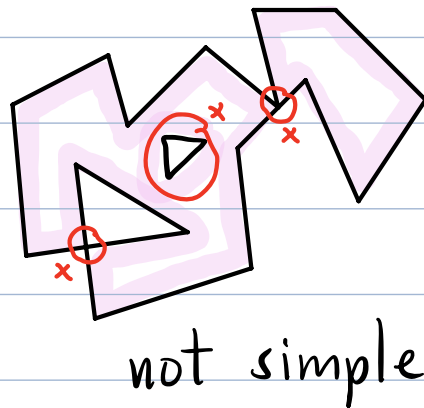
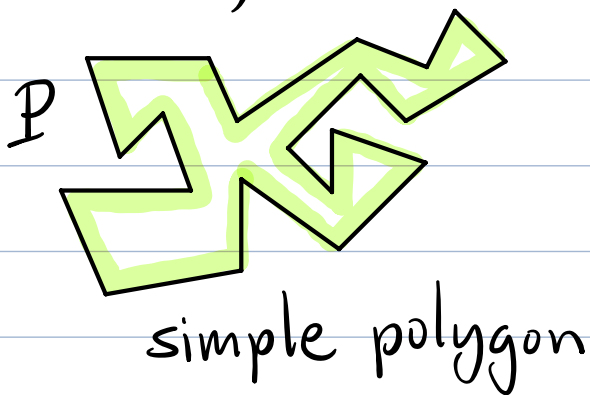


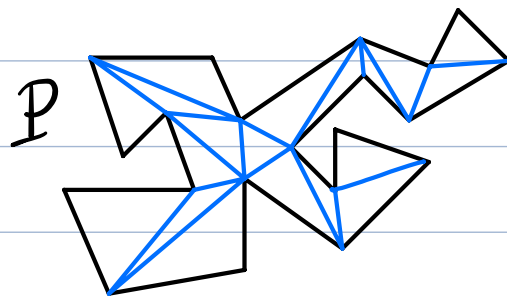
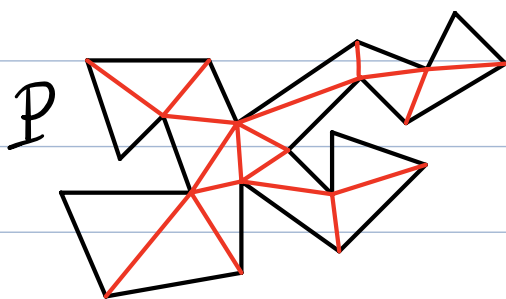
CMSC 754 - Computational Geometry

Lecture 5: Polygon Triangulation

Polygon Triangulation: Given a **simple polygon** P (that is, a simple, closed polygonal chain)...



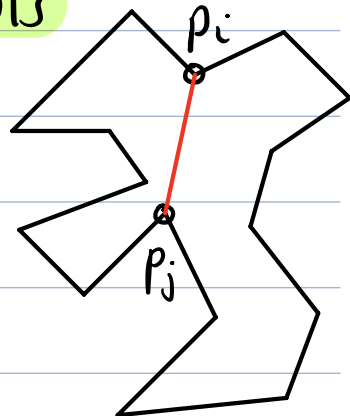
subdivide the interior of P into triangles
(vertices drawn from P 's vertices)



Notes: - P given as a **cyclic seq. of pts**

- Vertices $p_i + p_j$ are **visible** if
open segment $\overline{p_i p_j} \subseteq \text{int}(P)$

- If $p_i + p_j$ visible, segment $\overline{p_i p_j}$
called a **diagonal**



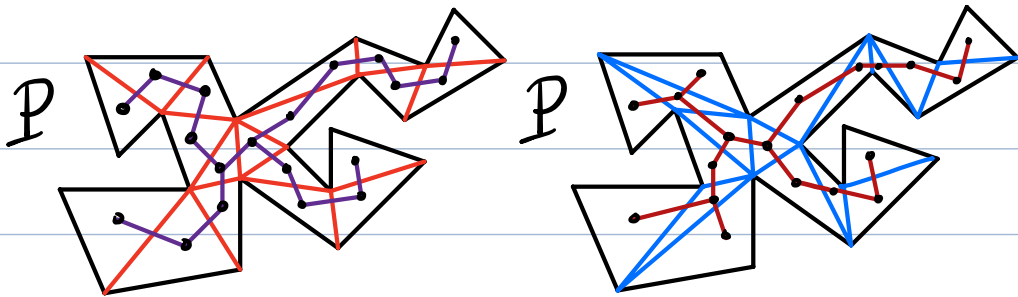
Lemma: Given any n -vertex simple polygon ($n \geq 3$)

- A triangulation exists
- Any triangulation has $n-3$ diagonals
- Any triangulation has $n-2$ triangles

Dual Graph: A triangulation defines a graph:

Vertices \leftarrow triangles

Edges \leftarrow adjacent (share common edge)



The dual graph of a polygon triangulation is **connected + acyclic** \Rightarrow **tree**

History of Polygon Triangulation:

$O(n^2)$ - Easy (find a diagonal + recurse)

$O(n \log n)$ - We'll present this

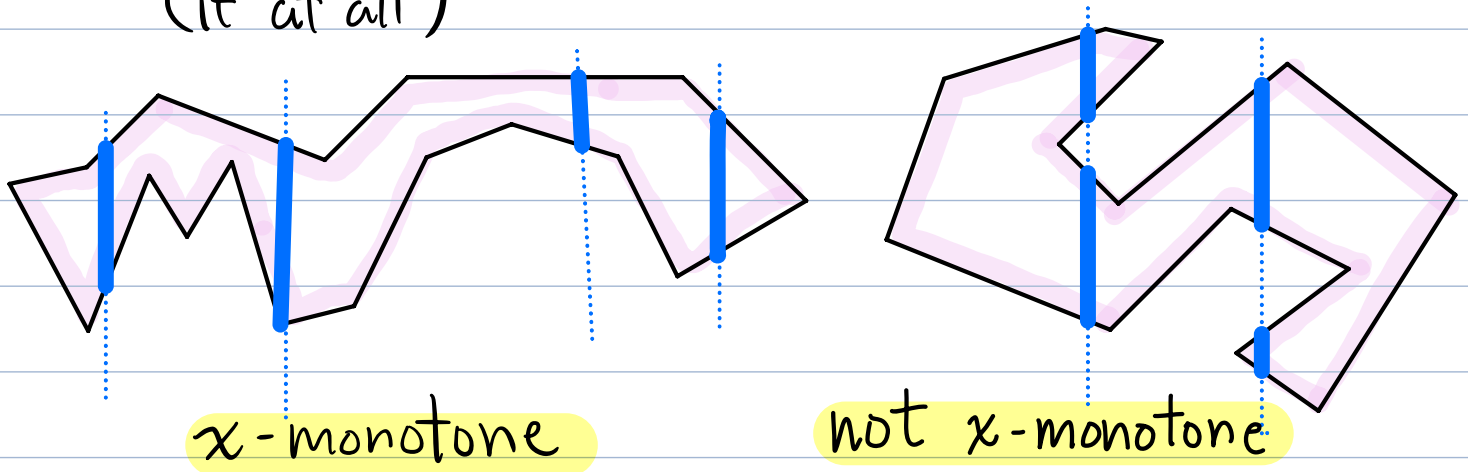
$O(n)$ - Chazelle 1991 (very complicated!)

Two steps:

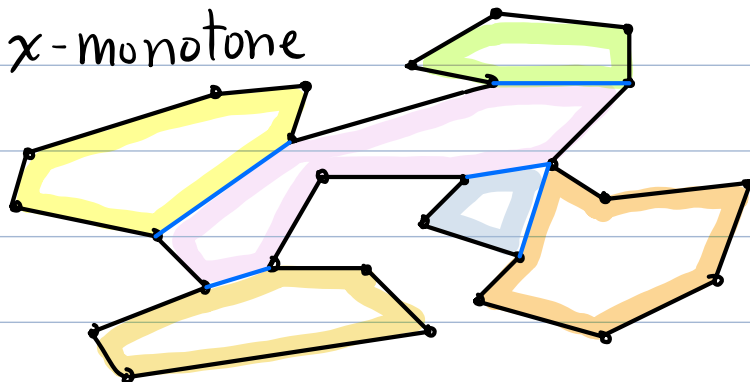
- ① Decompose the polygon into (simpler) polygons
- monotone polygons - $O(n \log n)$
- ② Triangulate each monotone polygon - $O(n)$

Output: Graph structure, called a doubly-connected edge list (DCEL) see text

Def: A polygon is **x-monotone** if any vertical intersects the polygon in a single segment (if at all)



Monotone Decomposition - Add (non-intersecting) diagonals so that connected components are all x-monotone

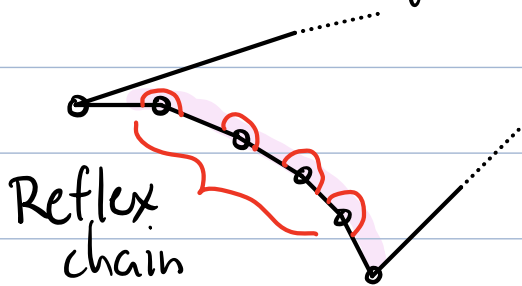


Triangulating a Monotone Polygon:

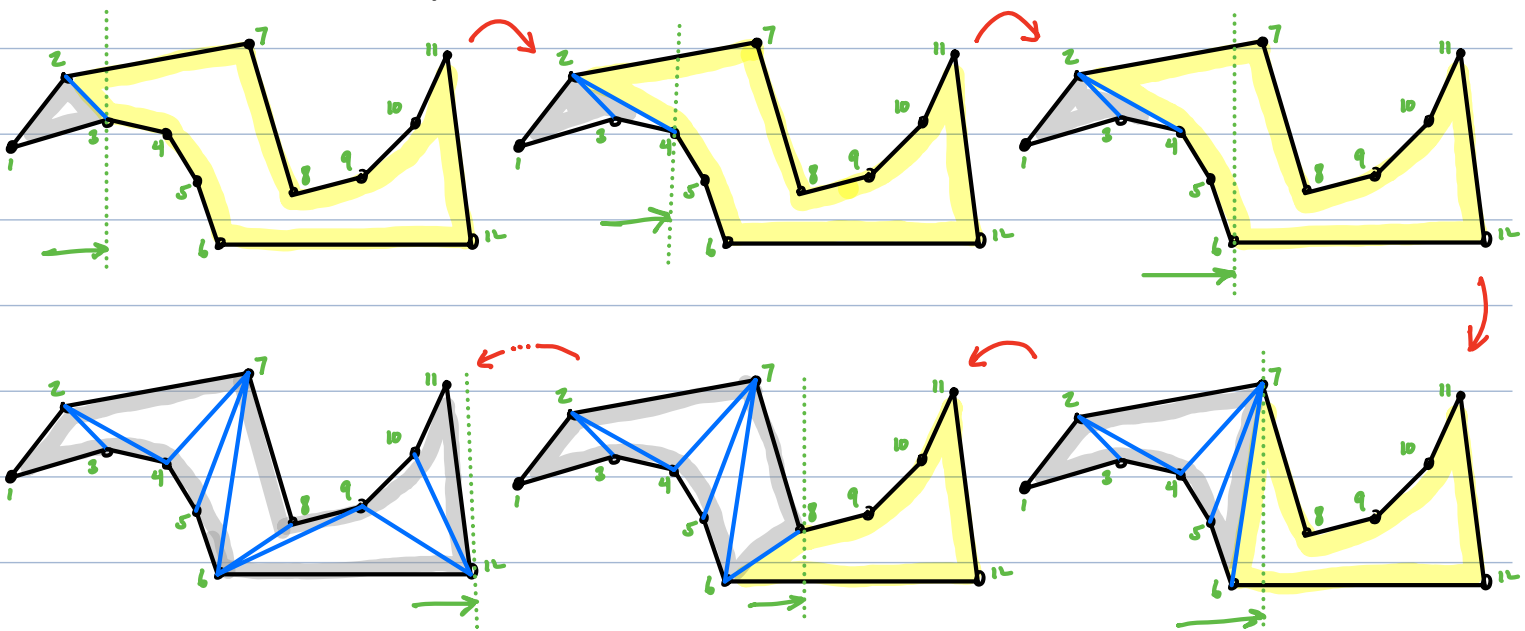
General position: No duplicate x-coords
(no vertical edges)

Reflex Vertex: Internal angle $\geq \pi$

Reflex Chain: Sequence of reflex vertices

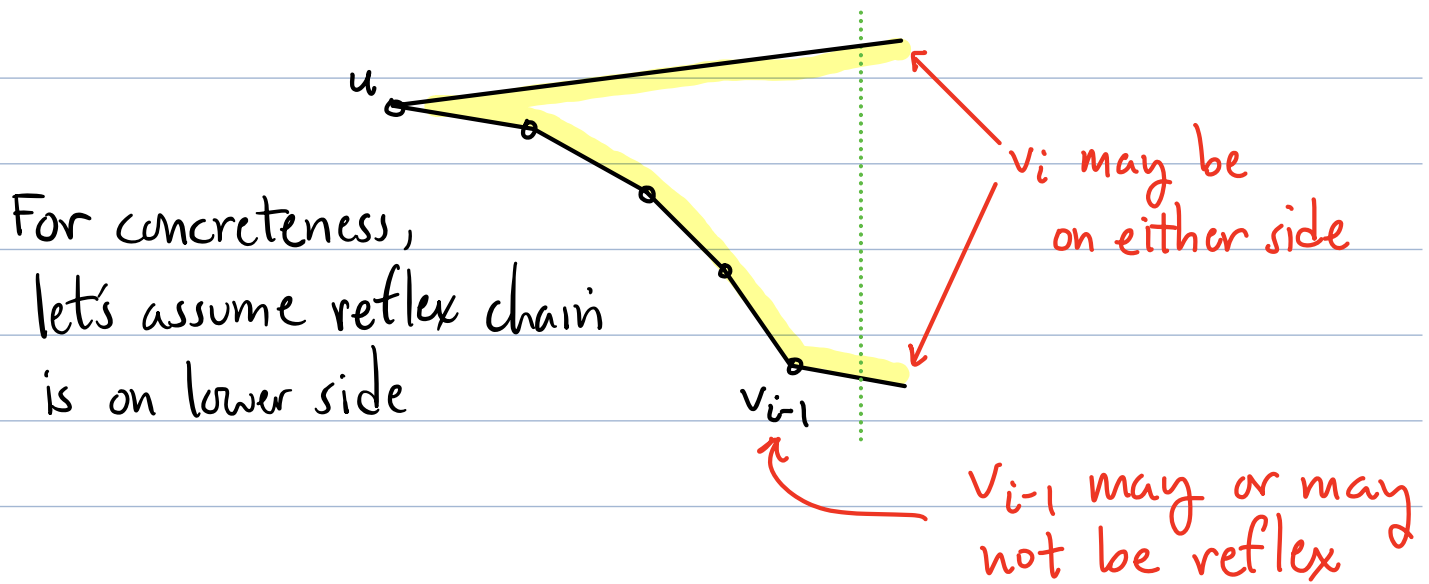


General approach: Sweep from left to right
+ triangulate as much as we can behind us.



What's the loop invariant?

Lemma: For $i \geq 2$, let v_i be the next vertex to process. The untriangulated region to left of v_i consists of two x -monotone chains starting from a common vertex u . One chain is a single edge, and the other is a reflex chain (of one or more edges).

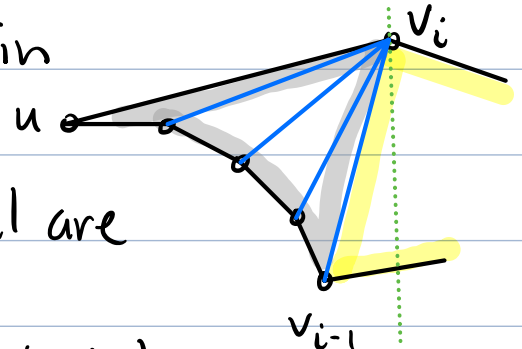


Case 1: (v_i lies on upper chain)

- add diagonals between v_i and all vertices of the chain

[By monotonicity, all are visible to v_i]

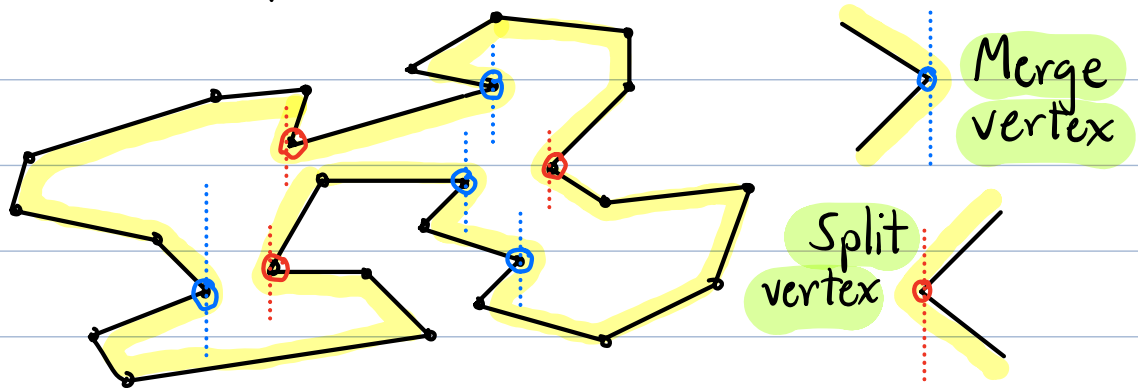
Now $u = v_{i-1}$. Reflex chain has just one edge.



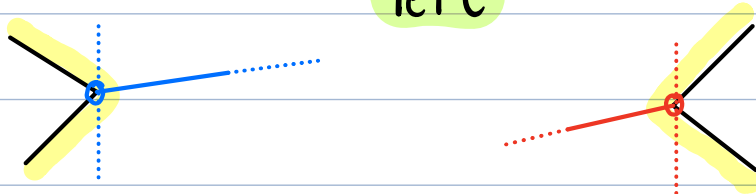
Monotone Subdivision:

Recall: Add diagonals to create x-monotone

Where? Scan reflex vertex: Reflex vertex where both edges on same side of vertical line.



Add a diagonal to right side of each merge
" " " " left " " " split

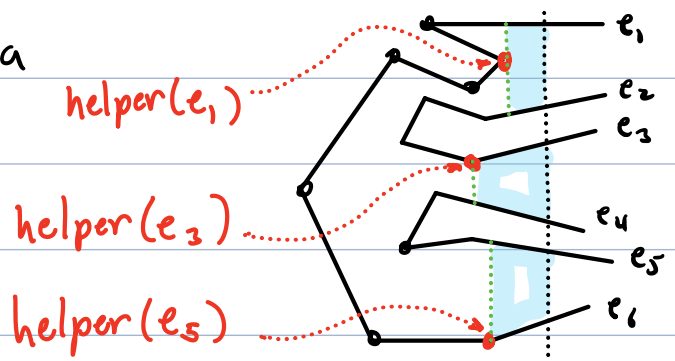


Plane-sweep Approach:

Need auxiliary info to help with diagonals
For each edge e_a of sweep line with int(\mathcal{L}) below:

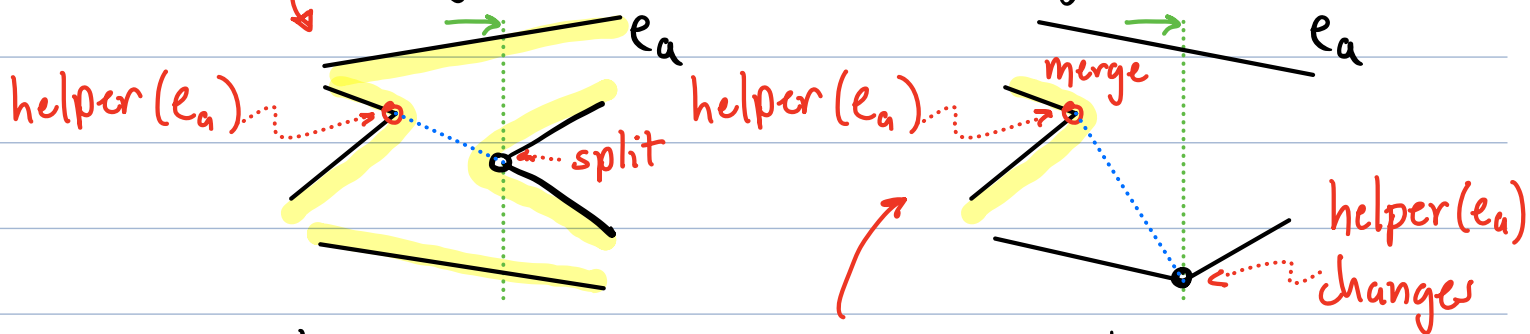
helper(e_a) = rightmost vertically visible

vertex on or below e_a
to left of sweep line



Why is the helper helpful?

- When we see a **split vertex**, we **add diagonal to helper of edge above**



- When we see a **merge vertex**, it is the helper of edge above + we **connect it to next vertex where helper(e_a) changes**

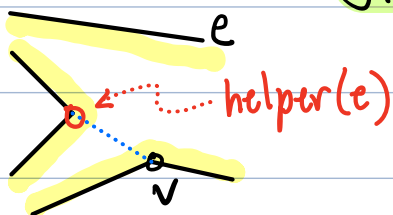
Events: Polygon vertices (sorted by x)

Sweep-line status: Edges intersecting the sweep line (ordered dictionary)

Event processing: There are many cases!

Utility:

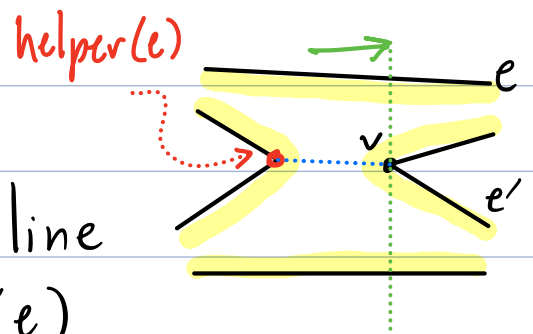
fix-up(v, e):



if (helper(e) is a merge vertex)
add diagonal v to helper(e)

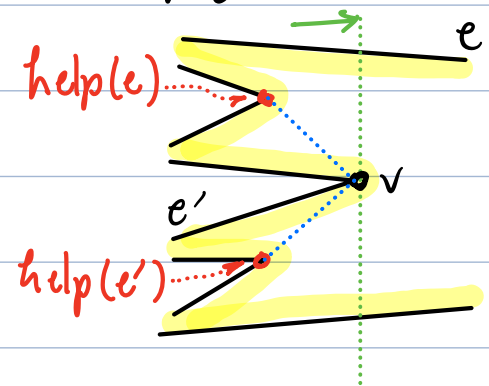
Split Vertex (v):

- $e \leftarrow$ edge above v in sweep line
- add diagonal v to $\text{helper}(e)$
- insert edges incident to v into sweep line
- letting e' be lower, set $\text{helper}(e') \leftarrow v$



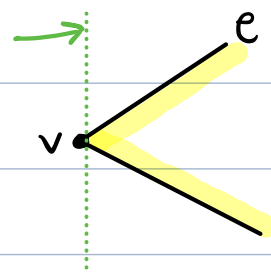
Merge Vertex (v):

- Consider two edges incident to v + let e' be lower one
- Delete both from sweep line
- Let e be edge above v
- $\text{fix-up}(v, e) + \text{fix-up}(v, e')$



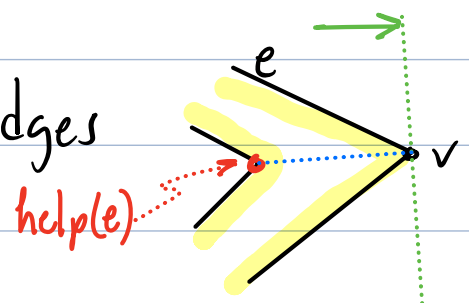
Start vertex (v):

- Insert v 's incident edges into sweep line
- Letting e be upper edge, $\text{helper}(e) \leftarrow v$



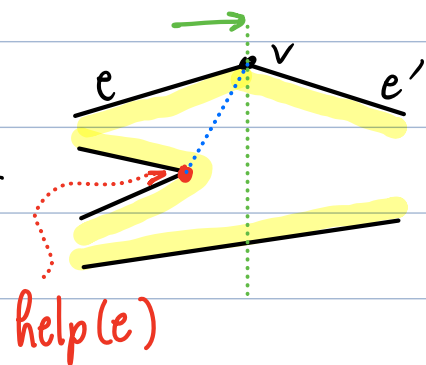
End vertex (v):

- Consider the two incident edges + let e be upper edge
- Delete both from sweep line
- $\text{fix-up}(v, e)$



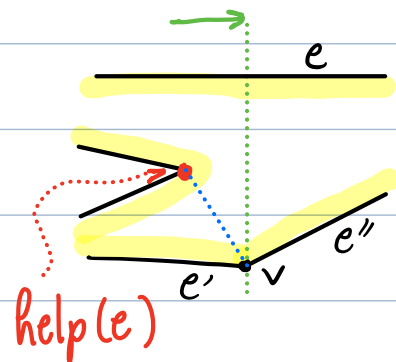
Upper-chain vertex (v):

- Let e be edge to left, e' to right
- $\text{fix-up}(v, e)$
- Replace e with e' in sweep line
- $\text{helper}(e') \leftarrow v$



Lower-chain vertex (v):

- Let e be edge above
- $\text{fix-up}(v, e)$
- Let e' be edge to left, e'' to right
- Replace e' with e'' in sweep line



Summary:

- Simple polygons
- Triangulation - Existence
 - Size
- Triangulating an x -monotone polygon $O(n)$
- Subdividing a polygon into x -monotone pieces - $O(n \log n)$
- Both based on plane sweep