CMSC 250
Discrete Structures

Relations

Relations

- The most basic relation is "=" (e.g. $x = y$)
- Generally $x\ R\ y \rightarrow$ TRUE or FALSE
  - $R(x,y)$ is a more generic representation
  - R is a binary relation between elements of some set A to some set B, where $x \in A$ and $y \in B$

Relations

- Binary relations: xRy
  On sets $x \in X$ $y \in Y$ $R \subseteq X \times Y$
- Example:
  "less than" relation from A={0,1,2} to B={1,2,3}

<u>Use traditional notation</u>

$0 < 1$, $0 < 2$, $0 < 3$, $1 < 2$, $1 < 3$, $2 < 3$

$1 \geq 1$, $2 \geq 1$, $2 \geq 2$

<u>Or use set notation</u>

$A \times B$={(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)}

R={(0,1),(0,2),(0,3), (1,2),(1,3), (2,3)}

<u>Or use Arrow Diagrams</u>

Formal Definition

- (Binary) relation from A to B

  where $x \in A$, $y \in B$, $(x,y) \in A \times B$ and $R \subseteq A \times B$

  $xRy \leftrightarrow (x,y) \in R$
- Finite example: $A=\{1,2\}$, $B=\{1,2,3\}$
- Infinite example: $A = Z$ and $B = Z$

  $aRb \leftrightarrow a-b \in Z^{even}$

Example

- Let $A = \{2,3\}$, $B = \{1,3,6\}$

- Define a relation R from A to B such that:

  $xRy \leftrightarrow x - y$ is odd
- How could this explicitly be represented as tuples?
  - $R = \{(2,2),(2,3),(3,6)\}$
- What if A and B were the set of all integers?
  - $R = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid \exists k \in \mathbf{Z}$ such that $x - y = 2k + 1\}$

Properties of Relations

- Reflexive

$$R \text{ is Reflexive} \leftrightarrow \ \forall x \in A,\ xRx$$

- Symmetric

$$R \text{ is Symmetric} \leftrightarrow \ \forall x, y \in A,\ xRy \rightarrow yRx$$

- Transitive

$$R \text{ is Transitive} \leftrightarrow \ \forall x, y, z \in A,\ xRy \wedge yRz \rightarrow xRz$$

Example

- Define a relation of A called R
  - $A = \{2,3,4,5,6,7,8\}$
  - $R = \{(4,4),(4,7),(7,4),(7,7),(2,2),(3,3),(3,6),(3,9),(6,6),(6,3),(6,9),(9,9),(9,3),(9,6)\}$

- Draw the arrow diagram
- Is R
    - Reflexive?
    - Symmetric?
    - Transitive?
- Could this arrow diagram represent a function?

Example

- A = {0,1,2,3}
- R over A = {(0,0),(0,1),(0,3),(1,0), (1,1),(2,3),(3,0),(3,3)}
- Draw the arrow diagram for it
- Is R
  - Reflexive?
  - Symmetric?
  - Transitive?

Proving Properties on Infinite Sets

- $m,n \in Z, \quad m \equiv_3 n$

- Reflexive
- Symmetric
- Transitive

Proving Properties on Infinite Sets

- Define R(x,y) R: $\mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ to be
  $\{(x,y) \in \mathbf{Z}^+ \times \mathbf{Z}^+ \mid x|y\}$

- Prove whether or not this is:
  - Reflexive?
  - Symmetric?
  - Transitive?

The Inverse of the Relations and the Complement of the Relation

- R: A$\to$B R $= \{(x,y) \in A \times B \mid xRy\}$
- R^{-1}:B$\to$A $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}$
- R': A$\to$B R' $= \{(x,y) \in A \times B \mid (x,y) \notin R\}$

$$\forall x \in X \forall y \in Y, (y,x) \in R^{-1} \leftrightarrow (x,y) \in R$$

$$\forall x \in X \forall y \in Y, (x,y) \in R' \leftrightarrow (x,y) \notin R$$

- D: $\{1,2\} \to \{2,3,4\}$

  D=$\{(1,2),(2,3),(2,4)\}$ D' = ? D^{-1}=?
- S = $\{(x,y) \in R \times R \mid y = 2*|x|\}$ S^{-1}=?

Closures over the Properties

- Reflexive Closure
- Symmetric Closure
- Transitive Closure

✓R^{xc} has property x

✓$R \subseteq R^{xc}$

✓R^{xc} is the minimal addition to R

(if S is any other transitive relation that contains R, $R^{xc} \subseteq S$)

Example

- Given the relation $R = \{(0,1),(1,2),(2,3)\}$

- What is R^{RC}?
- What is R^{SC}?
- What is R^{TC}?

Matrix Representation of a Relation

- $M_R = [m_{ij}]$
  - m_{ij}={1 iff $(i,j) \in R$ and 0 iff $(i,j) \notin R$}
- Example:
  - R : {1,2,3} $\rightarrow${1,2}   R = {(2,1),(3,1),(3,2)}

$$M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

Union, Intersection, Difference and Composition

- R: A→B and S: A→B

$$R \bigcup S = \{(x, y) \in A \times B \mid (x, y) \in R \vee (x, y) \in S\}$$
$$R \bigcap S = \{(x, y) \in A \times B \mid (x, y) \in R \wedge (x, y) \in S\}$$
$$R - S = \{(x, y) \in A \times B \mid (x, y) \in R \wedge (x, y) \notin S\}$$
$$S - R = \{(x, y) \in A \times B \mid (x, y) \in S \wedge (x, y) \notin R\}$$

- R: A→B and S: B→C

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B, (a, b) \in R \wedge (b, c) \in S\}$$

Example

- Let ID = set of student IDs
- Let Class = set of classes offered
- Define relation Summer2007

  $\{(x,y) \in ID \times Class \mid \text{student } x \text{ is registered for class } y\}$

- Can we do this?
- Relational databases …

Binary Relation Induced by a Partition

- Let A1,A2,…,An be a partition of A
- Let relation R: $A \times A$ be defined as:

  $\{(x,y) \in A \times A \mid \exists i\ 1 \leq i \leq n\ x \in A_i \wedge y \in A_i\}$
- $A = \{1,2,3,4,5,6,7\}$
- $A_1 = \{1,2,3\}$; $A_2 = \{4,5\}$; $A_3 = \{6,7\}$
- R = {
  (1,1),(2,2),(3,3),(1,2),(1,3),(2,3),(2,1),(3,2),(3,1),
  (4,4),(4,5),(5,4),(5,5),(6,6),(6,7),(7,6),(7,7)}
- Note that this (or any partition-induced relation) is reflexive, symmetric, and transitive

Equivalence Relations

- Any binary relation that is:
  - Reflexive
  - Symmetric
  - Transitive

- Partitions are called Equivalence Classes
  - [a] = equivalence class containing a
  - $[a] = \{x \in A \mid xRa\}$

Examples

- R: $X \to X$ $X=\{a,b,c,d,e,f\}$
- $\{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),$
  $(a,e),(a,d),(d,a)(d,e),(e,a),(e,d),(b,f),(f,b)\}$
- Lemma 10.3.3 If A is a set and R is an equivalence relation on A and x and y are elements of A, then either

$$[x]\cap[y] = \emptyset \quad \text{or} \quad [x]=[y]$$

Example

- $R(x,y)$; $R: \mathbf{Z} \to \mathbf{Z}$

  $\{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid 3 | (x - y)\}$
- How many equivalence classes are there?
- $[a] = \{x \in \mathbf{Z} \mid R(x,a)\}$

  $= \{x \in \mathbf{Z} \mid \exists n \in \mathbf{Z} \text{ such that } x = 3n + a\}$
- Classes are: [0], [1], [2]

Other Properties

Reflexive $\forall a \in A, aRa$

Irreflexive $\forall a \in A, a \not{R} a$

Symmetric $\forall a, b \in A, aRb \rightarrow bRa$

Antisymmetric $\forall a, b \in A, aRb \wedge bRa \rightarrow a = b$

Asymmetric $\forall a, b \in A, aRb \rightarrow b \not{R} a$

Non-symmetric

$$\forall a, b \in A, a \neq b \rightarrow (aRb \leftrightarrow b \not{R} a)$$

Transitive

$$\forall a, b, c \in A, aRb \wedge bRc \rightarrow aRc$$

Example

- Let R_1 be the divides relation on $\mathbf{Z}^+$
- Let R_2 be the divides relation on $\mathbf{Z}$

- Is R_1 antisymmetric? Prove or give counterexample.
  - $a\ R_1\ b$ and $b\ R_1\ a \rightarrow a = b$
  - True
- Is R_2 antisymmetric? Prove or give counterexample.
  - Counterexample ($a = 2$, $b = -2$)

Partial Order Relation

- R is a Partial Order Relation if and only if
  - R is Reflexive, Antisymmetric and Transitive
- Partial Order Set (POSET)

  (S,R) = R is a partial order relation on set S
- Examples
  - $(Z, \leq)$
  - $(Z^+, |)$ {note: | symbolizes divides}
  - $(S, \subseteq)$ {note: S indicates and set}

Total Ordering

- When all pairs from the set are "comparable" it is called a Total Ordering

- a and b are <u>comparable</u>
  - If and only if a R b or b R a

- a and b are <u>non-comparable</u>
  - If and only if a $\not R$ b and b $\not R$ a

Hasse Diagram

1. Take the digraph
   (since it represents the same relation)
2. Arrange verticies so all arrows go upward
   (since it is antisymmetric we know this is possible)
3. Remove the reflexive loops

   (since we know it is reflexive these are not necessary)
4. Remove the transitive arrows

   (since we know it is transitive, these are not necessary)
5. Make the remaining edges non-directed

   (since we know they are all going upward, the direction is not necessary)

Hasse Diagram Example

- The POSET $(\{1,2,3,9,18\},|)$
- The POSET $(\{1,2,3,4\},\leq)$
- The POSET $(P\{a,b,c\},\subseteq)$

- Draw complete digraph diagrams of these relations
- Derive Hasse Diagram from those

Terminology

- Maximal : $a \in A$ is maximal
  $\leftrightarrow \forall b \in A$ ($bRa \vee$ a and b are not comparable)
- Minimal : $a \in A$ is minimal
  $\leftrightarrow \forall b \in A$ ($aRb \vee$ a and b are not comparable)
- Greatest : $a \in A$ is greatest $\leftrightarrow \forall b \in A \ (bRa)$
- Least : $a \in A$ is least $\leftrightarrow \forall b \in A \ (aRb)$

Topological Sorting

1. Select any minimal
   a) put it into the list
   b) remove it from the Hasse diagram)
2. Repeat until all members are in the list

Example: $(\{2,3,4,6,18,24\},|)$