CMSC 754 Quiz 3

This quiz is closed-book and closed-notes, but you may use one sheet of notes, front and back. You may use any algorithms or results given in class. The total point value is 50 points. Good luck!

Problem 1: (15 points) Let us consider how to construct the Delaunay triangulation of a set of sites $P = \langle p_1, p_2, \ldots, p_n \rangle$ that are in convex position (given, say, in counterclockwise order). We start with the triangle defined by three random sites from P. The remaining sites are inserted one by one in random order. Each new site p_i is connected to the convex hull by adding its two tangent edges, thus creating a new triangle (see Fig. 1(b)). We then perform edge-flips until all the incircle tests succeed (see Fig. 1(c)).

(a) (8 points) Show that the expected number of edge-flips performed with each insertion is at most 2. (We will give partial credit if you show that the expected number is $O(1)$. **Hint:** You may use the fact that a planar triangulation with v vertices and h hull edges has $e = 3v - h - 3$ edges.)

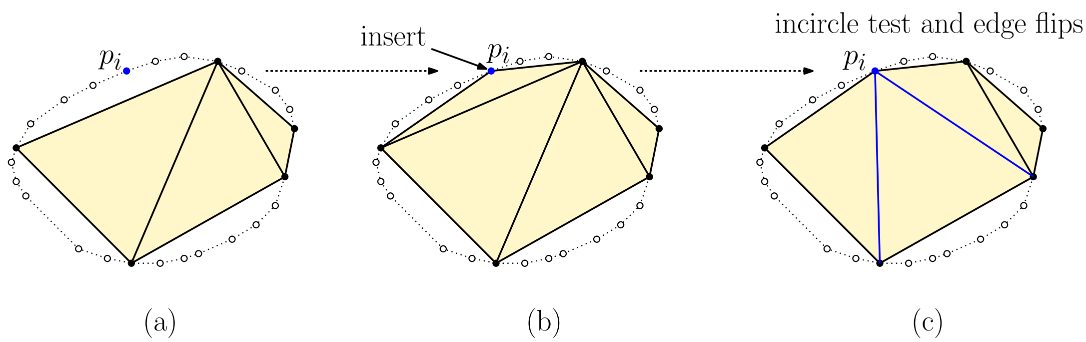

Figure 1: Delaunay triangulation of a convex polygon.

(b) (7 points) Whenever a new site is added, we need to determine where along the current convex hull it is to be added. Explain how to do this through bucketing. Prove that the probability that a fixed site is rebucketed during the ith insertion is at most $2/i$, and from this show that the overall rebucketing work is $O(n \log n)$. (We will give partial credit if you show that the expected probability is $O(1/i)$.)

Problem 2. (10 points) You are given a set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathbb{R}^2$. Define the *critical radius* to be the *smallest* radius r^* such that, if we place a Euclidean ball of radius r^* centered at each point of P, the entire set of balls forms a connected set. Present an efficient algorithm which, given P, computes r^*. Justify your algorithm's correctness and derive its running time. (**Hint:** Aim for a running time of $O(n \log n)$.)

Problem 3. (10 points) Let P be a set of n point sites in the plane in general position. In class, we showed that P's Euclidean minimum spanning tree is a subgraph of its Delaunay graph, that is $\text{EMST}(P) \subseteq \text{DT}(P)$. In this problem, we will consider another such graph.

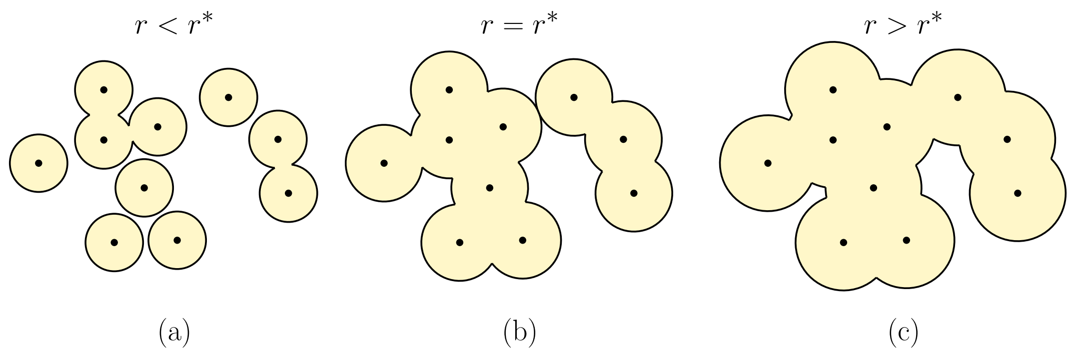

Figure 2: Critical radius.

Define the *relative neighborhood graph* of P, denoted $\mathrm{RNG}(P)$ as follows: Its vertex set is P and two sites $p, q \in P$, are adjacent in $\mathrm{RNG}(P)$ if and only if there is no other site of P that is simultaneously closer to p and q than they are to each other. That is, there exists no $r \in P$ such that $\max(\|r - p\|, \|r - q\|) < \|p - q\|$. (Recall that $\|x - y\|$ denotes the Euclidean distance between x and y.)

(a) (5 points) Given sites $p, q \in P$, define $\mathrm{lune}(p, q)$ to be the intersection of two Euclidean disks centered at p and q whose radii are equal to $\|p - q\|$ (see Fig. 3). Prove that (p, q) is an edge of $\mathrm{RNG}(P)$ if and only if the interior of $\mathrm{lune}(p, q)$ contains no other site of P.

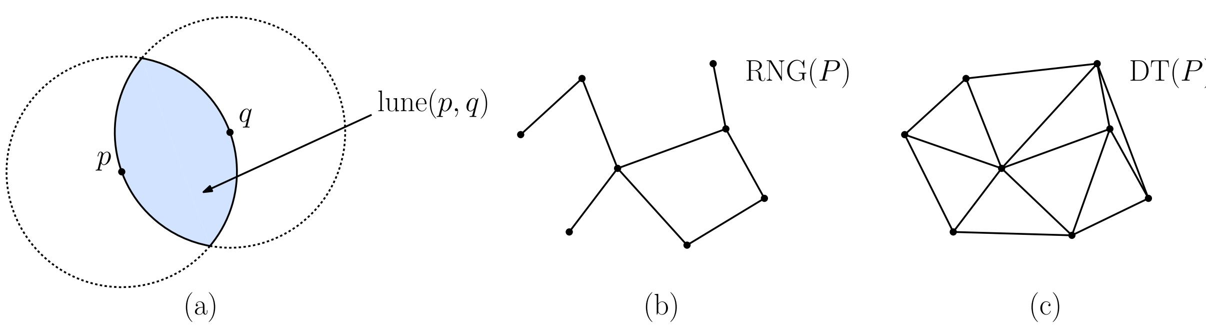

Figure 3: Lunes and the relative neighborhood graph.

(b) (5 points) Prove that $\mathrm{RNG}(P) \subseteq \mathrm{DT}(P)$. (**Hint:** Prove that if (p, q) is an edge of $\mathrm{RNG}(P)$ then there exists a circle passing through p and q that lies entirely within $\mathrm{lune}(p, q)$.)

Problem 4. (15 points) You are given two point sets R and B in $\mathbb{R}^2$ in general position, where $|R| = m$ and $|B| = n$, where both m and n are odd (see Fig. 4(a)). The *ham-sandwich cut problem* (HSC) computes a line ℓ that simultaneously bisects both sets (see Fig. 4(b)). That is, ℓ passes through a point of R and a point of B, and the numbers of points of R (resp., B) above and below ℓ are equal to $(m-1)/2$ (resp., $(n-1)/2$).

(a) (5 points) Formulate HSC as a computational problem in the dual setting. Briefly explain.

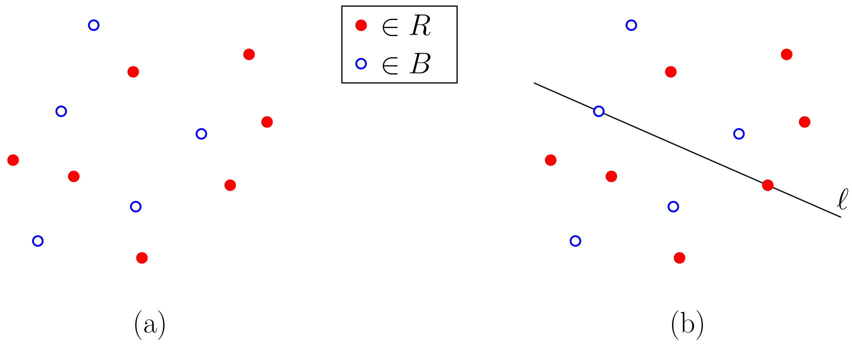

Figure 4: Ham-sandwich cut problem with $m = 7$ and $n = 5$.

(b) (10 points) Explain how to solve HSC by performing a plane-sweep through a line arrangement in the dual setting. Explain the contents of the sweep-line status, what the events are, and how the events are processed. Ignoring the time for scheduling events, your algorithm should run in $O((m+n)^2)$ time and $O(m+n)$ space.