CMSC 754: Lecture 9
Planar Point Location (using Trapezoidal Maps)

Reading: Chapter 6 of the 4M's.

Point Location: In *planar point location* we are given a polygonal subdivision of the plane, and the objective is to preprocess this subdivision into a data structure so that given a query point q, it is possible to efficiently determine which face of the subdivision contains q (see Fig. 1(a)). For example, the subdivision might represent government subdivisions, such as countries, states, or counties, and we wish to identify the country, state, or county of a point given its GPS coordinates.

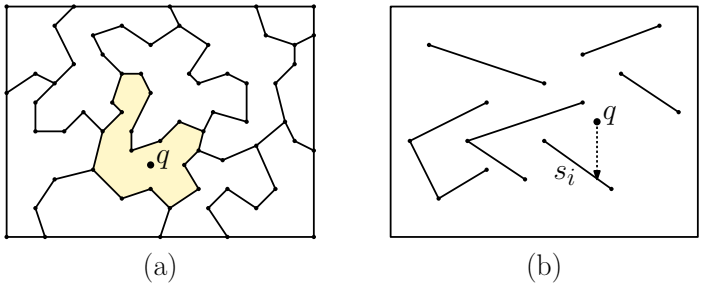

Fig. 1: (a) point location and (b) vertical ray-shooting queries.

It will be useful to generalize the above problem. Rather than assuming that the input is a subdivision of space into cells (what is commonly referred to as a *cell complex*), we will assume that the input is merely a set of n line segments $S = \{s_1, \ldots, s_n\}$. The objective is to answer *vertical ray-shooting queries*, which means, given a query point q, what line segment s_i (if any) lies immediately below the query point (see Fig. 1(b)). Observe that the ability to answer vertical ray-shooting queries implies that point-location queries can be answered. We simply label each segment with the identity of the subdivision cell that lies immediately above it.

We will make the usual general-position assumptions. In particular, except when two segments share a common endpoint, no two segment endpoints share the same x-coordinate. Also, we will assume no vertical segments. Let us also assume that the query point does not have the same x-coordinate of any segment endpoint.

Our objective is to obtain a data structure with $O(n)$ space and $O(\log n)$ query time, which we can build in $O(n \log n)$ time. This is asymptotically optimal (assuming a data structure based on comparisons).

History: When the problem was first considered, a number of solutions had been proposed, but they were all suboptimal by a log factor either in the query time or the space. The first optimal solution was proposed by David Kirkpatrick in the early 1980's. The algorithm involved a very clever incremental approach based on iteratively decimating a triangulation. The approach described here is both simpler and more efficient, but many of the ideas were inspired by Kirkpatrick's data structure. The method we will described is based on the randomized

construction of trapezoidal maps, and so the construction time holds in the expected case. (In contrast, Kirkpatrick's solution was deterministic.)

Recap of Trapezoidal Maps: Our point-location data structure will be based on the randomized trapezoidal map construction from the previous lecture. In that lecture we showed that a trapezoidal map of $O(n)$ space could be constructed in (randomized) $O(n \log n)$ expected time. In this lecture we show how to modify the construction so that, as a by product, we obtain a data structure for answering vertical ray-shooting queries. The preprocessing time for the data structure will also be $O(n \log n)$ in the expected case, the space required for the data structure will be $O(n)$, and the query time will be $O(\log n)$. The latter two bounds will hold unconditionally.

Let us recap some of the concepts from the previous lecture. Recall that the input as a set of segments in the plane $S = \{s_1, \ldots, s_n\}$, which are assumed to have been randomly permuted, and which are enclosed in an axis-aligned bounding rectangle. Recall that the trapezoidal map, denoted $\mathcal{T}(S)$, is a subdivision generated by shooting bullet paths both upwards and downwards from each segment endpoint until first striking another segment (or hitting the bounding box of the input). Let $S_i = \{s_1, \ldots, s_i\}$, and let $\mathcal{T}_i = \mathcal{T}(S_i)$ denote the trapezoidal map of S_i, and $\mathcal{T} = \mathcal{T}_n$.

Recall from the previous lecture that each time we add a new line segment, it may result in the creation of the collection of new trapezoids, which are said to *depend* on this line segment. We showed that (under the assumption of the random insertion order) the expected number of new trapezoids that are created with each stage is $O(1)$. This fact will be used later in this lecture.

Point Location Data Structure: Typical search structures (red-black trees, AVL trees, etc.) are rooted binary trees. Our point location data structure will be a rooted binary tree with *shared* subtrees, or equivalently, a rooted directed acyclic graph (DAG). Each node will have either zero or two outgoing edges. Nodes with zero outgoing edges are called *leaves*. The leaves will be in 1–1 correspondence with the trapezoids of the map. The other nodes are called *internal nodes*, and they are used to guide the search to the leaves. (Note that subtree sharing is important for our space efficiency.)

There are two types of internal nodes, *x-nodes* and *y-nodes*:

x-Nodes: Each x-node contains a reference to an endpoint of some segment. Its two children correspond to the points lying to the left and to the right of the vertical line passing through p (see Fig. 2(a)).

y-Nodes: Each y-node contains a reference to a line segment of the subdivision, and the left and right children correspond to whether the query point is above or below the line containing this segment, respectively (see Fig. 2(b)). (Don't be fooled by the name—y-node comparisons depend on both the x and y values of the query point.)

Note that the search will reach a y-node only if we have already verified that the x-coordinate of the query point lies within the vertical slab that contains this segment.

Our construction of the point location data structure mirrors the incremental construction of the trapezoidal map, as given in the previous lecture. In particular, if we freeze the

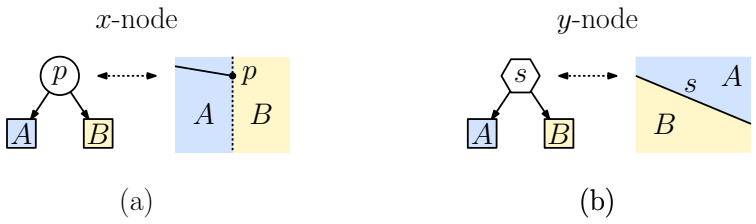

Fig. 2: (a) x-node and (b) y-node.

construction just after the insertion of any segment, the current structure will be a point location structure for the current trapezoidal map.

In Fig. 3 below we show a simple example of what the data structure looks like for two line segments. For example, if the query point is in trapezoid D, we would first detect that it is to the right of endpoint p_1 (right child), then left of t_1 (left child), then below s_1 (right child), then right of p_2 (right child), then above s_2 (left child).

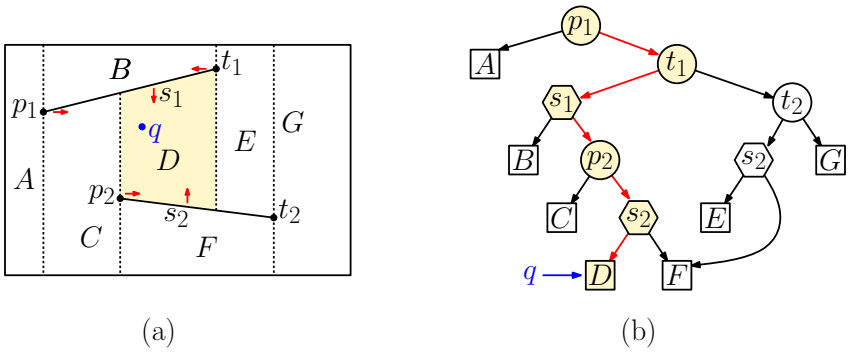

Fig. 3: Trapezoidal map point location data structure.

Incremental Construction: The question is how do we build this data structure incrementally? First observe that when a new line segment is added, we only need to adjust the portion of the tree that involves the trapezoids that have been deleted as a result of this new addition. Each trapezoid that is deleted will be replaced with a search structure that determines the newly created trapezoid that contains it.

Suppose that we add a line segment s. This results in the replacement of an existing set of trapezoids with a set of new trapezoids. As a consequence, we will replace the leaves associated with each such deleted trapezoid with a local search structure, which locates the new trapezoid that contains the query point. There are three cases that arise, depending on how many endpoints of the segment lie within the current trapezoid.

Single (left or right) endpoint: A single trapezoid A is replaced by three trapezoids, denoted B, C, and D. Letting p denote the endpoint, we create an x-node for p, and one child is a leaf node for the trapezoid B that lies outside vertical projection of the segment. For the other child, we create a y-node whose children are the trapezoids C and D lying above and below the segment, respectively (see Fig. 4(a)).

Two segment endpoints: This happens when the segment lies entirely inside an existing trapezoid A, which is replaced by four trapezoids, E, B, C, and D. Letting p and t denote the left and right endpoints of the segment, we create two x-nodes, one for p and the other for t. We create a y-node for the line segment, and join everything together (see Fig. 4(b)).

No segment endpoints: This happens when the segment cuts completely through a trapezoid A. This trapezoid is replaced by two trapezoids, one above and one below the segment, denoted C and D. We replace the leaf node for the original trapezoid with a y-node whose children are leaf nodes associated with C and D (see Fig. 4(c)).

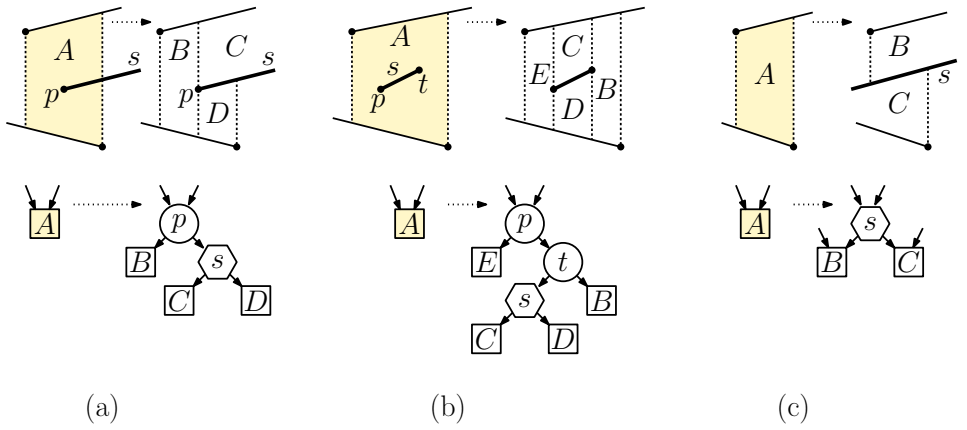

Fig. 4: Line segment insertion and updates to the point location structure. The single-endpoint case (left) and the two-endpoint case (right). The no-endpoint case is not shown.

It is important to notice that (through sharing) each trapezoid appears exactly once as a leaf in the resulting structure. How does this sharing occur? Whenever we add a segment, the wall trimming that results can result in two distinct trapezoids being merged into one (see trapezoid C in Fig. 5(a) and B and C in Fig. 5(b)). When this happens, the various paths leading into merged trapezoid are joined to a common node. An example showing the complete transformation to the data structure after adding a single segment is shown in Fig. 5 below.

Analysis: We claim that the size of the point location data structure is $O(n)$ and the query time is $O(\log n)$, both in the expected case. As usual, the expectation depends only on the order of insertion, not on the line segments or the location of the query point.

To prove the space bound of $O(n)$, observe that the number of new nodes added to the structure with each new segment is proportional to the number of newly created trapezoids. Last time we showed that with each new insertion, the expected number of trapezoids that were created was $O(1)$. Therefore, we add $O(1)$ new nodes with each insertion in the expected case, implying that the total size of the data structure is $O(n)$.

Analyzing the query time is a little subtler. In a normal probabilistic analysis of data structures we think of the data structure as being fixed, and then compute expectations over random queries. Here the approach will be to imagine that we have exactly one query point

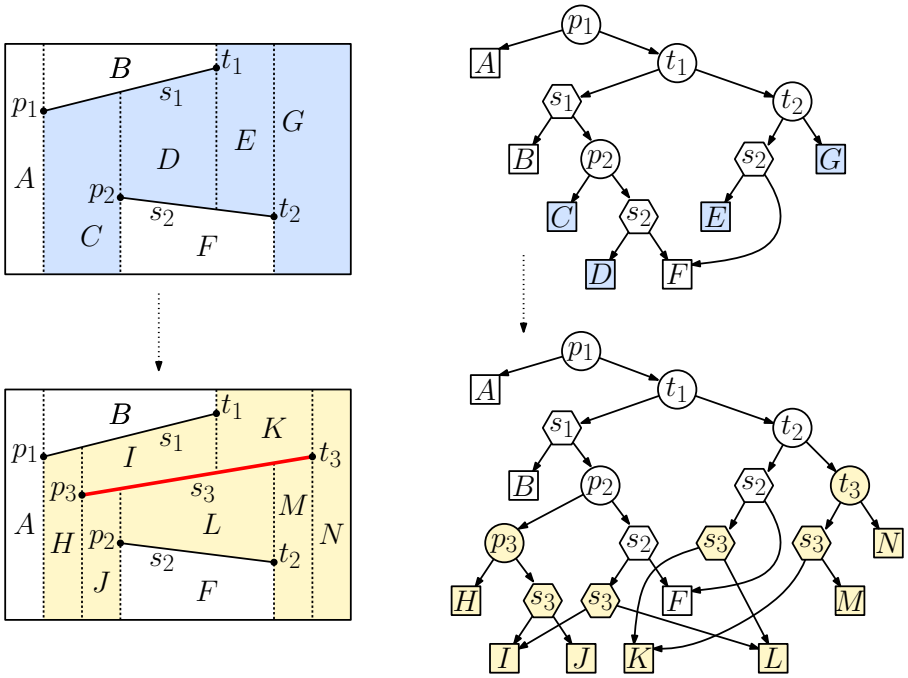

Fig. 5: Line segment insertion.

to handle. The query point can be chosen arbitrarily (imagine an adversary that tries to select the worst-possible query point) but this choice is made without knowledge of the random choices the algorithm makes. We will show that, given a fixed query point q, the expected search path length for q is $O(\log n)$, where the expectation is over all segment insertion orders. (Note that this does not imply that the expected maximum depth of the tree is $O(\log n)$. We will discuss this issue later.)

Let q denote the query point. Rather than consider the search path for q in the final search structure, we will consider how q moves incrementally through the structure with the addition of each new line segment. Let Δ_i denote the trapezoid of the map that q lies in after the insertion of the first i segments. Observe that if $\Delta_{i-1} = \Delta_i$, then insertion of the ith segment did not affect the trapezoid that q was in, and therefore q will stay where it is relative to the current search structure. (For example, if q was in trapezoid B prior to adding s_3 in Fig. 5 above, then the addition of s_3 does not incur any additional cost to locating q.)

However, if $\Delta_{i-1} \neq \Delta_i$, then the insertion of the ith segment caused q's trapezoid to be replaced by a different one. As a result, q must now perform some additional comparisons to locate itself with respect to the newly created trapezoids that overlap Δ_{i-1}. Since there are a constant number of such trapezoids (at most four), there will be $O(1)$ work needed to locate q with respect to these. In particular, q may descend at most three levels in the search tree after the insertion. The worst case occurs in the two-endpoint case, where the query point falls into one of the trapezoids lying above or below the segment (see Fig. 4(b)).

Since a point can descend at most three levels with each change of its containing trapezoid, the expected length of the search path to q is at most three times the number of times that q

changes its trapezoid as a result of each insertion. For $1 \leq i \leq n$, let $X_i(q) = 1$ if q changes its trapezoid after the ith insertion 0 otherwise. Let $E(X_i(q))$ denote its expected value, which is equivalent to $\text{Prob}(\Delta_i(q) \neq \Delta_{i-1}(q))$. Letting $D(q)$ denote the average depth of q in the final search tree, we have

$$D(q) \;\leq\; 3\sum_{i=1}^{n} E(X_i(q)) \;=\; 3\sum_{i=1}^{n} \text{Prob}(\Delta_i(q) \neq \Delta_{i-1}(q)).$$

What saves us is the observation that, as i becomes larger, the more trapezoids we have, and the smaller the probability that any random segment will affect a given trapezoid. In particular, we will show that $\text{Prob}(X_i(q)) \leq 4/i$. We do this through a backwards analysis. Consider the trapezoid Δ_i that contained q *after* the ith insertion. Recall from the previous lecture that each trapezoid is dependent on at most four segments, which define the top and bottom edges, and the left and right sides of the trapezoid. Clearly, Δ_i would have changed as a result of insertion i if any of these four segments had been inserted last. Since, by the random insertion order, each segment is equally likely to be the last segment to have been added, the probability that one of Δ_i's dependent segments was the last to be inserted is at most $4/i$. Therefore, $\text{Prob}(X_i(q)) \leq 4/i$.

From this, it follows that the expected path length for the query point q is at most

$$D(q) \;\leq\; 3\sum_{i=1}^{n} \frac{4}{i} \;=\; 12\sum_{i=1}^{n} \frac{1}{i}.$$

Recall that $\sum_{i=1}^{n} \frac{1}{i}$ is the Harmonic series, and for large n, its value is very nearly $\ln n$. Thus we have

$$D(q) \;\approx\; 12 \cdot \ln n \;=\; O(\log n).$$

Guarantees on Search Time: (Optional) One shortcoming with this analysis is that even though the search time is provably small in the expected case for a given query point, it might still be the case that once the data structure has been constructed there is a single very long path in the search structure, and the user repeatedly performs queries along this path. Hence, the analysis provides no guarantees on the running time of all queries.

It is far from trivial, but it can be shown that by repeated application of the randomized incremental construction, it is possible to achieve worst-case search time of $O(\log n)$, worst-case size of $O(n)$, and expected-case construction time is $O(n \log n)$.[1] The idea is to engineer the constants so that the probability of failure along any search path is extremely small (say $1/n^c$, for some constant $c \geq 1$). It follows that all the possible search paths will have the desired $O(\log n)$ depth with at least a constant probability. While we might be unlucky on any given execution of the algorithm, after a constant number of attempts, we expect one of them to succeed.

Summary: In the previous lecture, we showed that a randomized incremental algorithm allows to construct the trapezoidal map of any set of nonintersecting lines segments in expected time

[1] M. Hemmer, M. Kleinbort, and D. Halperin. Optimal randomized incremental construction for guaranteed logarithmic planar point location. *Comput. Geom.*, 58:110–123, 2016.

$O(n)$. Further, if the line segments have m intersections, this can be generalized to $O(n+m)$. However, we ignored the question of how to locate the trapezoid containing the left endpoint of each segment. In this lecture, we have shown that this can be done in $O(\log n)$ time per endpoint, for a total of $O(n \log n)$ time. (Note that when the segments intersect, the time is actually $O(\log(n+m))$, but since $m = O(n^2)$, this is the same as $O(\log n)$.)

This data structure applies more generally to vertical ray-shooting queries for any set of line segments. As observed earlier, we can solve the point-location problem in *any* planar subdivision of line segments with $O(n)$ space, $O(\log n)$ query times, with expected-case preprocessing of $O(n \log n)$.

Note that the randomization is not a necessity. It can be eliminated, but at the expense of higher constant factors and a more complicated algorithm. Also note that we can generalize this to curved objects, as long as they are x-monotone, or can be subdivided into a relatively small number of x-monotone pieces.

Finally, note that this data structure does not generalize to higher dimensions. Indeed, there is no known data structure for point location in subdivisions in dimensions 3 and higher. There are data structures that achieve $O(\log n)$ worst-case query time, but the space generally grows as $O(n^{O(d)})$. Conversely, if the space is constrained to $O(n)$, then the worst-case query time grows to a polynomial function of n. There are good heuristic data structure (such as quadtrees) that may achieve good performance in "typical" instances, but they cannot guarantee good performance for all inputs.