CMSC 754 - Computational Geometry
Lecture 6: Halfplane Intersection + Duality

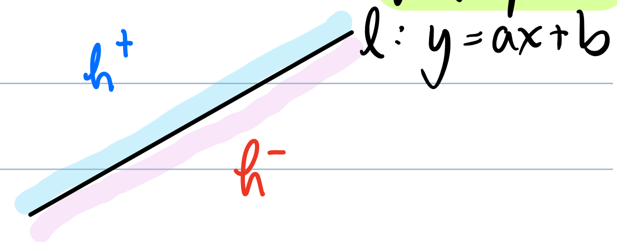
Halfplane Intersection:

Recall, each line in plane defines two halfspaces

$\ell: y = ax + b$

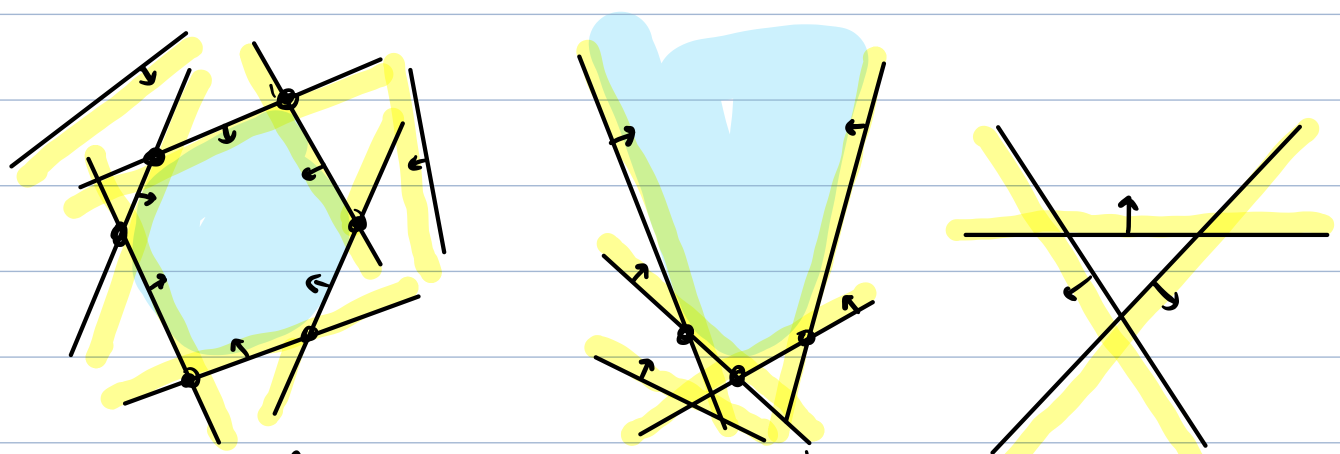
$h^+: y \geq ax + b$

$h^-: y \leq ax + b$



A halfspace is an (unbounded) convex set

Given a set of halfspaces: $H = \{h_1, \ldots, h_n\}$ their intersection $\bigcap_{i=1}^{n} h_i$ is a (possibly unbounded / possibly empty) convex polygon

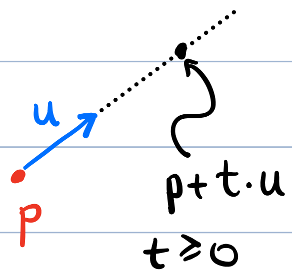


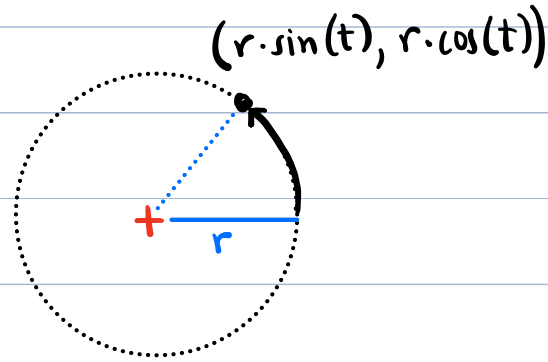
Bounded   Unbounded   Empty

Representing lines (and more):

	$\mathbb{R}^2$ (Line)	$\mathbb{R}^d$ (Hyperplane)
Explicit: $y=f(x)$	$y = ax + b$	$x_d = \sum_{i=1}^{d-1} a_i x_i + b$
Implicit: $f(x,y)=0$	$f(x,y) = ax+by+c$	$f(x_1,\dots,x_d) = \sum_{i=1}^{d} a_i x_i + b$

Parametric: $(x(t), y(t))$



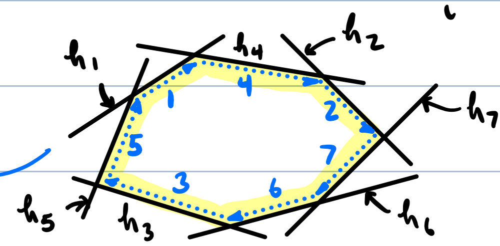


Halfplane Intersection:

Given halfplanes $H = \{h_1, \dots, h_n\}$ construct $\mathcal{H} = \bigcap_i h_i$

Output: Sequence of edges

$\langle 5,1,4,2,7,6,3 \rangle$



Divide and Conquer Algorithm: $O(n \log n)$

Intersect (H) {

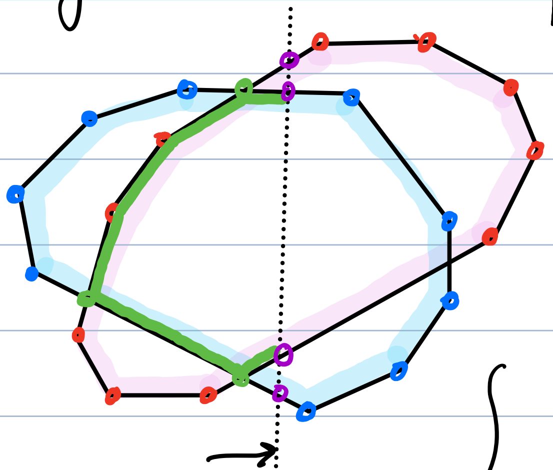
- if$(|H| = 1)$ return h_1 [single halfspace]
- else

  partition $H \to H_1, H_2$ $\quad |H_i| \le n/2$

  $I_1 \leftarrow$ Intersect(H_1); $I_2 \leftarrow$ Intersect(H_2)

  return merge(I_1, I_2) ← How?

How to merge? Plane sweep



- At most 4 segments hit sweep line
- $\leq n_1 + n_2$ end pt events
  $n_i = |H_i|$
- $\leq 2(n_1+n_2)$ intersection events
- Boundaries are already sorted

$O(n_1+n_2)$ time

Overall Running Time:

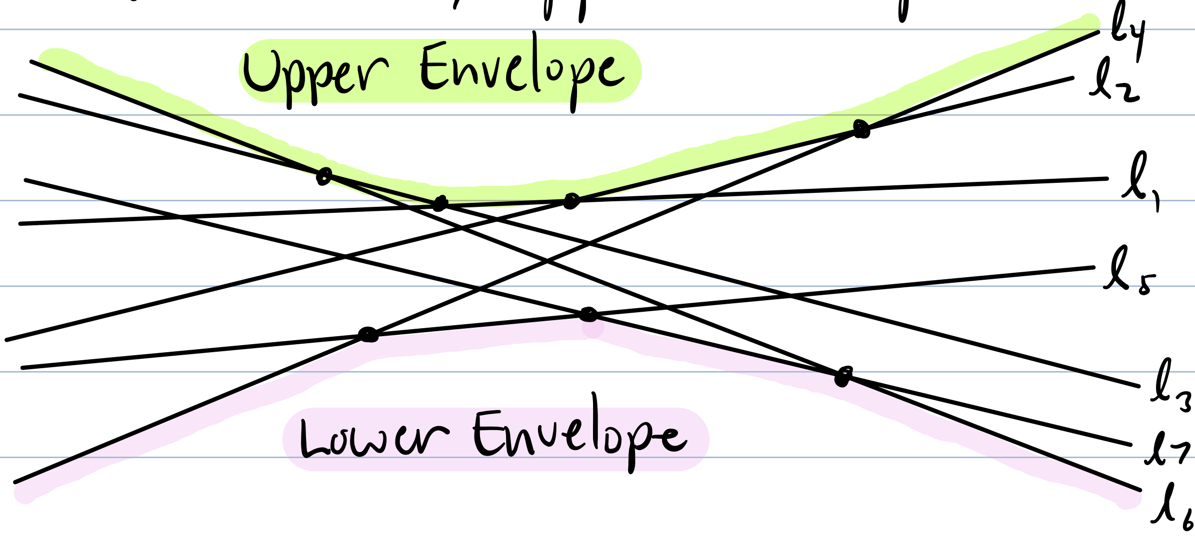
$$T(n) = 2T(n/2) + n$$

2 recursive calls on $n/2$ halfspaces

merge in linear time

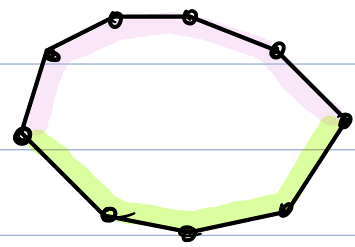
$$= O(n \log n)$$ [see, e.g., CLRS]

Special Case: Lower/Upper Envelopes



Envelopes of lines ~ Hull of points

Related?

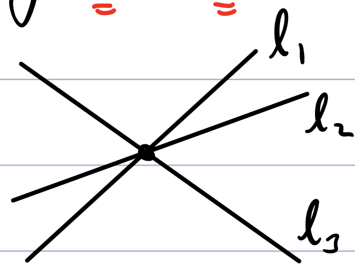


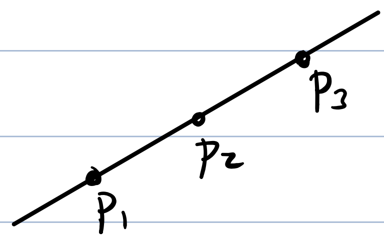
Point-Line Duality

Lines in $\mathbb{R}^2$ are a lot like points:

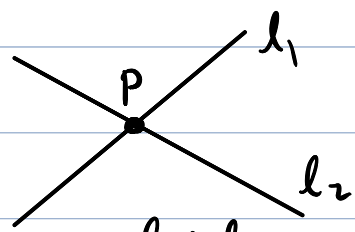
2 degrees of freedom: $y = a x + b$ $\qquad$ $p: (a, b)$

degeneracy:



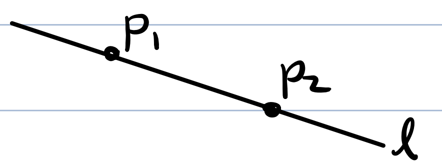


incidence:



$p = \ell_1 \wedge \ell_2$

Two lines meet at a point



$\ell = p_1 \vee p_2$

Two points join to form a line

Dual Operator:

Given point $p = (a, b)$ $\qquad$ $a, b \in \mathbb{R}$

line $\ell: y = cx - d$ $\qquad$ $c, d \in \mathbb{R}$

Dual p^* is the line $y = a \cdot x - b$

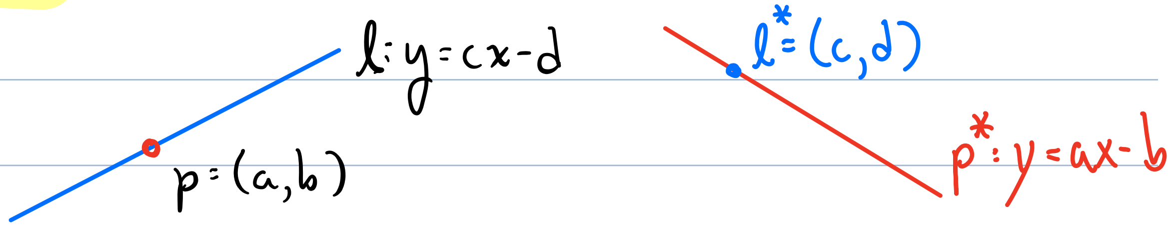
ℓ^* is the point (c, d)

Observations:

Self-inverse: $p^{**} = p \qquad \ell^{**} = \ell$

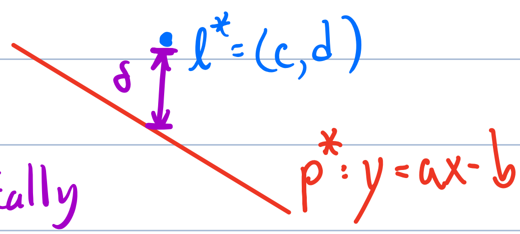
Incidence: p lies on ℓ **iff** ℓ^* lies on p^*

Proof: $b = c \cdot a - d \iff d = a \cdot c - b$



Order reversing: p is at dist δ above ℓ **iff** ℓ^* is at dist δ above p^*

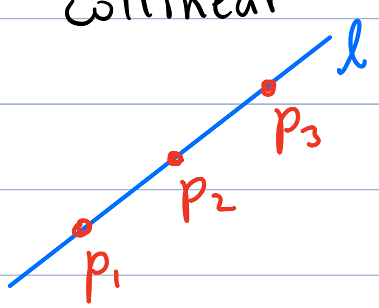
Proof: $b = c \cdot a - d + \delta \iff d = a \cdot c - b + \delta$

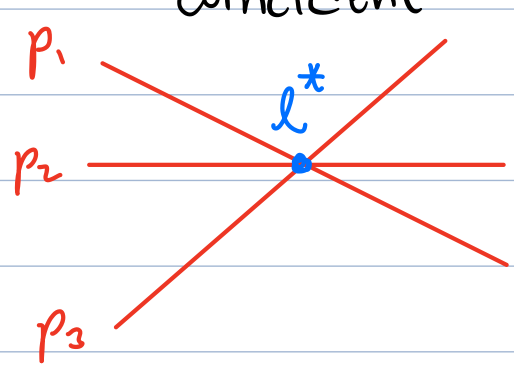


Note: Distance measured vertically

Degeneracy:

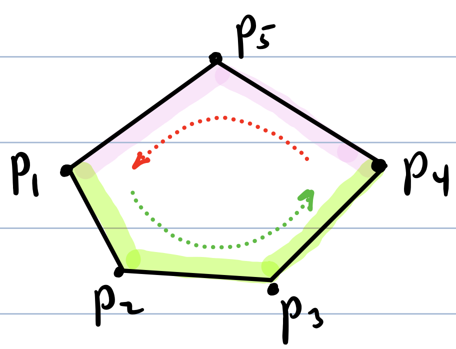
p_1, p_2, p_3 are collinear **iff** p_1^*, p_2^*, p_3^* are coincident

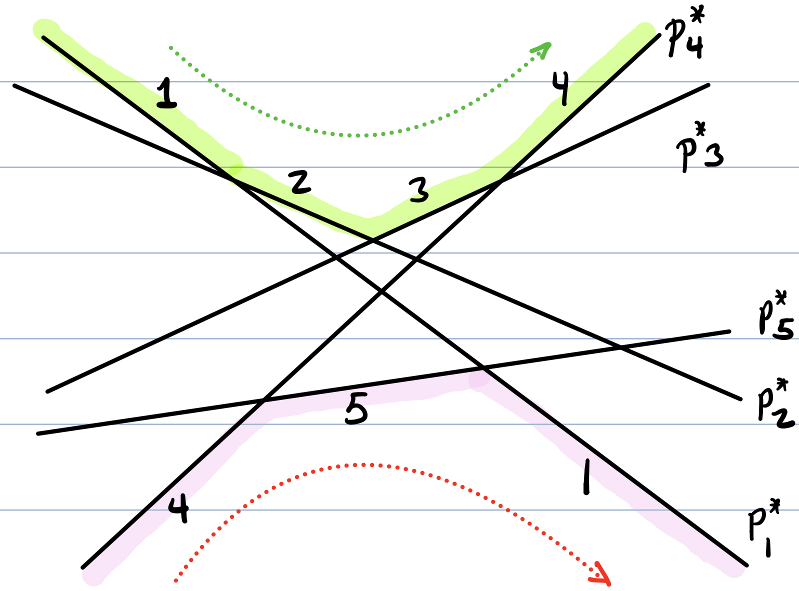




Hulls and Envelopes:

Lemma:
Given a set $P = \{p_1, \dots, p_n\}$ in $\mathbb{R}^2$, point p_i is on upper (lower) hull of P iff p_i^* contributes an edge to lower (upper) envelope of P^*.
Further - CCW order of vertices on hull $\equiv$ left-to-right order of edges on env.

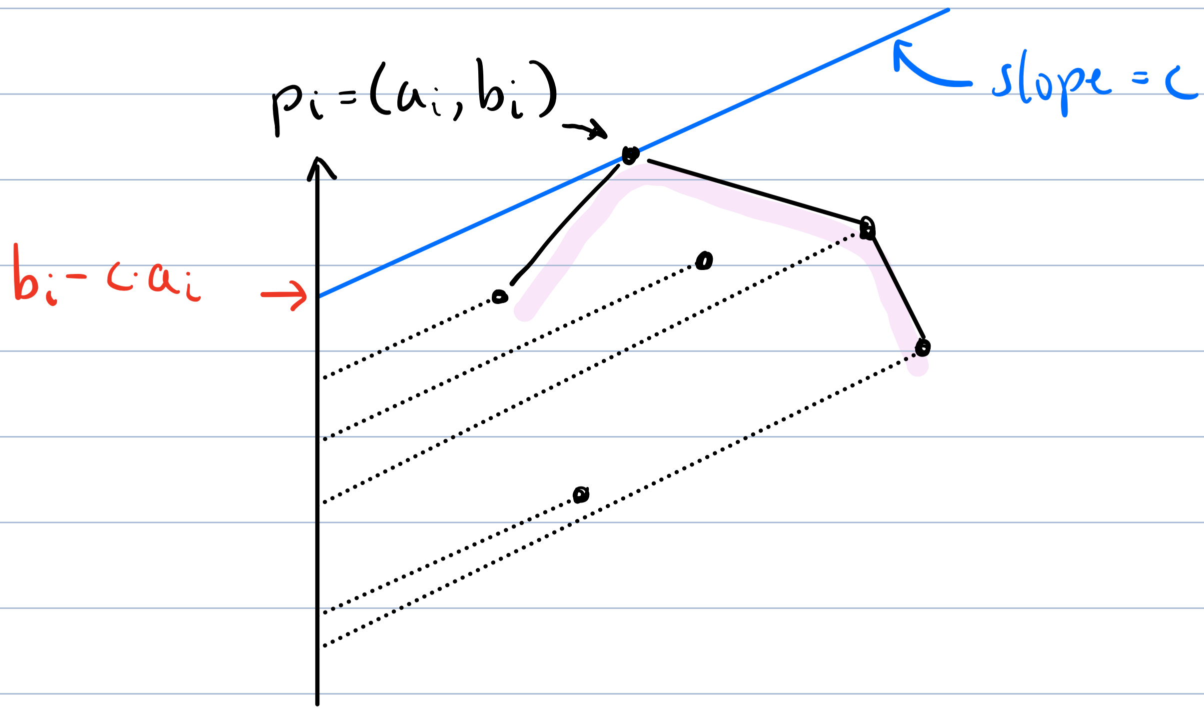




Proof: We'll show p_i on upper hull iff p_i^* contributes edge to lower envelope

(other case is symmetrical)

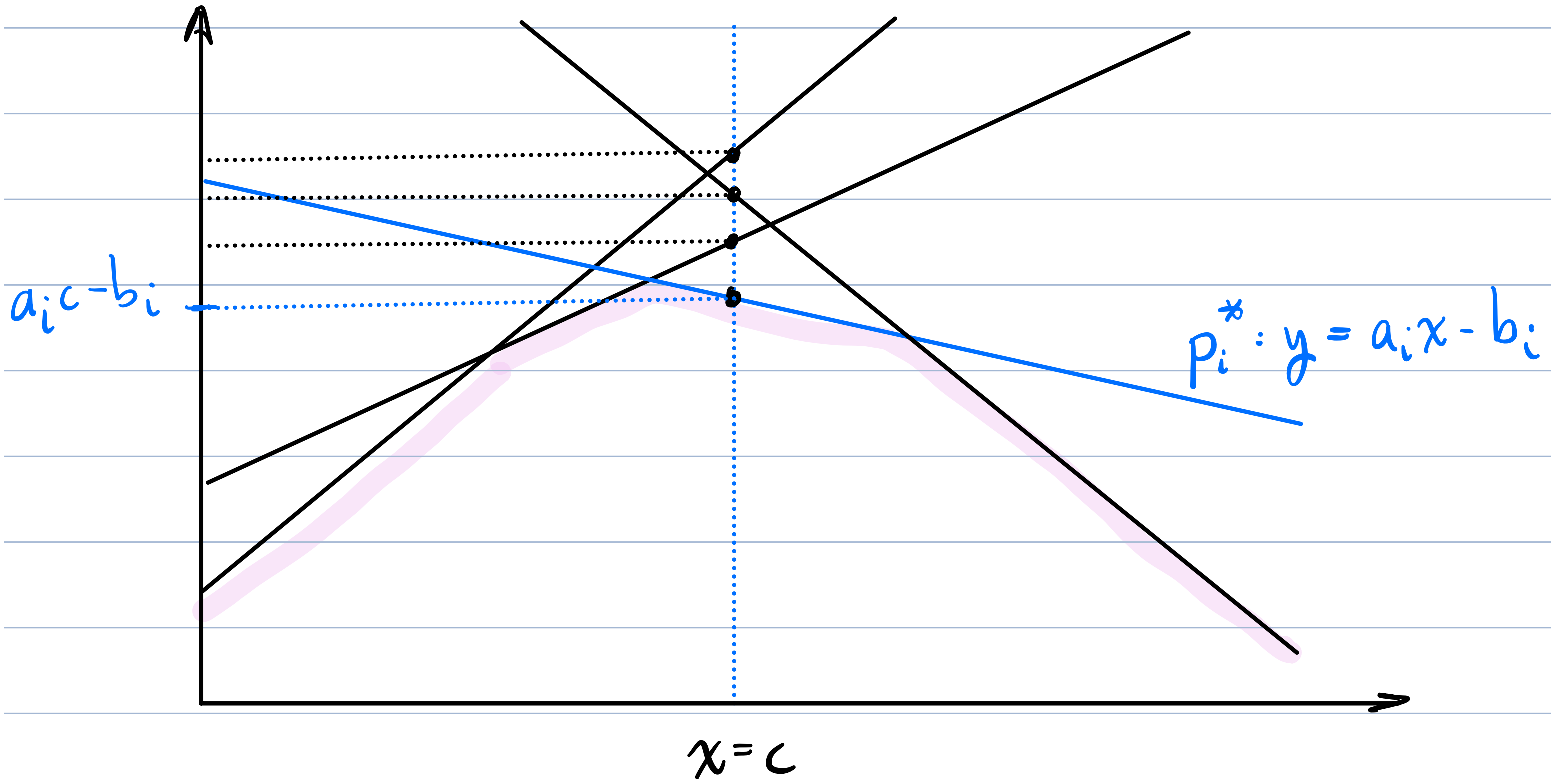
- p_i is vertex of upper hull iff there is a supporting line ℓ of some slope c through p_i



$\iff$ The line with slope c through p_i has max y-intercept

$$\iff \quad \boxed{i = \underset{1 \le j \le n}{\operatorname{argmax}} \left(b_j - c \cdot a_j \right)} \qquad (1)$$

- p_i^* contributes an edge to lower envelope iff p_i^* is lowest line to hit the vertical line $x = c$ for some c.

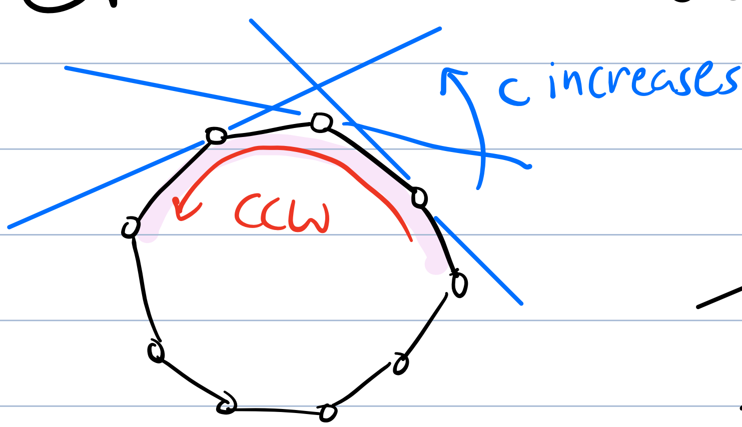


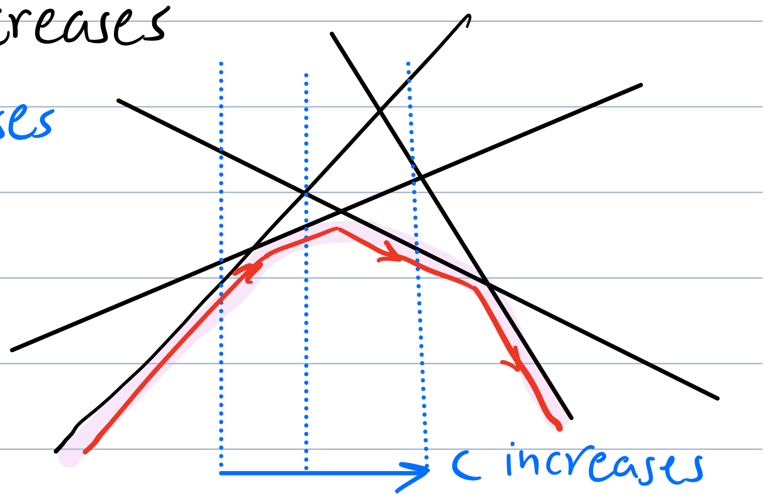
$$\iff \quad i = \underset{1 \le j \le n}{\arg\min} \; a_j c - b_j$$

$$\boxed{i = \underset{1 \le j \le n}{\arg\max} \,(b_j - c a_j)} \qquad (2)$$

But $(1) \equiv (2)$

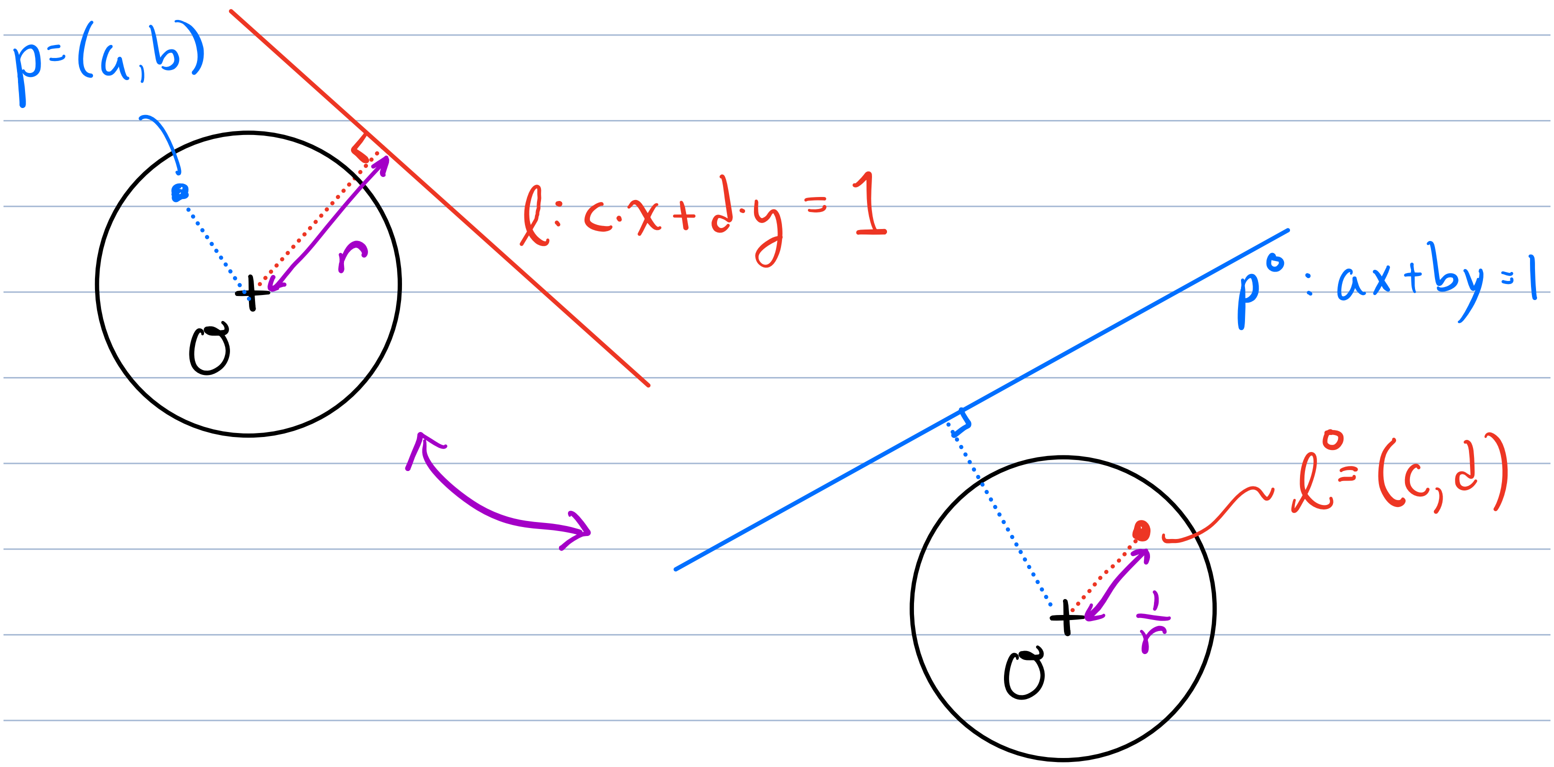
Order? As c increases



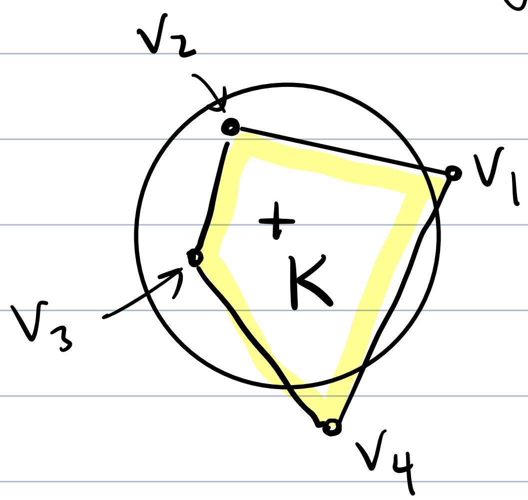


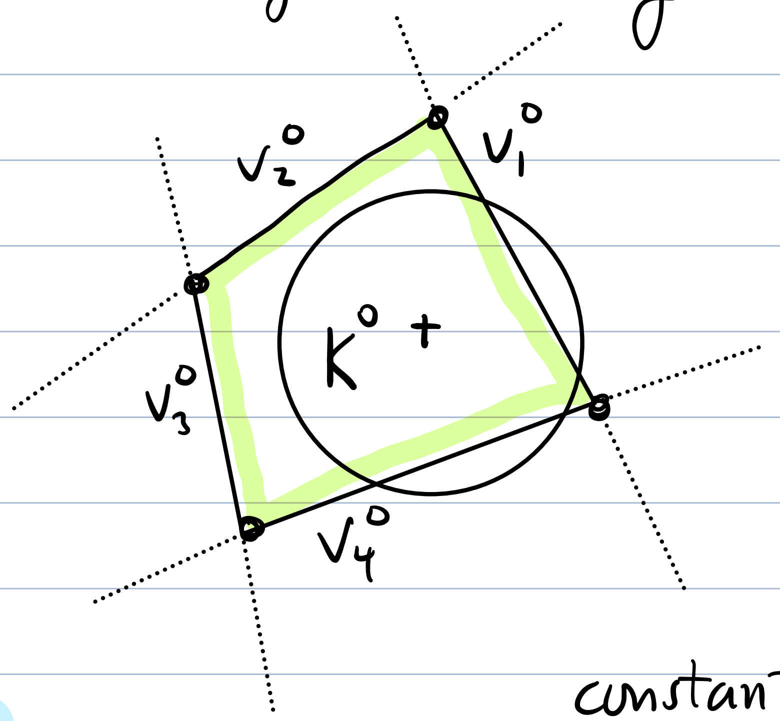
□

Polar Dual: Another dual transform



If K is a convex body containing the origin





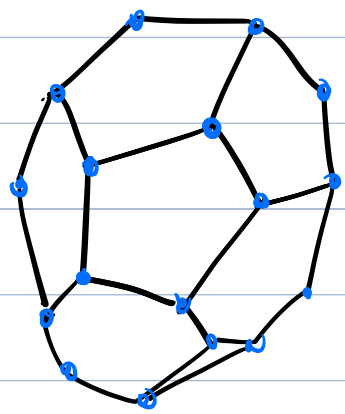
Mahler Volume:

$$\mathrm{vol}(K)\cdot\mathrm{vol}(K^\circ) = \Theta(1)$$

constant depend. on dimension

Higher Dimensions:

Let K be a polytope in $\mathbb{R}^d$



For $0 \le i \le d-1$
f_i = num. of faces of dim i

$i = 0$ – vertices
$= 1$ – edges
$\vdots$
$= d-1$ – facets

McMullen's Upper-Bound Thm:

- A polytope in $\mathbb{R}^d$ with n vertices may have $O(n^{\lfloor d/2 \rfloor})$ facets
- .... n facets ... $O(n^{\lfloor d/2 \rfloor})$ vertices

Tight – Cyclic Polytopes

Summary:

- Halfplane intersection - $O(n \log n)$
- Special case - Upper/Lower envelopes
- Point-line duality
  - Incidence/Order properties
- Upper/Lower hulls $\equiv$ Lower/Upper envelopes
- Polar duality
- Polytopes in $\mathbb{R}^d$