CMSC 250
Discrete Structures

Introduction &
Propositional Logic

Knights and Knaves

- There is an island containing two types of people
  - Knights who always tell the truth, and
  - Knaves who always lie
- You visit the island and are approached by two natives who speak to you as follows:
  - A says: B is a knight.
  - B says: A and I are of opposite types.
- What are A and B?

Raymond Smullyan, *What is the Name of This Book?*

Course Content

- Propositional Logic (and circuits)
- Predicate Calculus (quantification)
- Number Theory
- Mathematical Induction
- Counting – combinations and probability
- Functions
- Relations
- Graph Theory

Motivation

- Why learn this material?
  - Some things can be "directly applied"
  - Some things are "good to know"
  - Mathematical maturity
    - A way of thinking and expressing yourself
    - Mathematical basis to understand future analyses
- Overall Theme – Proofs

CMSC 250

- Syllabus
- www.cs.umd.edu/class/summer2007/cmsc250/

- Lecture Section: MTWTh 9:30-10:50 in CSI 2120
- Discussion Section Friday 9:30-10:50 in CSI 2120
- Every Week:
  - 2 Homework assignments
  - 1 Quiz
- 2 Exams + Final – as noted on Syllabus

Statement / Proposition

- Declarative
  - Makes a statement
  - Can be understood to be either true or false in an interpretation
- Symbolized by a letter

- Examples:
  - Today is Wednesday.
  - 5 + 2 = 7
  - 3 * 6 > 18
  - The sky is blue.
  - Why is the sky blue?
  - Brett Favre
  - This sentence is false.

Symbols & Definitions for Compound Statements

- Conjunction
  - AND — symbolized by $\wedge$
- Disjunction
  - OR — symbolized by $\vee$
- Negation
  - NOT — symbolized by **~ / ¬**
- Truth Tables for these operators
  - Alone
  - Combined

p	q	$p \wedge q$	$p \vee q$	~p
1	1	1	1	0
1	0	0	1	0
0	1	0	1	1
0	0	0	0	1

Translation of English to Symbolic Logic Statements

- The sky is blue.
  - One simple (atomic) statement – assign to a letter i.e. b
- The sky is blue and the grass is green.
  - One statement
  - Conjunction of two atomic statements
  - Each single statement gets a letter i.e. b g
  - And join with ^ i.e. b ^ g
- The sky is blue or the sky is purple.
  - One statement
  - Disjunction of two atomic statements
  - Each single statement gets a letter i.e. b p
  - And join with $\vee$ i.e. $b \vee p$

Trickier Translation 1

- The sky is blue or purple.

- Two statements (two concepts)
  - The sky is blue (assign this to b)
  - The sky is purple (assign this to p)
- Still a disjunction
  - The sky is blue or the sky is purple
  - b v p

Trickier Translation 2

- The sky is blue but not dark.

- Two statements
  - The sky is blue   assign this to   b
  - The sky is dark   assign this to   d
- Conjunction with negation
  - The sky is blue **and** the sky is not dark
  - The sky is blue and **it is not the case that** the sky is dark
  - "it is not the case that the sky is dark" is ~d
  - b ^ ~ d

Trickier Translation 3

- $2 \le x \le 6$

- English:
  - x is greater than or equal to 2 and
  - less than or equal to 6
- Two statements:
  - x is greater than or equal to 2 assign this to p
  - x is less than or equal to 6 assign this to q
- Becomes
  - p ^ q

… Continued $2 \leq x \leq 6$

- p is actually a compound statement
  - x is greater than 2 or x is equal to 2 r v s
  - x is greater than 2 is symbolized by r
  - x is equal to 2 is symbolized by s
- q is actually a compound statement
  - x is less than 6 or x is equal to 6 m v n
  - x is less than 6 is symbolized by m
  - x is equal to 6 is symbolized by n
  - p ^ q becomes (r v s) ^ (m v n)

Truth Table Examples

- Happy baby
  - f – Fed
  - d – Needs a new diaper
  - s – Sleeping

- Quick examples:
  - $(p \wedge q) \vee \sim r$
  - $(p \wedge q) \vee (\sim p \wedge \sim q)$
  - $(p \wedge q) \vee \sim(p \vee q)$
  - $(p \vee q) \wedge \sim(p \wedge q)$

More about Operators

- Exclusive or:
  - p, q: p or q but not both
  - $p \oplus q$
  - same as $(p \vee q) \wedge \sim(p \wedge q)$
- Precedence between the operators
  - **~** (NOT) highest precedence
  - **^** (AND) / v (OR) have equal precedence
  - Use parentheses to override default precedence
- $a \wedge b \vee c$

Special Results in the Truth Table

- **Tautological Proposition**
  - A tautology is a statement that can never be false
  - When all of the lines of the truth table have the result "true"
- **Contradictory Proposition**
  - A contradiction is a statement that can never be true
  - When all of the lines of the truth table have the result "false"
- **Logical Equivalence of two propositions**
  - $p \equiv q$
  - Two statements are logically equivalent if they will be true in exactly the same cases and false in exactly the same cases
  - When all of the lines of one column of the truth table have all of the same truth values as the corresponding lines from another column of the truth table

Logical Equivalences Theorem 1.1.1 – Page 14

- Double Negative:
  - $\sim(\sim p) \equiv p$
- Commutative:
  - $p \vee q \equiv q \vee p$, and
  - $p \wedge q \equiv q \wedge p$
- Associative:
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$, and
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive:
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$, and
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

More Logical Equivalences

- Idempotent:
  - $p \wedge p \equiv p$, and
  - $p \vee p \equiv p$
- Absorption:
  - $p \vee (p \wedge q) \equiv p$, and
  - $p \wedge (p \vee q) \equiv p$
- Identity:
  - $p \wedge t \equiv p$, and
  - $p \vee c \equiv p$
- Negation:
  - $p \vee {\sim}p \equiv t$, and
    $p \wedge {\sim}p \equiv c$
- Universal Bound:
  - $p \wedge c \equiv c$, and
  - $p \vee t \equiv t$
- Negations of t and c:
  - ${\sim}t \equiv c$, and
  - ${\sim}c \equiv t$

Simplification Examples

- $(\sim p \vee (\sim q \wedge (z \vee f))) \vee (p \wedge (p \vee q)) \equiv ?$
- $(\sim p \vee (\sim p \wedge q)) \wedge (\sim p \wedge (\sim p \vee q)) \equiv ?$

DeMorgan's Laws

- $\sim(p \vee q) \equiv \sim p \wedge \sim q$
- $\sim(p \wedge q) \equiv \sim p \vee \sim q$

- It is not the case that Pete or Quincy went to the store. ≡ Pete did not go to the store and Quincy did not go to the store.
- It is not the case that both Pete and Quincy went to the store. ≡ Pete did not go to the store or Quincy did not go to the store.

Prove by Truth Tables & by Rules

- $\sim(p \vee \sim q) \vee (\sim q \wedge \sim p) \equiv \sim p$
- $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$
- $(p \vee q) \wedge \sim(p \wedge q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Conditional Statements

- Hypothesis $\rightarrow$ Conclusion
- If this, then that; Hypothesis implies Conclusion
- $\rightarrow$ has lowest precedence ($\sim$ / ^ $\vee$ / $\rightarrow$)
- Examples
  - If it is raining, I will carry my umbrella.
  - If you don't eat your dinner, you will not get desert.

p	q	p $\rightarrow$ q
1	1	1
1	0	0
0	1	1
0	0	1

Converting: $\rightarrow$ to $\vee$

- $p \rightarrow q \equiv \sim p \vee q$
- Show with Truth Table
- $\sim(p \rightarrow q) \equiv p \wedge \sim q$
- Show with Truth Table and Rules

Contrapositive

- $p \rightarrow q \equiv \sim q \rightarrow \sim p$
- Negate both the conclusion and the hypothesis
- Use the negated Conclusion as the new Hypothesis and the negated Hypothesis as the Conclusion

- Example 1
  - If Paula is here, then Quincy is here.
  - If Quincy is not here, then Paula is not here.
- Example 2
  - If I turn in my homework late, I will not get credit.
  - If I get credit for my homework, I turned it in on time.

Converse and Inverse

- $p \rightarrow q$
  - If Paula is here, then Quincy is here.

- **Converse:**
  - $q \rightarrow p$
  - Swap the hypothesis and the conclusion
  - If Quincy is here, then Paula is here.
- **Inverse:**
  - $\sim p \rightarrow \sim q$
  - Negate the hypothesis and negate the conclusion
  - If Paula is not here, then Quincy is not here.

Only If

- Translation to if-then form
  - p only if q
  - p can be true only if q is true
  - if q is not true then p cannot be true
  - if not q then not p ($\sim q \to \sim p$)
  - if p then q ($p \to q$)

- Translation in English
  - You will graduate in CS **only if** you pass this course.
    - G only if P
  - **If** you do **not** pass this course **then** you will **not** graduate in CS.
    - $\sim P \to \sim G$
  - **If** you graduate in CS **then** you passed this course.
    - $G \to P$

Biconditional

- p if and only if q
- $p \leftrightarrow q$
- p iff q

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Other English Words for Implication

- **Sufficient Condition**
  - "if r, then s" $r \rightarrow s$
  - The truth of r is sufficient to ensure the truth of s
  - Means r is a **sufficient** condition for s
- **Necessary Condition**
  - Equivalent to "if not r, then not s" $\sim r \rightarrow \sim s$
  - If r does not occur, then s cannot occur either
  - The truth of r is necessary if s is true
  - Means r is a **necessary** condition for s
- **Sufficient and Necessary Condition**
  - r if, and only if s $r \leftrightarrow s$
  - The truth of r is enough to ensure the truth of s and vice versa

Argument

- A sequence of statements where
  - The **last in the sequence** is the **Conclusion**
  - All **others** are **Premises** (Assumptions, Hypotheses)
  - (premise1 ^ premise2 ^ …premise*N*) $\rightarrow$ conclusion

- Critical rows of the truth table
  - Where all of the premises are true
    - Only one premise being false makes the conjunction false
    - A false hypothesis on a conditional can never make the whole false
- The truth value of the conclusion in the critical rows
  - **Valid Argument** If and only if all Critical rows have true conclusion
  - **Invalid Argument** If any single Critical row has a false conclusion

Rules of Inference (Table 1.3.1- Page 39)

Modus Ponens $p \to q$ p $\therefore q$	**Modus Tollens** $p \to q$ $\sim q$ $\therefore \sim p$	Disjunctive Syllogism $p \vee q$ | $p \vee q$ $\sim q$ | $\sim p$ $\therefore p$ | $\therefore q$
Disjunctive Addition p | q $\therefore p \vee q$ | $\therefore p \vee q$	Conjunctive Simplification $p \wedge q$ | $p \wedge q$ $\therefore p$ | $\therefore q$	Rule of Contradiction $\sim p \to c$ $\therefore p$
Hypothetical Syllogism $p \to q$ $q \to r$ $\therefore p \to r$	Conjunctive Addition p q $\therefore p \wedge q$	Dilemma $p \vee q$ $p \to r$ $q \to r$ $\therefore r$

Proofs Using Rules of Inference

P1	$p \vee q$
P2	$q \rightarrow r$
P3	$\sim p$
$\therefore$	r

P1	$p \wedge q$
P2	$p \rightarrow s$
P3	$\sim r \rightarrow \sim q$
$\therefore$	$s \wedge r$

P1	$p \vee q$
P2	$\sim(q \vee r)$
P3	$p \rightarrow (m \rightarrow r)$
$\therefore$	$\sim m$

Conditional Worlds

- Making assumptions – only allowed if you go into a "conditional world"

List of statements that are true in all worlds

```
|-------------
|  Assume anything
|  List of statements true in the worlds
|       where the assumption is true
| ---------------
```

Assumption → anything from the conditional world

Conditional World Assumption Leads to Contradiction

- Make an assumption, but that assumption leads to a contradiction in the conditional world.

  List of statements that are true in all worlds
  |-------------
  | Assume anything
  | List of statements true in the worlds
  | where the assumption is true
  | A contradiction with something else known true
  | ---------------
  Assumption must be false in all possible worlds

Prove Using "Conditional World Method"

P1	$(p \vee q) \rightarrow s$
P2	$r \rightarrow p$
$\therefore$	$r \rightarrow s$

P1	$m \rightarrow s$
P2	$s \rightarrow (q \wedge r)$
P3	$q \rightarrow \sim r$
$\therefore$	$\sim(m \wedge p)$

Use both conditional world methods

P1	$\sim m \vee p$
P2	$p \rightarrow (q \vee s)$
P3	$\sim(s \vee \sim x)$
P3	$q \rightarrow \sim r$
$\therefore$	$\sim(m \wedge r)$

Knights and Knaves

- There is an island containing two types of people
  - Knights who always tell the truth, and
  - Knaves who always lie
- You visit the island and are approached by two natives who speak to you as follows:
  - A says: B is a knight.
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