prove all of them are NP-complete.

Clique

A **clique** in a graph $G = (V, E)$ is a set of vertices all of whose pairs are connected by edges. Decision problem:

$$\mathrm{CLIQUE} = \{\langle G, k\rangle : G \text{ has a clique of size at least } k\}.$$

Clique is NP-complete

Theorem. The Clique problem is NP-complete.

We shall prove:

1. the Clique problem is in NP,
2. 3-SAT reduces to the Clique problem.

$\mathrm{CLIQUE} \in \mathrm{NP}$

Certificate: a list of k vertices. Verifier:

1. check that the list contains k distinct vertices,
2. check that every pair among them is connected by an edge.

This takes polynomial time, so $\mathrm{CLIQUE} \in \mathrm{NP}$.

Reduction Idea: 3-$\mathrm{SAT} \leq_m^p$ CLIQUE

Let

$$\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

be a 3-SAT formula. Construct a graph with:

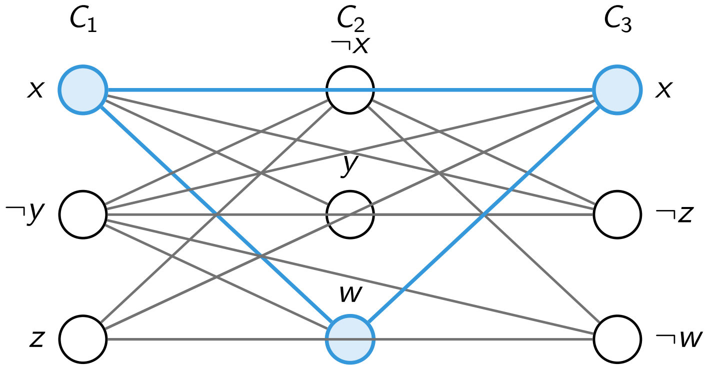
1. one vertex for each literal occurrence in each clause,
2. edges only between vertices from different clauses,
3. no edge between contradictory literals such as x and $\neg x$.

Then ask whether the graph has a clique of size m.

Example Gadget for 3-SAT $\leq_m^p$ CLIQUE

Let

$$\varphi = (x \vee \neg y \vee z) \wedge (\neg x \vee y \vee w) \wedge (x \vee \neg z \vee \neg w).$$



blue edges show a clique of size 3

Why a Satisfying Assignment Gives a Clique

Suppose φ is satisfiable. Choose one true literal from each clause.

Because all chosen literals are simultaneously true:

1. no two chosen literals contradict each other,
2. they come from different clauses,

Therefore every chosen pair is connected by an edge and the chosen vertices form a clique of size m.

Why a Clique Gives a Satisfying Assignment

Now suppose the graph has a clique of size m.

Edges only go between different clauses $\Rightarrow$ the clique must pick exactly one vertex from each clause.

All chosen vertices are pairwise adjacent $\Rightarrow$ no two chosen literals are contradictory.

Therefore we can assign truth values that make all chosen literals true and therefore satisfy the assignment

Independent Set

An **independent set** is a set of vertices with no edges between them.
Decision problem:

$$\mathrm{INDSET} = \{\langle G, k \rangle : G \text{ has an independent set of size at least } k\}.$$

Independent Set is NP-complete

Theorem. The INDSET problem is NP-complete.

A proposed independent set can be checked in polynomial time, so the problem is in NP.

From Clique to Independent Set

Let $\overline{G}$ be the complement graph of G. Key fact:

$$G \text{ has a clique of size } k \iff \overline{G} \text{ has an independent set of size } k.$$

A set of vertices is pairwise adjacent in G exactly when it is pairwise non-adjacent in $\overline{G}$.

Vertex Cover

A **vertex cover** of a graph is a set $S \subseteq V$ such that every edge has at least one endpoint in S. Decision problem:

$$\mathrm{VC} = \{\langle G, k\rangle : G \text{ has a vertex cover of size at most } k\}.$$

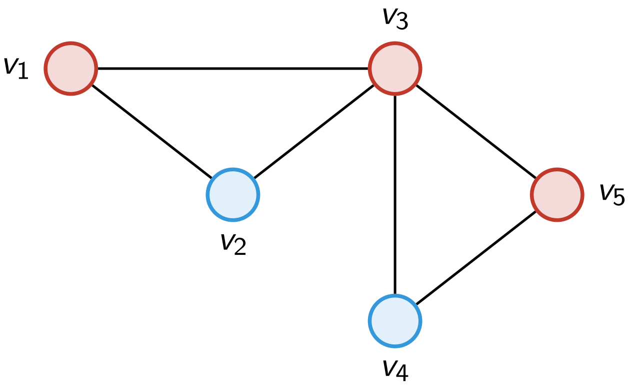
Vertex Cover is NP-complete

Theorem. The VC problem is NP-complete.

Again the problem is in NP because a proposed cover can be checked quickly.

From Independent Set to Vertex Cover

Red vertices form a vertex cover. Their complement, shown in blue, is an independent set.



S is an independent set $\iff$ $V - S$ is a vertex cover.