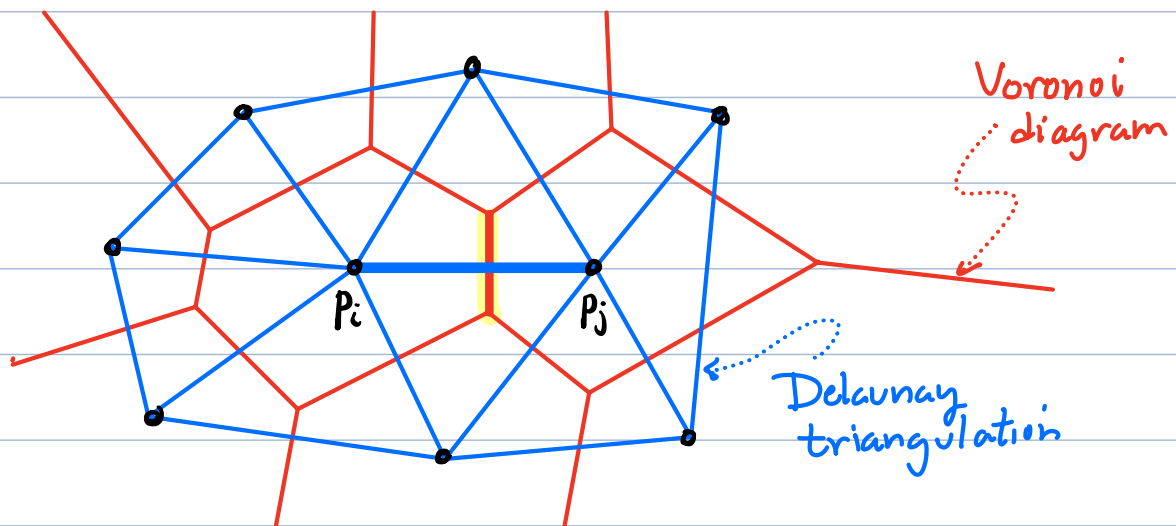


CMSC 754 - Computational Geometry

Lecture II: Delaunay Triangulations (Properties)

Last lecture - Voronoi Diagrams
This - The dual structure - Delaunay Triangulations



Delaunay Triangulation:

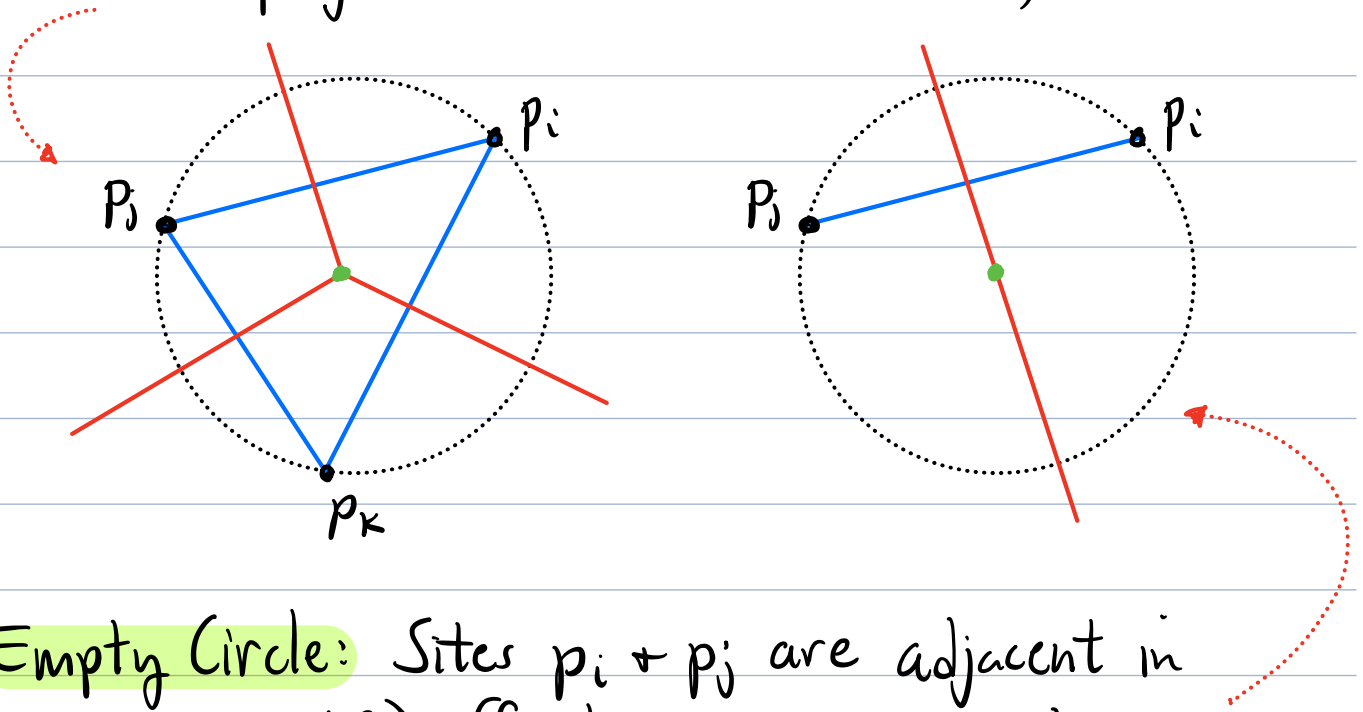
Given a set $P = \{p_1, \dots, p_n\}$ of sites in \mathbb{R}^2 , the **Delaunay Triangulation** is the cell complex whose **vertices are sites** & there is an **edge $\overline{p_i p_j}$** iff **$V(p_i) \cap V(p_j)$** share a common edge. Called **DT(P)**

Properties:

Triangulation: If **general position** (no four sites cocircular), the internal faces are all **triangles**

Hull: The boundary of the external face is the boundary of $\text{conv}(P)$

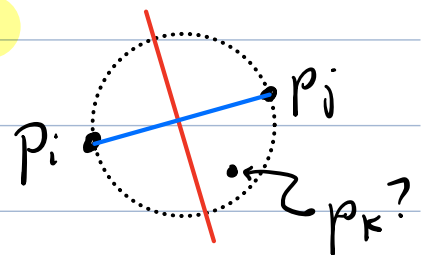
Circumcircle: The circumcircle of any triangle is empty (no sites in its interior)



Empty Circle: Sites p_i + p_j are adjacent in $\text{DT}(P)$ iff there is an empty circle through p_i + p_j .

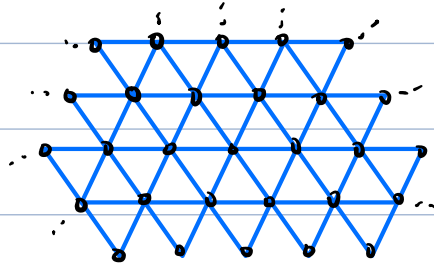
Closest Pair: The **closest pair** of sites are **Delannay neighbors**

- Consider the circle with diameter $\overline{p_i p_j}$.
- No site p_k can lie within
- Apply empty circle prop.



Combinatorial Complexity:

By applying Euler's formula, there are at most $2n$ triangles and at most $3n$ edges



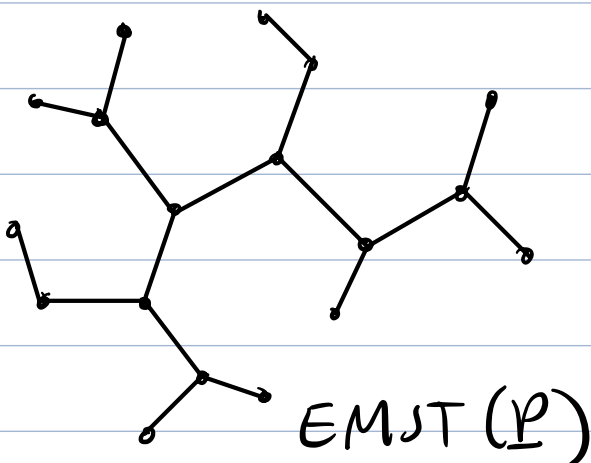
[In \mathbb{R}^d , size is $\mathcal{O}(n^{\lceil d/2 \rceil})$]

Euclidean Minimum Spanning Tree: (EMST)

Euclidean graph: Complete graph on vertex set $P = \{p_1, \dots, p_n\}$, where edge weight is Euclidean distance ($w(p_i, p_j) = \|p_i - p_j\|$)

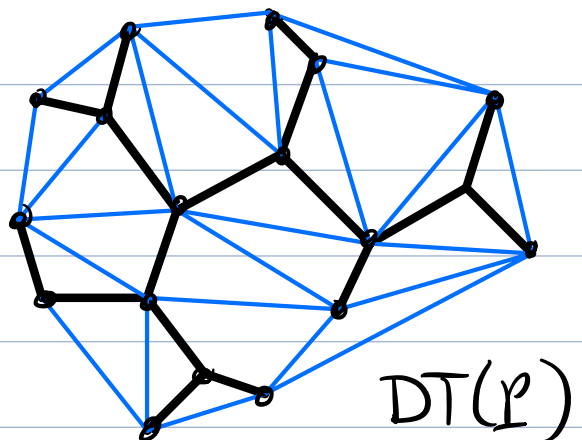
EMST(P) = MST of Euclidean graph
(lowest weight tree spanning P)

Thm: EMST(P) \subseteq DT(P)



EMST(P)

\subseteq



DT(P)

Proof: (Contradiction)

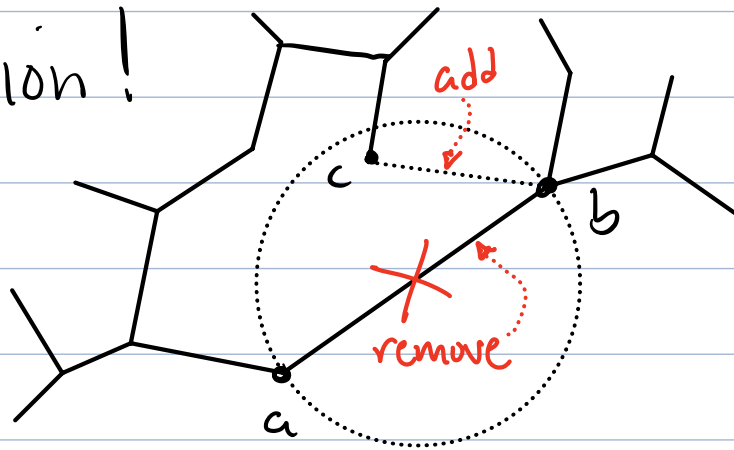
- Suppose some edge $\overline{ab} \in \text{EMST}(P)$
but not in $\text{DT}(P)$

- Empty circle \Rightarrow circle with diameter \overline{ab} contains site c

- $\|ac\| < \|ab\|$
 $\|bc\| < \|ab\|$

- Can remove \overline{ab} from EMST + replace
with either \overline{ac} or \overline{bc} to produce
a spanning tree of lower weight

- Contradiction!



Minimum Weight Triangulation: No!

$\text{MWT}(P)$ = triangulation of P whose
sum of edge lengths is minimum

Generally $\text{MWT}(P) \neq \text{DT}(P)$

Notation: Given graph $G=(V,E)$ and vertices $u,v \in V$, let $d_G(u,v)$ = shortest path distance in G from u to v .

Spanner Properties:

Given a graph G and $t \geq 1$, a t -spanner is a subgraph G' of G on same vertex set s.t. $\forall u,v \in V$,

$$d_{G'}(u,v) \leq t \cdot d_G(u,v)$$

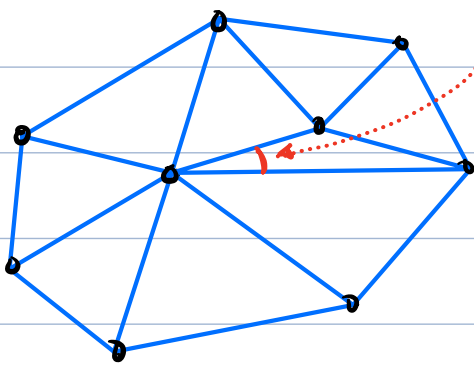
(Path lengths don't stretch too much)

Theorem (Keil + Gutwin, '92) Given a set P of sites in the plane, $DT(P)$ is a $4\pi\sqrt{3}/9 \approx 2.418$ spanner of the Euclidean graph. That is, $\forall p,q \in P$

$$d_{DT(P)}(p,q) \leq \frac{4\pi\sqrt{3}}{9} \cdot \|p-q\|$$

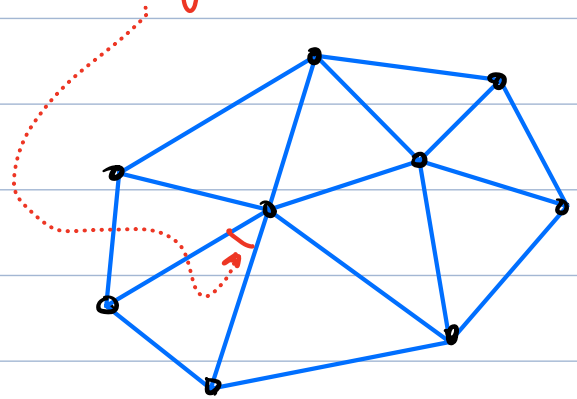
Avoids skinny triangles:

Let P be a set of sites in the plane. Among all possible triangulations of P , $DT(P)$ maximizes the size of the smallest angle.



other triangulation

smallest angle

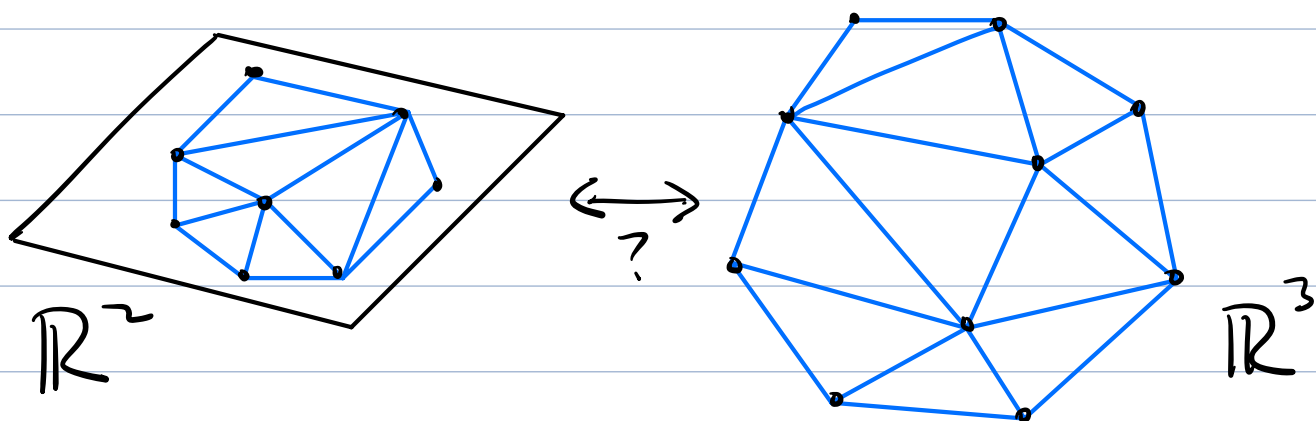


$DT(P)$

Thm: If all angles of all triangulations are ordered small to large, $DT(P)$ is the largest lexicographically compared to all triangulations of P .

(See full lecture notes)

Relationship to polytopes in \mathbb{R}^{d+1}



Delaunay triangulation in \mathbb{R}^d is the projection of a lower convex hull in \mathbb{R}^{d+1}

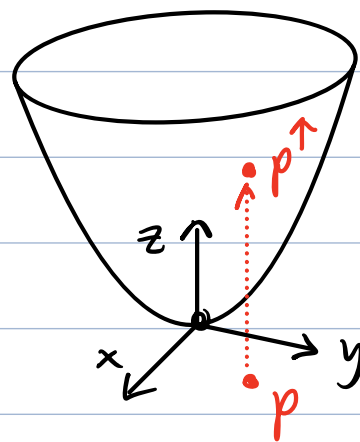
Voronoi diagram in \mathbb{R}^d is the projection of a lower envelope of hyperplanes in \mathbb{R}^{d+1}

→ We'll prove the first only: $\mathbb{R}^2 \rightarrow \mathbb{R}^3$

Consider the paraboloid:

$$z = f(x, y) = x^2 + y^2$$

Given $p = (p_x, p_y)$, define p^\uparrow to be $(p_x, p_y, p_x^2 + p_y^2)$

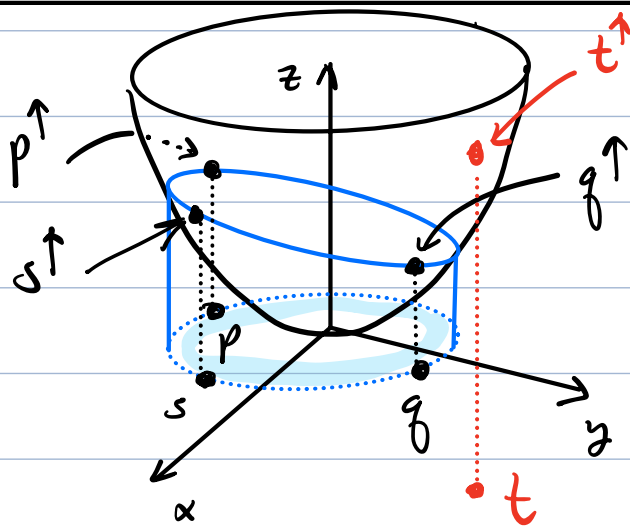
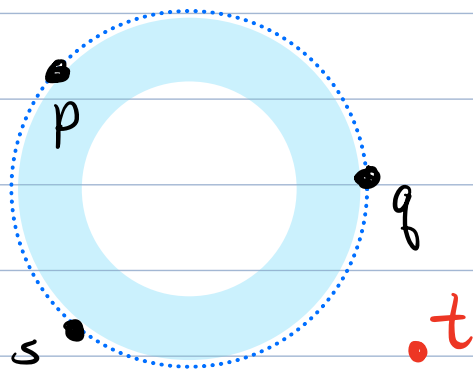


Lemma:

Three pts $p, q, s \in \mathbb{R}^2$
have an empty
circumcircle w.r.t. P



Three pts $p^\uparrow, q^\uparrow, s^\uparrow$
lie on plane h
with all pts of P^\uparrow
above h

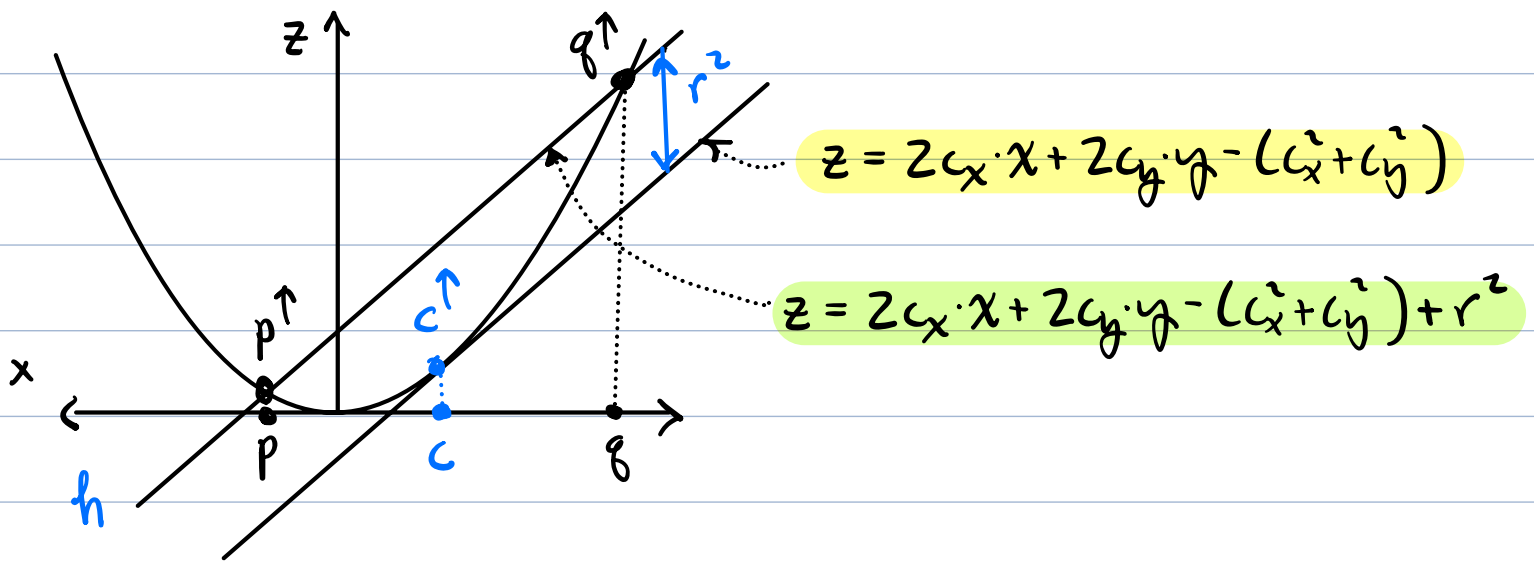


- Let $c = (c_x, c_y)$ be center of circumcircle through p, q, s & let r be its radius
- The plane tangent to paraboloid at c^\uparrow is:

$$z = 2c_x \cdot x + 2c_y \cdot y - (c_x^2 + c_y^2)$$

- Shift this plane up by distance r^2 :

$$h: z = 2c_x \cdot x + 2c_y \cdot y - (c_x^2 + c_y^2) + r^2$$



- All 3 lifted pts lie on this plane:

$$p_x \text{ on circle: } (p_x - c_x)^2 + (p_y - c_y)^2 = r^2$$

$$\Leftrightarrow (p_x^2 - 2p_x c_x + c_x^2) + (p_y^2 - 2p_y c_y + c_y^2) = r^2$$

$$\Leftrightarrow p_x^2 + p_y^2 = 2c_x \cdot p_x + 2c_y \cdot p_y - (c_x^2 + c_y^2) + r^2$$

$$\Leftrightarrow p_z^\uparrow = 2c_x \cdot p_x^\uparrow + 2c_y \cdot p_y^\uparrow - (c_x^2 + c_y^2) + r^2$$

$$\Leftrightarrow p^\uparrow \text{ lies on plane } h$$

- Can also show that t outside/inside circle $\Leftrightarrow t^\uparrow$ above/below h

⇒ Δpqs is Delaunay

⇔ circumcircle pqs is empty

⇔ \exists supporting hyperplane
through $p^\uparrow q^\uparrow s^\uparrow$

⇔ $p^\uparrow q^\uparrow s^\uparrow$ define facet of
convex hull of P^\uparrow

⇒ Delaunay triangulation of P

≡

Convex hull of P^\uparrow

Summary:

Delaunay triangulations

- Dual of Voronoi diagram
- Empty circumcircle condition
- $EMST \subseteq DT$
- Not min weight triangulation
- Maximizes min angle (in \mathbb{R}^2)
- Is a ~ 2.42 spanner
- Convex hull of lifted points