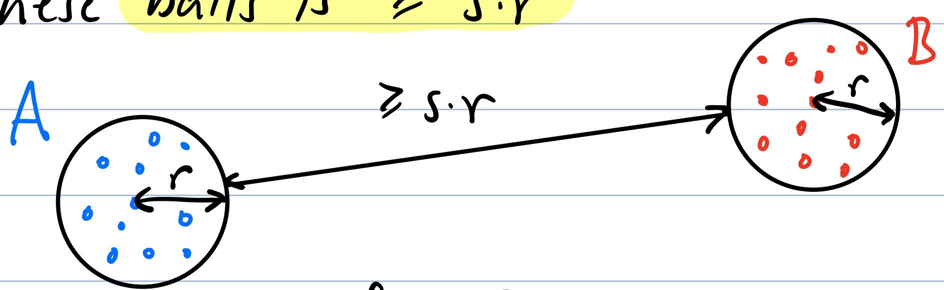
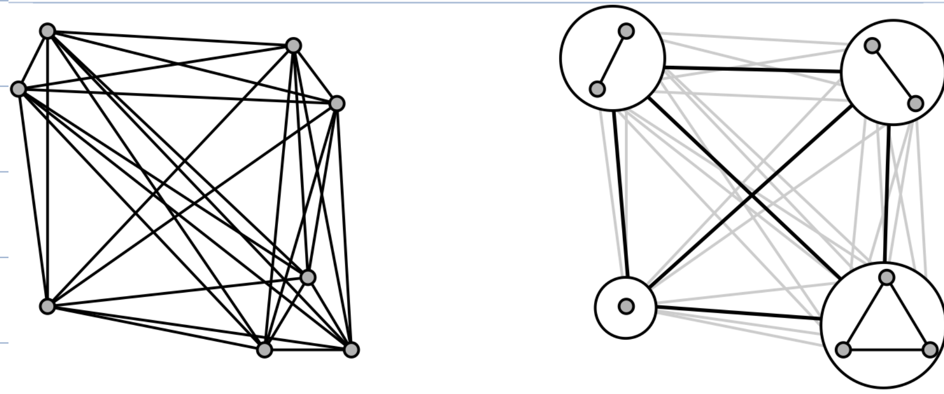
CMSC 754 - Computational Geometry
Lecture 15: Applications of WSPDs

Review of WSPDs:

- Given a point set $P = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^d$ (d a fixed constant) and separation factor $s > 0$, two sets A + B are s-well separated if they can be contained in two balls of some radius r s.t. the distance between these balls is $\geq s \cdot r$

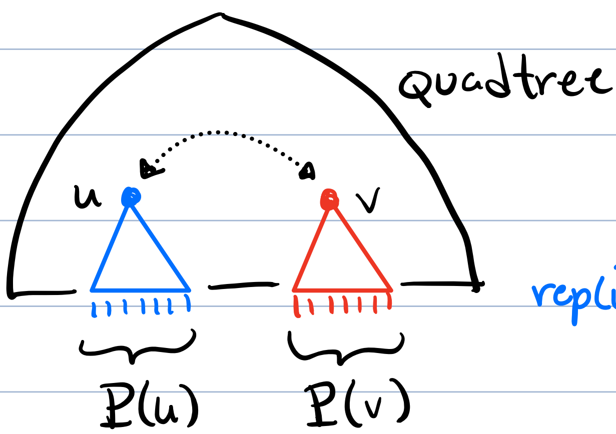


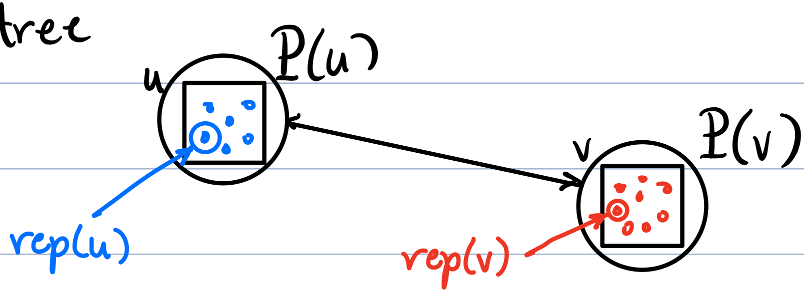
- An s-WSPD for P is a collection: $\{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ such that
  - $A_i, B_i \subseteq P$
  - $A_i \cap B_i = \emptyset$ (disjoint)
  - $\bigcup_i A_i \otimes B_i = P \otimes P$ (cover all pairs)
  - A_i + B_i are s-well separated
- Given P + $s \geq 1$, in time $O(n \log n + s^d \cdot n)$ we can construct an s-WSPD for P of size $O(s^d \cdot n)$.



28 pairs | 11 well-separated pairs

- Construction is based on d-dim quadtree
  - Given nodes u, v in tree let
    - $P(u)$ - points in u's subtree
    - $rep(u)$ - an arbitrary pt of $P(u)$ (u's representative)



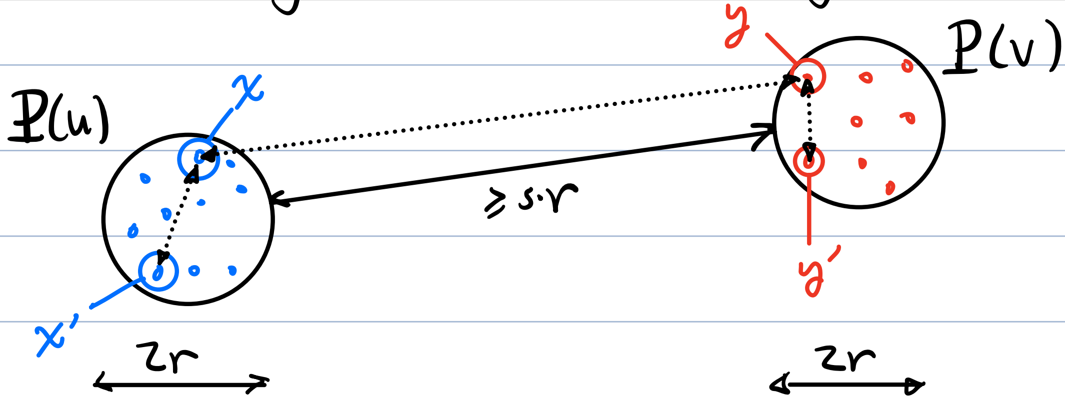


The WSP $\{P(u), P(v)\}$ is represented by the pair $\{u, v\}$

Utility Lemma: Given an s-WSP $\{P(u), P(v)\}$ and $x, x' \in P(u)$ + $y, y' \in P(v)$:

(i) $\|x - x'\| \le 2/s \cdot \|x - y\|$

(ii) $\|x' - y'\| \le (1 + 4/s) \cdot \|x - y\|$



Intuition: (i) Same side closer than cross side

(ii) Cross side dists similar

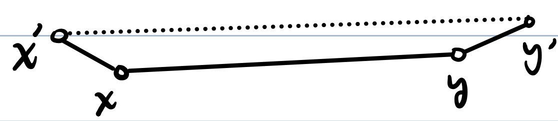
Proof: (i) $\|x - x'\| \le 2 \cdot r$

$$= 2 \cdot r \frac{s \cdot r}{s \cdot r} \le \frac{2 \cdot r}{s \cdot r} \|x - y\|$$

$$= \left(\frac{2}{s}\right) \|x - y\| \quad \checkmark$$

(ii) Observe: $\|x - y\| \ge s \cdot r \Rightarrow 4 \cdot r \le \frac{4}{s} \|x - y\|$

By the triangle inequality:



$$\begin{aligned}
\|x' - y'\| &\le \|x' - x\| + \|x - y\| + \|y - y'\| \\
&\le 2r + \|x - y\| + 2r \\
&\le \|x - y\| + 4r \\
&\le \|x - y\| + 4/s \, \|x - y\| \\
&= (1 + 4/s) \|x - y\| \quad \checkmark
\end{aligned}$$

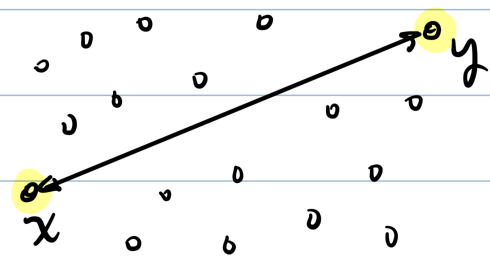
Applications:

- $(1+\varepsilon)$ approx to diameter (farthest pair)
- exact closest pair
- Computing a t-spanner (for any $t>1$)
- $(1+\varepsilon)$ approx to Euclidean MST

$(1+\varepsilon)$ Approx Diameter: in time $O(n \log n + n/\varepsilon^d)$

Given $P = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^d$

$$\operatorname{diam}(P) = \max_{x,y \in P} \|x - y\|$$



Exact:

In $\mathbb{R}^2$: Can compute in $O(n \log n)$
[Convex hull + rotating calipers]

$\mathbb{R}^d$: (Nearly)? quadratic in n

$(1+\varepsilon)$-Approx:

- Set $s = 4/\varepsilon$
- Compute an s-WSPD for P → $O(n \log n + n/\varepsilon^d)$
- for each WSP $\{u, v\}$: → $O(n/\varepsilon^d)$

  $\text{dist}_{u,v} = \|\text{rep}(u) - \text{rep}(v)\|$
- return $\max \text{dist}_{u,v}$ as approx diam

Correctness:

Plan:

① Since reps $\subseteq P$, approx diam $\leq$ diam(P)

② We will show

$*$: $\exists$ WSP u, v s.t.

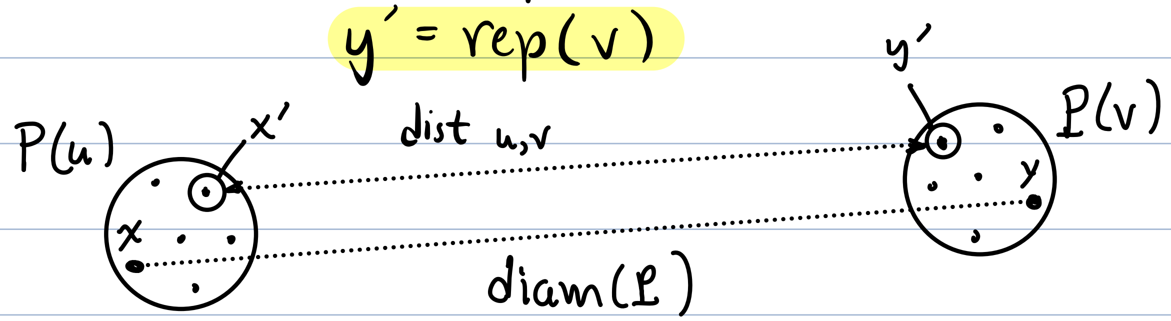
$\text{dist}_{u,v} \geq \text{diam}(P)/(1+\varepsilon)$

$\Rightarrow \max \text{dist}_{u,v} \geq \text{diam}(P)/(1+\varepsilon)$

① + ② $\Rightarrow$ $\dfrac{\text{diam}(P)}{1+\varepsilon} \leq$ approx diam $\leq$ diam(P) ✓

Need to show $*$

- Let x, y be diameter pair
- $\exists$ WSP $\{u, v\}$ s.t. $x \in P(u)$ $y \in P(v)$
- Let $x' = \text{rep}(u)$
  $y' = \text{rep}(v)$



By WSPD utility lemma:

$$\text{diam}(P) = \|x - y\| \leq \left(1 + \frac{4}{s}\right)\|x' - y'\|$$

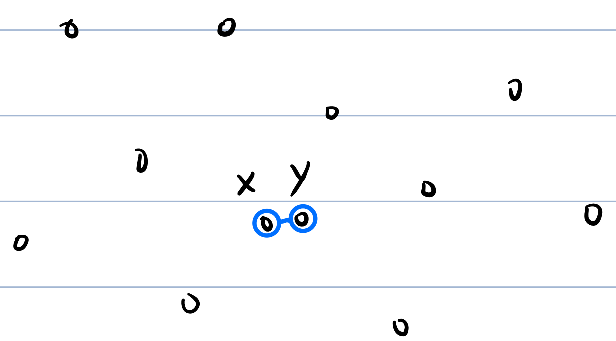
$$= (1+\varepsilon)\|x' - y'\| \qquad (s = 4/\varepsilon)$$

$$= (1+\varepsilon)\, d_{u,v} \Rightarrow \text{dist}_{u,v} \geq \text{diam}(P)/(1+\varepsilon)$$

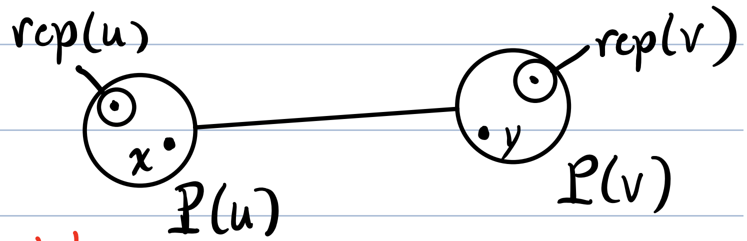
(Exact) Closest Pair: in time $O(n \log n)$

Given $P = \{p_1, \dots, p_n\}$ in $\mathbb{R}^d$ find $x, y \in P$

$$\min_{x,y \in P} \|x - y\|$$



Intuition: Some WSP $\{u,v\}$ must cover the pair $\{x,y\}$



Huh? It looks like x + y not closest!

It must be that $rep(u) = x$ + $rep(v) = y$

Exact Closest Pair:

- Let $s > 2$ (eg. $s = 2.0001$)
- Build s-WSPD for P
- For each WSP $\{u,v\}$
  $dist_{u,v} = \| rep(u) - rep(v) \|$
- return $\min_{u,v} dist_{u,v}$ as closest dist

$O(n \log n + 2^d \cdot n) = O(n \log n)$

Correctness:

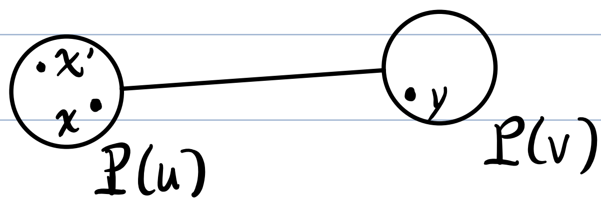
Follows directly from the following lemma:

> **Lemma:** If $s>0$ + x, y are closest pair in P, then any s-WSPD of P contains the pair $\{\{x\},\{y\}\}$

That is, x + y are singletons in WSPD

Proof:

- Suppose not.
- Let $\{u,v\}$ be WSP with $x \in P(u)$, $y \in P(v)$
- May assume w.l.o.g. that $P(u)$ has another pt x'



- By WSPD Utility Lemma:

$$\|x-x'\| \le \frac{2}{s} \cdot \|x-y\|$$

$$< \|x-y\| \qquad (\text{since } s>2)$$

$\Rightarrow$ x, y not closest pair

$\rightarrow$ contradiction

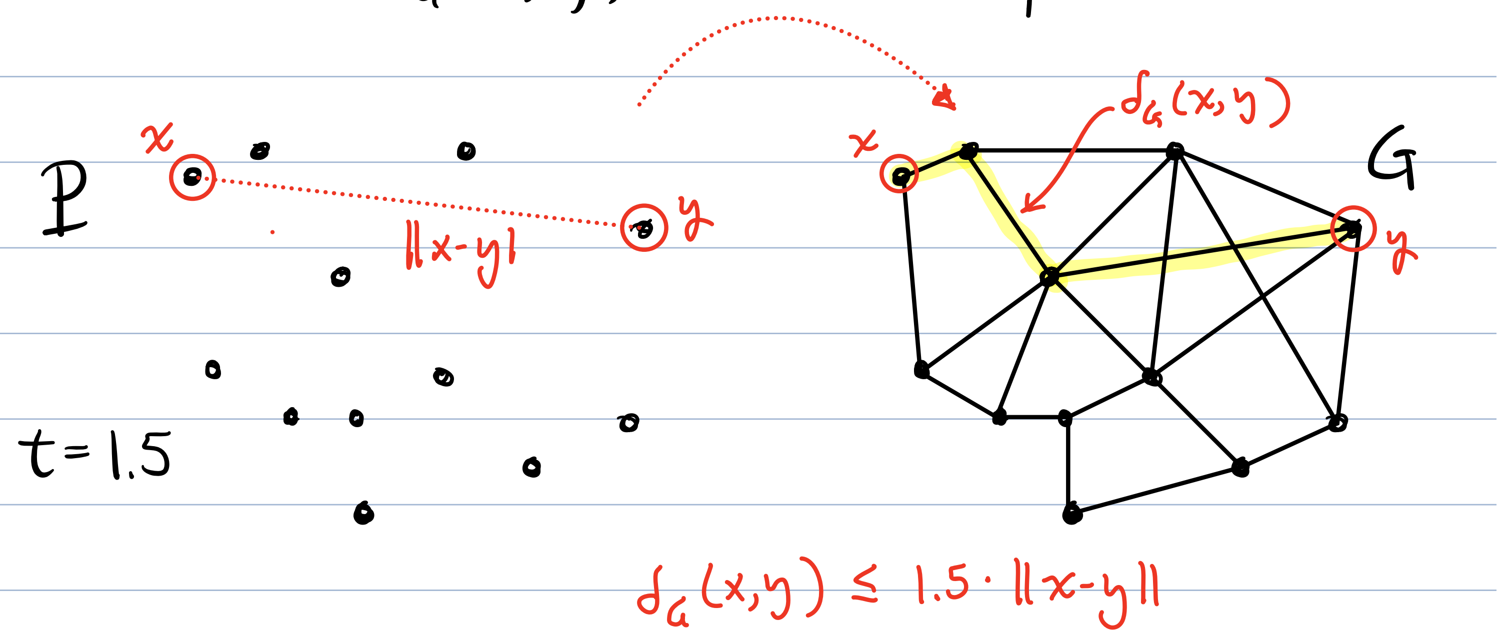
Spanners:

Recall def. of t-spanner (from lect. on Delaunay Tri.)

Given point set $P = \{p_1, \dots, p_n\}$ in $\mathbb{R}^d$ and $t \geq 1$, a t-spanner is a graph on P s.t. $\forall x, y \in P$:

$$\|x-y\| \leq d_G(x,y) \leq t \cdot \|x-y\|$$

where $d_G(x,y)$ is shortest path dist in G



We will show that given $P \subseteq \mathbb{R}^d$ + $t > 1$ can build a $(1+\varepsilon)$-spanner for P in time $O(n \log n + n/\varepsilon^d)$ consisting of $O(n/\varepsilon^d)$ edges

Spanner construction (Given P + $t > 1$)

- Let $s = \frac{4(t+1)}{t-1}$
- $G \leftarrow$ graph with vertex set P + no edges
- Build an s-WSPD for P
- for each WSP $\{u, v\}$:
    - add edge $(rep(u), rep(v))$ to G
- return G

Time: If $t = 1 + \varepsilon$, $s = O(1/\varepsilon)$ $[0 < \varepsilon < 1]$
$\Rightarrow O(n \log n + n/\varepsilon^d)$

Size: $O(n/\varepsilon^d)$ WSPs $\Rightarrow O(n/\varepsilon^d)$ edges

Correctness:

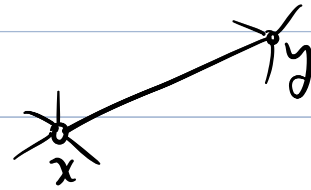
Will show that for all $x, y \in P$

$$\|x-y\| \overset{①}{\leq} \delta_G(x,y) \overset{②}{\leq} t \cdot \|x-y\|$$

① Trivially true since G is a subgraph of complete Euclidean graph

② Rest of the proof...
Induction on num. of edges in path from x to y in G

Basis: Edge (x,y) is in G

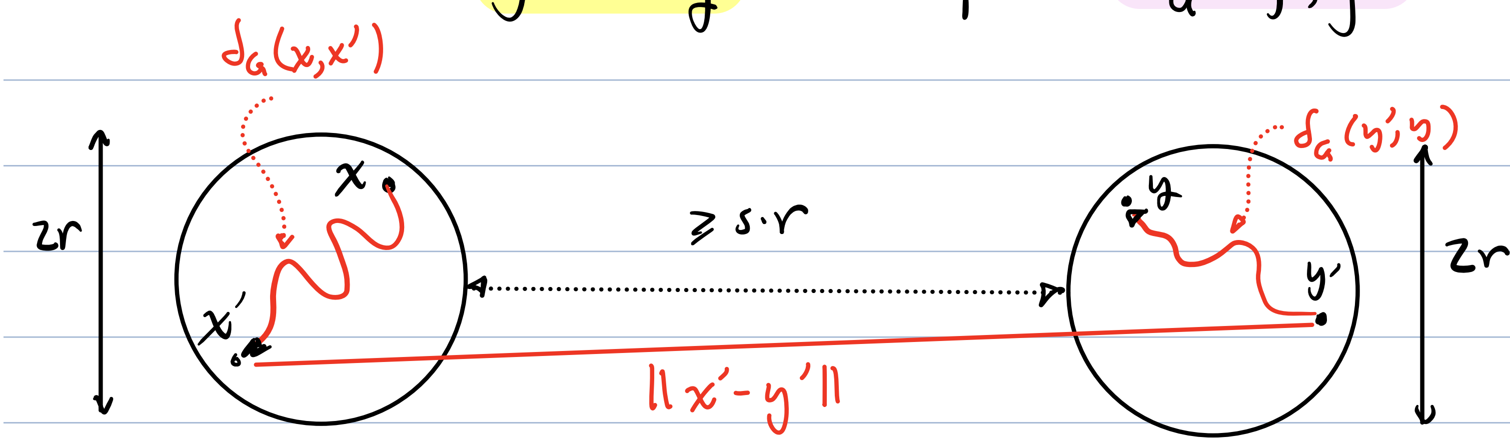


$\Rightarrow \delta_G(x,y) = \|x-y\| \leq t \cdot \|x-y\|$ ✓
(since $t > 1$)

Induction step:

- $\exists$ pair $\{u,v\}$ in WSPD that covers the pair (x,y)
- Let $x' = rep(u)$ $y' = rep(v)$
  (possibly $x'=x$ or $y'=y$)

- To get from x to y in G we can:
  - x to x' $\longrightarrow$ path $\delta_G(x,x')$
  - x' to y' $\longrightarrow$ direct edge: $\|x'-y'\|$
  - y' to y $\longrightarrow$ path $\delta_G(y',y)$



By the induction hyp: $\delta_G(x,x') \leq t \cdot \|x-x'\|$
$\delta_G(y',y) \leq t \cdot \|y'-y\|$

$$\Rightarrow \delta_G(x,y) \leq t \cdot \|x-x'\| + \|x'-y'\| + t \cdot \|y'-y\|$$
$$= t\left(\|x-x'\| + \|y'-y\|\right) + \|x'-y'\|$$

By WSPD Utility Lemma:

- $\|x-x'\| \leq \frac{2}{s}\|x-y\|$
- $\|y'-y\| \leq \frac{2}{s}\|x-y\|$
- $\|x'-y'\| \leq \left(1+\frac{4}{s}\right)\|x-y\|$

$$\Rightarrow \delta_G(x,y) \leq t\left(\frac{2}{s}\|x-y\| + \frac{2}{s}\|x-y\|\right) + \left(1+\frac{4}{s}\right)\|x-y\|$$
$$= \left(t\frac{4}{s} + 1 + \frac{4}{s}\right)\|x-y\|$$
$$= \left(1 + \frac{4(t+1)}{s}\right)\|x-y\|$$
$$= t\|x-y\| \qquad \left(\text{since: } s = \frac{4(t+1)}{t-1}\right)$$

$\square$

To obtain a $(1+\varepsilon)$-spanner, set $t = 1+\varepsilon$ + apply this construction

Approx. to Euclidean MST

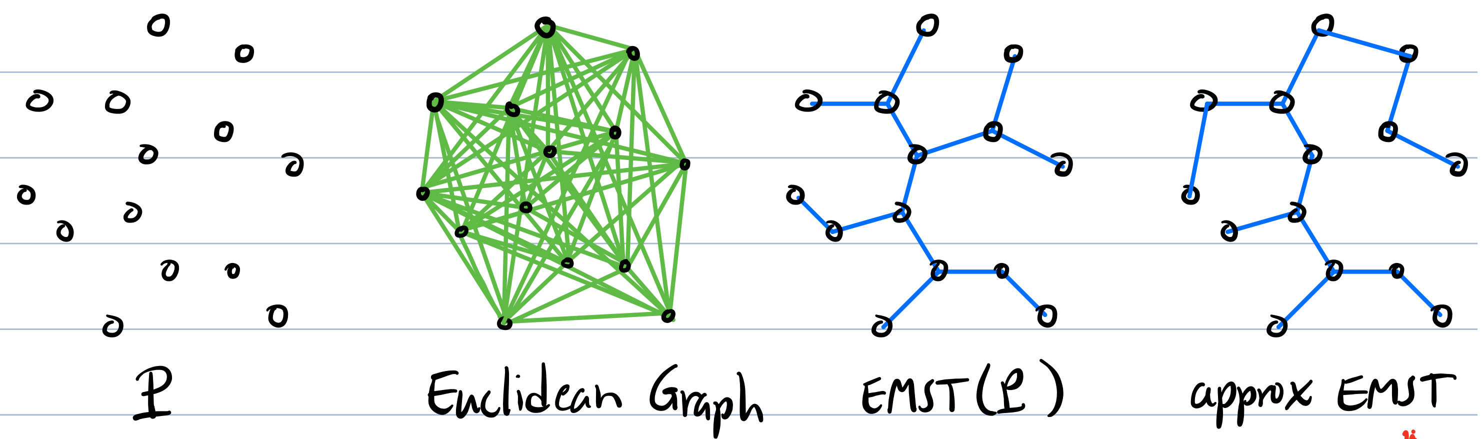
Given a point set $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$

define:

$EMST(P)$ = Min. spanning tree of complete Euclidean graph on P (where $w(u,v) = \|u-v\|$)

Let: $emst(P) = \sum_{(x,y) \in EMST(P)} \|x-y\|$

= total weight of $EMST(P)$



A graph H is an $(1+\varepsilon)$-approx EMST if:

(1) H is a spanning tree for P

(2) $w(H) \leq (1+\varepsilon) \cdot emst(P)$

where $w(H)$ = total weight of H's edges

We'll show how to compute an $(1+\varepsilon)$-approx EMST in time $O(n \log n + n/\varepsilon^d)$

approx-EMST(P, ε)
- $G \leftarrow (1+\varepsilon)$-spanner for P
- return MST(G)

Time: Compute G: $O(n\log n + n/\varepsilon^d)$

Compute MST(G):

- Can compute MST of a graph with v vertices + e edges in time $O(v \log v + e)$
- G has n vertices + n/ε^d edges
- MST(G) takes $O(n\log n + n/\varepsilon^d)$

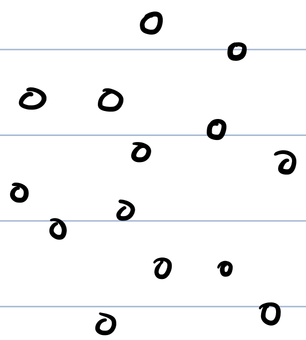
Correctness:

- We'll show that G has a connected subgraph that contains all pts of P ("spans P") and has weight $\leq (1+\varepsilon)\cdot \text{emst}(P)$
- If G has a spanning subgraph H of weight ϖ, then the weight of its MST is no larger

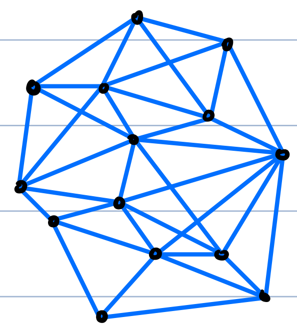
For each $x, y \in P$, let $\pi_G(x,y)$ be shortest path from x to y in G.

Let $\delta_G(x,y)$ be length of this path

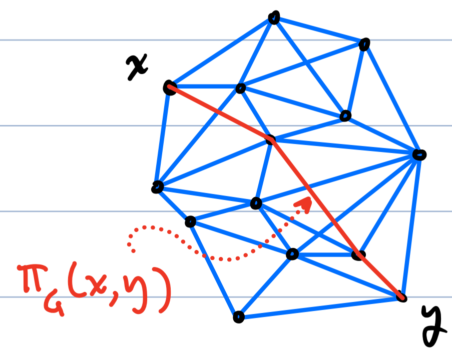
Know that $\delta_G(x,y) \le (1+\varepsilon)\,\|x-y\|$



$\mathcal{P}$



G



G

H:

for each $(x,y) \in EMST(\mathcal{P})$
add the edges of $\pi_G(x,y)$ to H

Obs:

① H is connected and spans all pts of $\mathcal{P}$

② Total weight:

$$\omega(H) \le \sum_{(x,y)\in EMST(\mathcal{P})} \delta_G(x,y)$$

$$\le \sum_{(x,y)\in EMST(\mathcal{P})} (1+\varepsilon)\cdot\|x-y\|$$

$$= (1+\varepsilon) \sum_{(x,y)\in EMST(\mathcal{P})} \|x-y\|$$

$$= (1+\varepsilon)\cdot emst(\mathcal{P})$$

$$① + ② \Rightarrow w(MST(G))$$

$$\leq w(H) \leq (1+\varepsilon) \cdot emst(P)$$

✓

Summary:

- WSPD Utility Lemma
  - A useful tool in WSPD applications
- $(1+\varepsilon)$ approx to diameter (farthest pair)
  - $O(n \log n + n/\varepsilon^d)$ time
- exact closest pair
  - $O(n \log n)$ time
- Computing a $(1+\varepsilon)$-spanner (for any $\varepsilon > 0$)
  - $O(n \log n + n/\varepsilon^d)$ time $O(n/\varepsilon^d)$ space
- $(1+\varepsilon)$ approx to Euclidean MST
  - $O(n \log n + n/\varepsilon^d)$ time