Problem Set #2

Quantum Error Correction
Instructor: Daniel Gottesman

Due Tuesday, Mar. 10, 2026, 11:59 PM

General instructions: No use of LLMs. If you collaborate or use any outside resources, remember to cite them in your solutions.

This problem set has problems on 2 pages.

Problem #1. Low-density parity check CSS codes (15 points)

A classical LDPC ("low density parity check") code is an $[n, k, d]$ linear code where each row of the parity check matrix has at most r 1's and each column of the parity check matrix has at most c 1's, with r and c of constant size (as n gets large). (Sometimes LDPC codes with r and c increasing sublinearly with n are also considered, but assume r and c are constant for the purposes of this problem.) Classical LDPC codes are interesting because they can achieve good values of k/n, d/n, and also generally have good decoding algorithms.

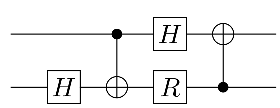
A quantum LDPC code is a stabilizer code for which each generator has low weight and each qubit appears in only a small number of generators. One might try to make good quantum LDPC codes using the CSS construction, based on pairs of classical LDPC codes $C_1(n)$ and $C_2(n)$. Suppose that one finds a family of such codes which produce $[[n, k, d]]$ quantum codes with k/n and d/n both constant as n gets large. Show that this family of quantum codes must be degenerate for large n.

[It took a long time to find such codes in the quantum case. The point of the problem is that, because degeneracy is important to find such codes, the quantum case is not a straightforward application of the CSS construction.]

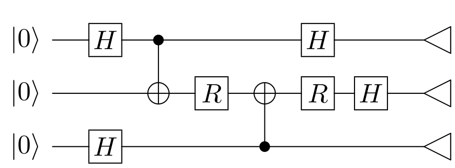
Problem #2. Analyzing Clifford group circuits (25 points)

In the following diagrams, $R = R_{\pi/4}$ is the matrix $\mathrm{diag}(1, i)$ and H is the Hadamard transform.

a) (10 points) For the following Clifford group circuit, compute the overall action on Paulis and use that to write down the 4×4 unitary matrix performed by the circuit:



b) (15 points) For the following Clifford group circuit, use Clifford simulation techniques to compute the full probability distribution of the 8 possible classical outputs after measuring all qubits in the computational basis:



Problem #3. Twirling (20 points)

Let $S(\rho)$ be a quantum operation (a completely positive trace-preserving map) taking n qubits to n qubits. **Hint:** (For both parts) Any $2^n \times 2^n$ matrix can be expanded in the basis of Pauli operators.

a) (10 points) Consider the following quantum operation: Choose a uniformly random $P \in \mathcal{P}_n/\{\pm I, \pm iI\}$ (i.e., a Pauli ignoring global phase). Apply $P^\dagger$, then S, then P (for the same P). Show that, averaging over P, the resulting quantum operation is a Pauli channel. A Pauli channel is any channel $\mathcal{S}$ such that $\mathcal{S}(\rho) = (1 - \sum_P p_P)\rho + \sum_P p_P P \rho P^\dagger$. (Where the sums are over non-trivial Paulis without phases, $\mathcal{P}_n/\{\pm 1, \pm i\} \setminus \{I\}$.)

b) (10 points) Now instead of choosing a random Pauli, choose a random Clifford and do the same thing, i.e., uniformly random $C \in \mathcal{C}_n/\{e^{i\phi} I\}$, apply $C^\dagger$, then S, then C. Show that, averaging over C, the resulting quantum channel is a depolarizing channel. A depolarizing channel is a channel $\mathcal{S}$ such that $\mathcal{S}(\rho) = [1 - (4^n - 1)p]\rho + \sum_P p P \rho P^\dagger$. (Where the sums are over non-trivial Paulis without phases, $\mathcal{P}_n/\{\pm 1, \pm i\} \setminus \{I\}$.) That is, an n-qubit depolarizing channel is a Pauli channel where all probabilities p_P are the same.