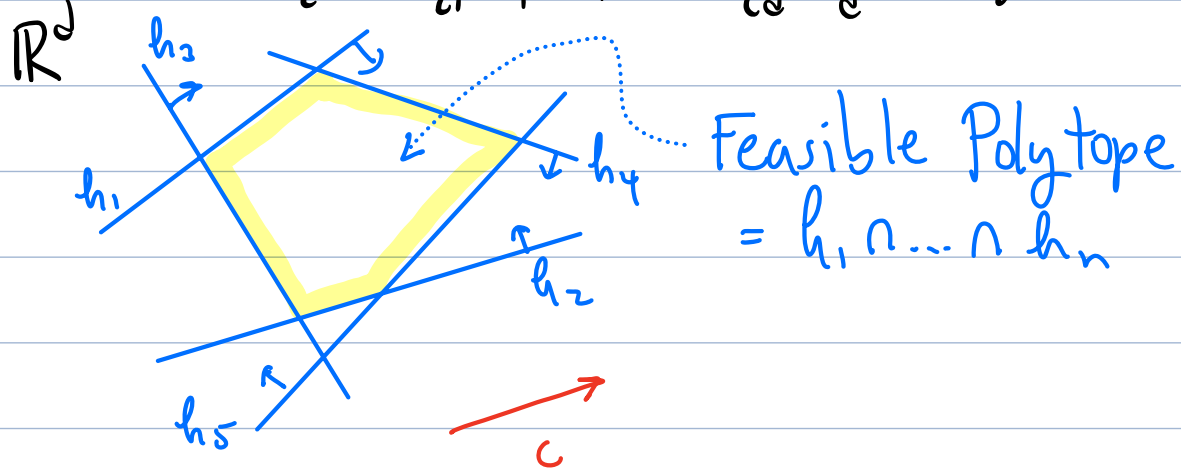
CMSC 754 - Computational Geometry
Lecture 7: Linear Programming

Linear Programming (LP):

- Fundamental optimization problem in $\mathbb{R}^d$
- Given a set of **n linear constraints** (halfspaces) $H = \{h_1, \ldots, h_n\}$

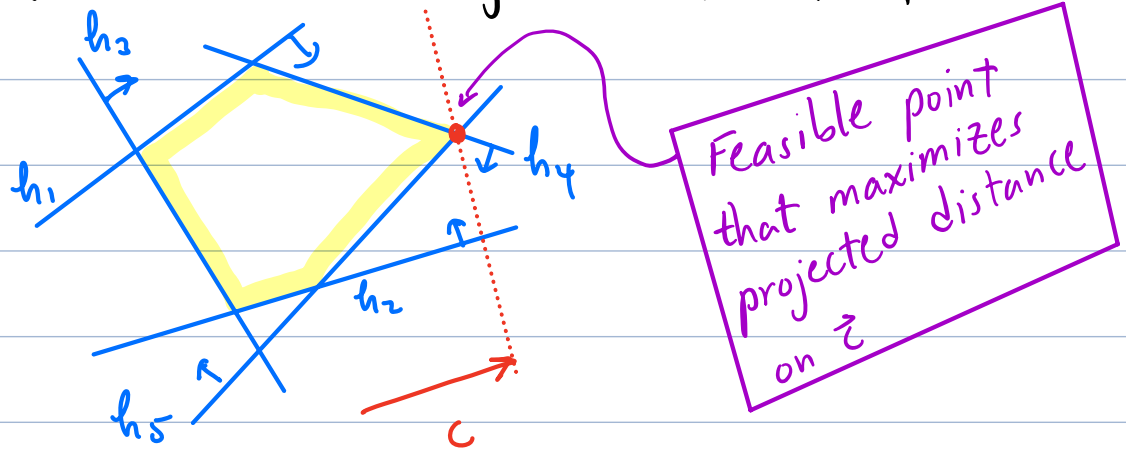
$$h_i : a_{i1}x_1 + \ldots + a_{id}x_d \leq b_i$$



- Given a **linear objective function**

$$f(\bar{x}) = c_1x_1 + \ldots + c_dx_d = c^Tx$$

LP: **Find the vertex of the feasible polytope that maximizes the objective function**



Matrix form:

Given $c \in \mathbb{R}^d$ and $n \times d$ matrix A and $b \in \mathbb{R}^n$
find $x \in \mathbb{R}^d$ to:

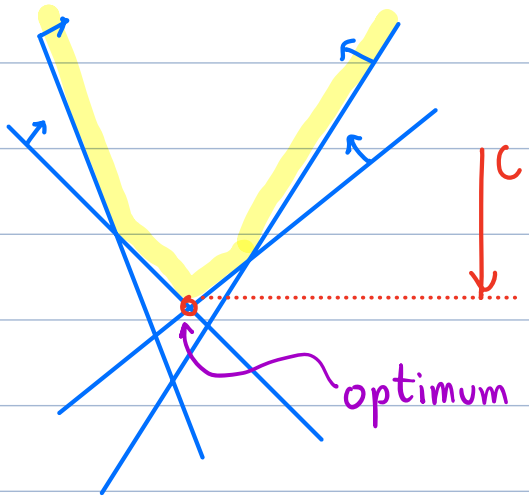
maximize: $c^T x$

subject to: $Ax \leq b$ ← i^{th} row of A corresponds to h_i

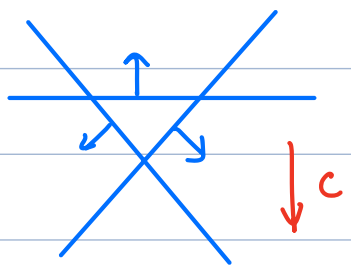
3 Possible Outcomes:

- ☺ **Feasible:** An optimal pt exists (gen'l position: a unique vertex of feasible polytope)
- ☹ **Infeasible:** No solution because feasible polytope is empty
- ☹ **Unbounded:** No (finite) solution because feasible polytope is unbounded in direction of objective fn.

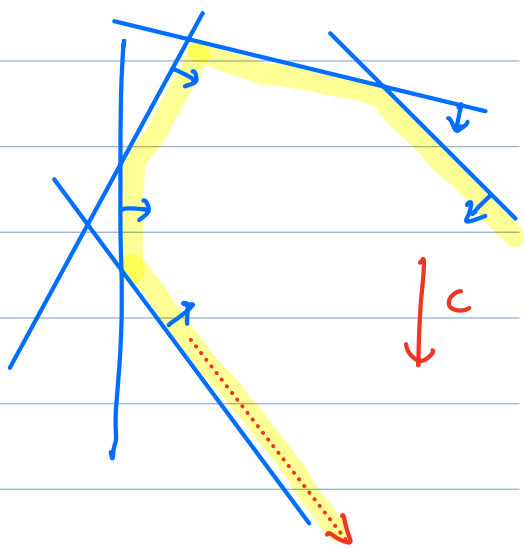
Feasible



Infeasible

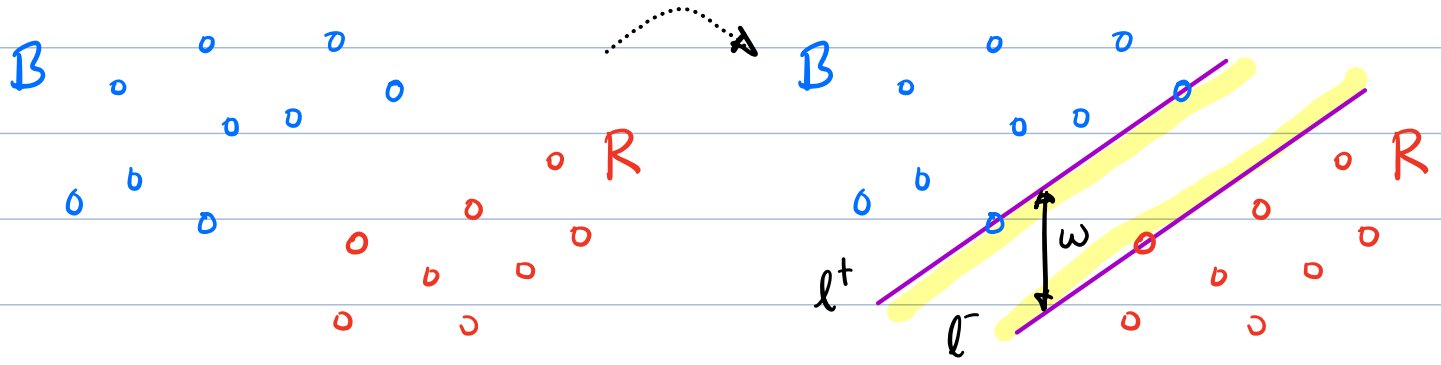


Unbounded



Example:

- Given two point sets $B + R$ in $\mathbb{R}^2$ find lines of max. vertical distance with B above both + R below both



- **Lines:** $\ell^+ : y = e \cdot x + f^+$ $\ell^- : y = e \cdot x + f^-$

- **Constraints:** $\forall p \in B,\ p_y \geq e \cdot p_x + f^+$ (above ℓ^+)
  $\forall p \in R,\ p_y \leq e \cdot p_x + f^-$ (below ℓ^-)

- **Objective:** maximize $\omega = f^+ - f^-$

LP in $\mathbb{R}^3$

Standard form: Find (e, f^+, f^-) ← c

to maximize $f^+ - f^- \equiv (0, 1, -1) \cdot (e, f^+, f^-)$

subject to:

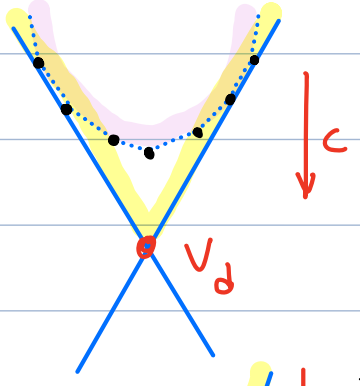
$$p_{ix} \cdot e + 1 \cdot f^+ + 0 \cdot f^- \leq p_{iy},\ \forall p_i \in B$$
$$-p_{jx} \cdot e + 0 \cdot f^+ - 1 \cdot f^- \leq -p_{jy},\ \forall p_j \in R$$

LP in constant-dimensional space

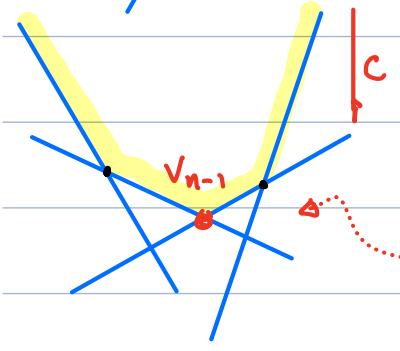
- Assume - n is large
  d is a constant
- We'll present a (randomized) algorithm with (expected) running time $O(d!n) = O(n)$

Incremental Approach:

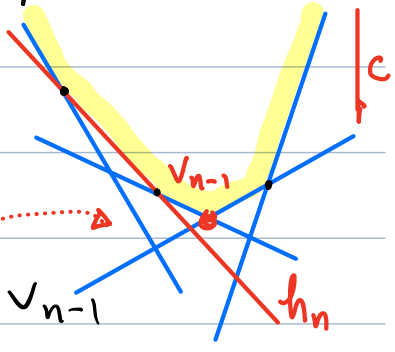
Overview:



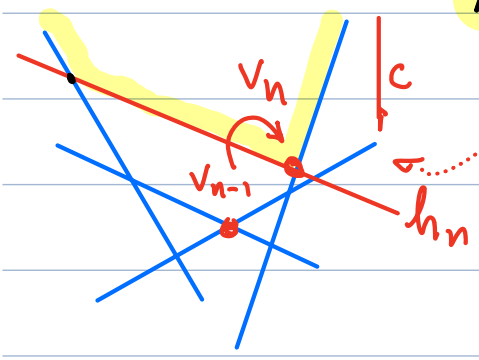
- Find d-halfspaces that define an initial vertex v_d (or report that LP is unbounded)
  → $O(dn)$ time (see our text)
- Remove halfspace h_n and recursively compute LP on $n-1$ halfspaces $h_1 \dots h_{n-1}$



If infeasible → return
else let v_{n-1} be opt

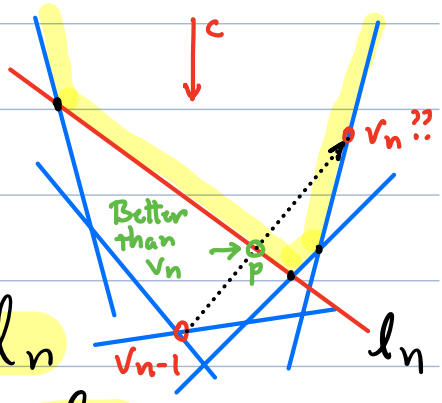


- Add back h_n
  - If ($v_{n-1} \in h_n$) return v_{n-1}
  - else ...



How to update opt. vertex?

Lemma: If $v_{n-1} \notin h_n$ then new opt vertex (v_n) lies on the hyperplane bounding h_n.



Proof: Let l_n be hyperplane bounding h_n. Assume c directed downwards.

v_{n-1} – not feasible $\Rightarrow$ below l_n

v_n – if not on $l_n \Rightarrow$ above l_n

Let $p = l_n \cap \overline{v_{n-1} v_n}$

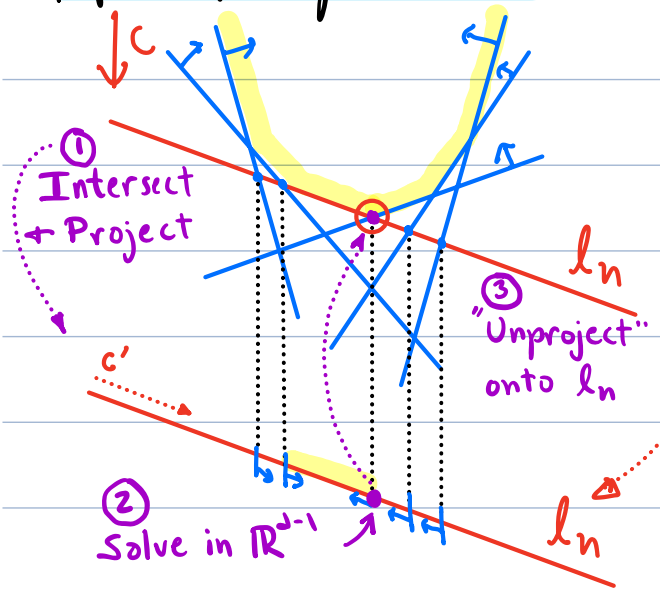
By convexity, $p \in$ feasible polytope

By linearity, obj. function gets progressively worse from $v_{n-1} \to v_n$

$\Rightarrow$ p is better solution than v_n

$\lightning$ contradiction!

How to update?



① Intersect $h_1, \ldots, h_{n-1}$ with l_n + project $\vec{c}$
[Yields an LP in $\mathbb{R}^{d-1}$ with n-1 constraints]

② Solve this (d-1)-dim LP recursively (If $d = 1$, solve by brute force $O(n)$)

③ "Unproject" solution back onto l_n

(See latex notes for details)

Running time? Pretty bad - $O(n^d)$

- Let $W_d(n)$ be worst-case complexity for n halfspaces in dim d
- **Recurrence:**

$$W_d(n) = W_d(n-1) + d + [dn + W_{d-1}(n-1)]$$

- solve LP on $h_1 \dots h_{n-1}$
- Test $v_{n-1} \in h_n$
- Project onto l_n
- Solve LP on l_n

Claim: $W_d(n) = O(n^d)$ ← Too slow!

How to fix this?

Easy! **Randomize the choice of h_n**

Why?

$$W_d(n) = W_d(n-1) + d + dn + W_{d-1}(n-1)$$

- This solves to $O(n)$
- Only applies if $v_{n-1} \notin h_n$
- This rarely happens!

Randomized Incremental Algorithm

Input: $H = \{h_1, \dots, h_n\}$ constraint halfspaces in $\mathbb{R}^d$
$c \in \mathbb{R}^d$ objective vector

Output: Optimum vertex v or error { unbounded or infeasible

(1) If $(d = 1)$ solve LP by brute force - $O(n)$

(2) Find initial subset $\{h_1, \ldots, h_d\}$ that provide initial optimum v_d (or return "unbounded") - $O(d \cdot n)$ (see text)

(3) Randomly select halfspace from $\{h_{d+1}, \ldots, h_n\}$ - call it h_n. Recursively solve LP on remaining $n-1$ halfspaces → Let v_{n-1} be result

(4) If $(v_{n-1} \in h_n)$ return v_{n-1} → $O(d)$

(5) else, project $\{h_1, \ldots, h_{n-1}\} + c$ onto h_n, → $O(dn)$ the bounding hyperplane for h_n.
Solve recursively, letting v_n be result. Return v_n

Expected Case Running Time:

- Running time depends on (random) choice, h_n
- Let $T_d(n)$ be the expected-case running time, over all choices of h_n.
- Let p_n = probability that $v_{n-1} \notin h_n$
- To simplify, assume all halfspaces chosen randomly ($h_1, \ldots, h_d$ aren't)

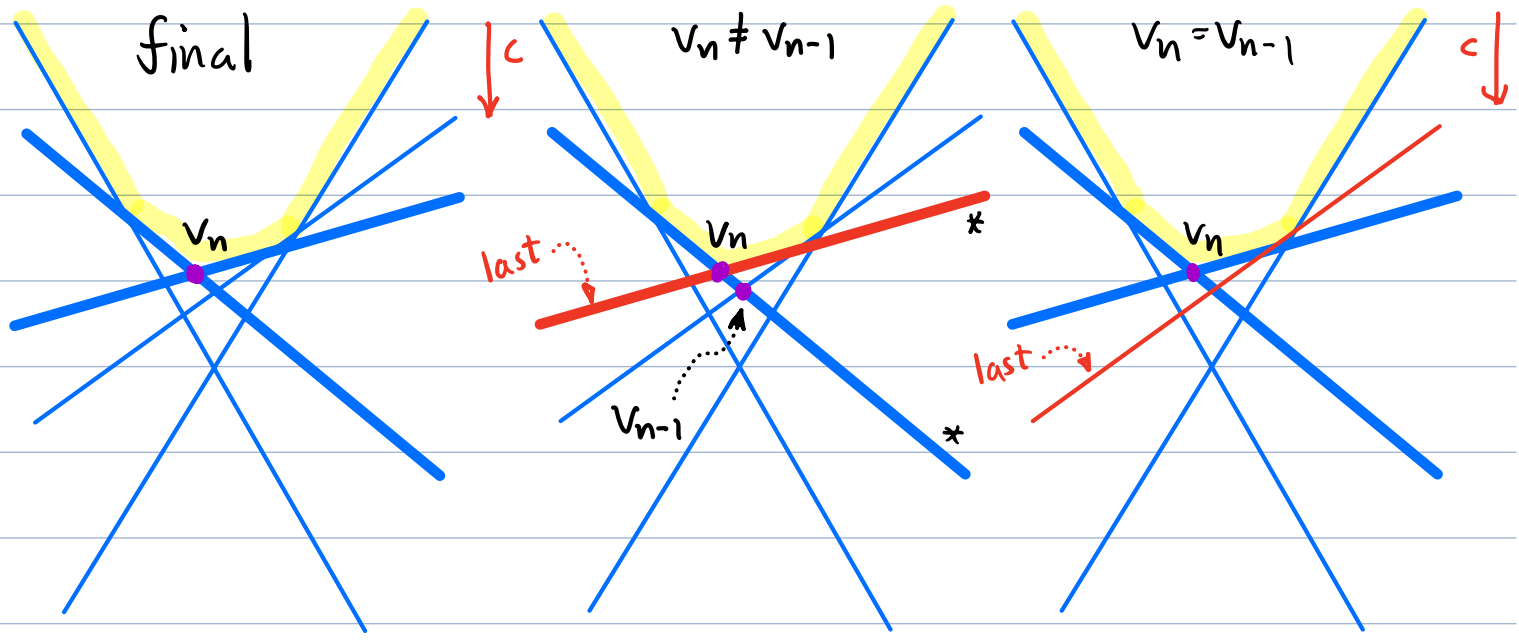
Recurrence:

$$T_d(n) = \begin{cases} 1 & \text{if } n = 1 \\ n & \text{if } d = 1 \\ T_d(n-1) + d + p_n\left(dn + T_{d-1}(n-1)\right) & \text{o.w.} \end{cases}$$

- $T_d(n-1)$: (3) Recursively compute v_{n-1}
- d: (4) test if $v_{n-1} \in h_n$
- p_n: if not
- dn: (5) project $h_1 \dots h_{n-1}$ onto h_n
- $T_{d-1}(n-1)$: (5) solve d-1 dim LP on projections

What is p_n? Backwards Analysis

- Let's consider the final configuration and ask - which halfspace came last and how does its choice affect things?



Obs: The optimum is determined by d halfspaces (assuming gen'l position)

- If h_n is any of these, $v_{n-1} \notin h_n$ + $v_n \neq v_{n-1}$
- Otherwise, $v_{n-1} \in h_n$ + $v_n = v_{n-1}$ 

$\Rightarrow$ $p_n = d/n$ If $n \gg d$, p_n very small + bad case unlikely

Why is it called "backwards"?

- We consider final config. and look backwards to our last random choice

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Lemma: $T_d(n) \leq \gamma_d \, d! \, n$, where γ_d is a constant depending on dimension

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Proof: Induction on $n + d$

simplify

$$T_d(n) = T_d(n-1) + d + p_n \left(dn + T_{d-1}(n) \right)$$

by I.H. + def of p_n

$$\leq \gamma_d \cdot d! \, (n-1) + d + \frac{d}{n} \left(d \cdot n + \gamma_{d-1} (d-1)! \, n \right)$$

$$= \gamma_d \, d! \, (n-1) + d + \left(d^2 + \gamma_{d-1} \, d! \right)$$

$$= \gamma_d \, d! \, n + \left(d + d^2 + \gamma_{d-1} \, d! - \gamma_d \, d! \right)$$

want:

$$\leq \gamma_d \, d! \, n$$

Suffices to select γ_d such that

$$d + d^2 + \gamma_{d-1} \, d! - \gamma_d \, d! \leq 0$$

$$\Leftrightarrow d! \, \gamma_d \geq d + d^2 + \gamma_{d-1} \, d!$$

We can satisfy this by setting:

$$\gamma_1 \leftarrow 1$$

$$\gamma_d \leftarrow \frac{d + d^2}{d!} + \gamma_{d-1}$$

$\Rightarrow \gamma_d$ is a constant depending on dim

□

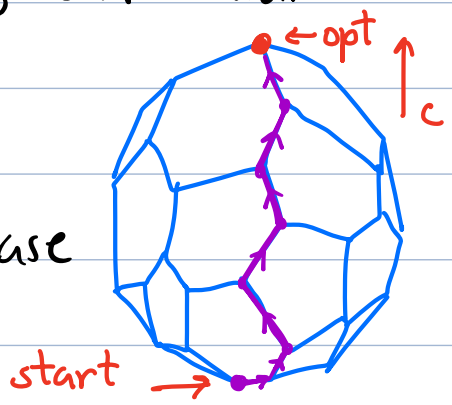
Summary:

- Randomized algorithm for LP
- Expected run time of LP is $O(d! \, n) = O(n)$ (since we assume d is constant)
- Variation depends on random choices, not input
- (Seidel) Prob of running slower extremely small

A Bit of History (Optional)

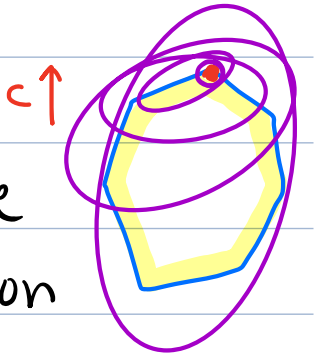
1940's: Used in operations research (Econ, Business)
Kantorovich, Dantzig, von Neuman

Dantzig - **Simplex algorithm** (1947)



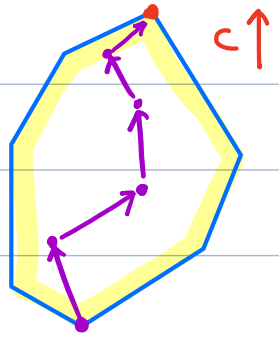
- fast in practice
- exponential in worst case
  - feasible polytope may have $O(n^{\lfloor d/2 \rfloor})$ vertices
- Karp - Not known to be NP-hard

Khachiyan - **Ellipsoid Algorithm** (1979)



- (Weakly) polynomial time
  - ↳ Time depends on precision
- Compute smaller & smaller ellipsoids containing optimum

Karmarkar - **Interior-Point Methods** (1984)



- Move through polytope's interior
- (Weakly) polynomial
- Practical

Open - Is there a poly. time (purely combinatorial) algorithm?