Pumping Lemma (Continued from last lecture)

Lecture 7

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What kind of language is NOT regular?

Two Methods of Proof

- **Method 1:** Run the DFA on many small words. By the **Pigeonhole Principle (PHP)**, two of the words must finish in the same state. Then use this to derive a contradiction.

- **Method 2 — Pumping Lemma:** Run the DFA on one long word. By the **PHP**, the word must visit the same state twice (creating a loop). Then use this to derive a contradiction.

The Pumping Lemma (Formal)

Pumping Lemma: If L is regular, then there exists a constant n (the pumping length) such that for all $w \in L$ with $|w| \geq n$, there exist x, y, z such that:

1. $w = xyz$ and $y \neq e$.
2. $|xy| \leq n$.
3. For all $i \geq 0$, $xy^i z \in L$.

Proof of pumping Lemma on whiteboard

How We Use the Pumping Lemma

To prove L is not regular:

1. Assume L is regular.
2. Let n be the pumping length.
3. Choose a specific string $w \in L$ with $|w| \geq n$.
4. Consider all possible ways to split $w = xyz$ with $|xy| \leq n$ and $y \neq e$.
5. Derive contradiction and show L is not regular.

Four examples we talked in the last lecture

- $L_1 = \{a^n b^n : n \geq 0\}$ revisited
- $L_3 = \{w : \#_a(w) \neq \#_b(w)\}$
- $L_4 = \{a^{n^2} : n \in \mathbb{N}\}$

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Since $p > 1$ and $1 + k > 1$, the result is composite. **Contradiction.**

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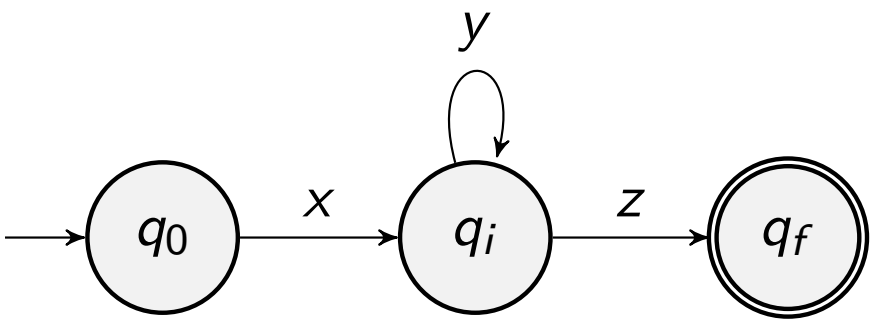
The $i = 0$ case: Pumping Down



xy^0z means we bypass the loop entirely and go straight from x to z.

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Lower Bounds: Looking Ahead

1. DFAs are simple enough that we can prove certain languages are not regular.
2. Context-Free Grammars are also simple enough to prove some languages are not context-free.
3. Proving P $\neq$ NP is hard because Turing Machines are very complex.
4. Proving undecidability is often easier because it depends on the fundamental logic of computation rather than model details.