CMSC 250
Discrete Structures

Final Exam Review

Propositional Logic

- Statements/propositions
- Operations
- Translation of symbolic statements
- Truth tables
- Properties, laws, rules
  - Commutative, associative, distributive …
  - DeMorgan's, idempotent, absorption, identity …
- Conditional statements (including biconditional)
- Contrapositive, converse, inverse
- Proofs – truth tables, inference rules and conditional worlds
- Circuits

Predicate Calculus

- Notation, quantifiers (single/multiple), etc.
- Translation
  - Informal to formal
  - Formal to informal
- Euler diagrams
- Inference rules
  - Universal instantiation, existential generalization
  - Universal modus ponens and modus tolens
- Converse/inverse error
- Direct proofs

Number Theory

- What proofs must have (next slide)
- Domains (Z,Q,R)
- Closure of operations (for Z)
- Definitions – even, odd, prime, composite
- Constructive proofs of existence
  - $\exists x \in$ **D**, such that $Q(x)$
- Proving universal statements
  - Exhaustion, general particular
- Divisibility, mod, congruence, etc. (Quotient Remainder Theorem)
- Proof by contradiction
- Unique factorization theorem
- Sqrt(2)$\notin$ Q
- Floor/ceiling operations

Proofs Must Have!

- Clear statement of what you are proving
- Clear indication you are starting the proof
- Clear indication of flow
- Clear indication of reason for each step
- Careful notation, completeness and order
- Clear indication of the conclusion and why it is valid.

- Suggest pencil and good erasure when needed

Summary of Proof Methods

- Constructive Proof of Existence

- Proof by Exhaustion
- Proof by Generalizing from the Generic Particular
- Proof by Contraposition
- Proof by Contradiction
- Proof by Division into Cases

Summations

- What is next in the series … $4, \frac{9}{4}, \frac{16}{9}, \frac{25}{16}, \_\_\_\_\_$
- General formula for a series $a_k = \frac{(k+1)^2}{k^2}, \quad k \geq 1$
- Identical series $a_k = \frac{k}{k+1}, \; k \geq 1 \quad b_i = \frac{i-1}{i}, \; i \geq 2$
- Summation and product notation $\sum_{k=1}^{6} 2^k \qquad \prod_{k=1}^{5} 2k$
- Properties (splitting/merging, distribution)
- Change of variables $\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}$
- Applications (indexing, loops, algorithms)

Mathematical Induction

- Definition
  - Used to verify a property of a sequence
  - Formal definition (next slide)
- What proofs must have
- We proved:
  - General summation/product
  - Inequalities
  - Strong induction

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\forall r \in R^{>1}, \forall a \in R, \forall n \in Z^{\geq 0}, \sum_{j=0}^{n} ar^{j} = \frac{ar^{n+1} - a}{r-1}$$

- Misc
  - Recurrence relations
  - Quotient remainder theorem
  - Correctness of algorithms (Loop Invariant Theorem)

Inductive Proof

- Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer.
- Suppose the following two statements are true.
  - $P(a)$ is true.
  - For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.
- Then the statement for all integers $n \geq a$, $P(n)$ is true.

Inductive Proofs Must Have

- Base Case (value)
  - Prove base case is true
- Inductive Hypothesis (value)
  - State what will be assumed in this proof
- Inductive Step (value)
  - Show
    - State what will be proven in the next section
  - Proof
    - Prove what is stated in the show portion
    - Must use the Inductive Hypothesis sometime

Sets

- Set
  - Notation – $\in$ versus $\subseteq$
  - Definitions – Subset, proper subset, partitions/disjoint sets
  - Operations ($\cup$, $\cap$, $-$, $'$, $\times$)
  - Properties and inference rules
  - Venn diagrams
  - Empty set properties
- Proofs
  - Element argument, set equality
  - Propositional logic / predicate calculus
  - Inference rules
  - Counterexample
  - Types – generic particular, induction, contra's, CW
- Russell's Paradox (The Barber's Puzzle) & Halting Problem

Counting

- Counting elements in a list
  - How many in list are divisible by x
- Probability – likelihood of an event
- Permutations – with and without repetition
- Multiplication rule
  - Tournament play
  - Rearranging letters in words
  - Where it doesn't work
- Difference rule – If A is a finite set and $B \subseteq A$, then $n(A - B) = n(A) - n(B)$
- Addition rule – If $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k = A$ and $A_1, A_2, A_3, \ldots, A_k$ are pairwise disjoint, then $n(A) = n(A_1) + n(A_2) + n(A_3) + \ldots + n(A_k)$
- Inclusion/exclusion rule
- Combinations – with and without repetition, categories
- Binomial theorem (Pascal's Triangle)

$$P(E) = \frac{n(E)}{n(S)}$$

$$\prod_{i=1}^{n} c(i)$$

$$P(n,r) = {}_rP_n = \frac{n!}{(n-r)!}$$

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}$$

$$\binom{r+n-1}{r}$$

Functions

- Definitions/terminology
  - Function
  - Domains, co-domain, range, etc.
  - One-to-one (injective), onto (surjective)
  - One-to-one correspondence (bijective)
- Pigeonhole principle
- Composition of functions
- Cardinality
  - Countably infinite ($\mathbf{Z}$, $\mathbf{Z}^+$, $\mathbf{Z}^{\geq 0}$, $\mathbf{Z}^{even}$, etc.)
  - Not countably infinite ($\mathbf{R}$) – by diagonalization

Relations

- Definitions/notation
  - Binary relations: xRy ($x \in X$, $y \in Y$, $R \subseteq X \times Y$)
- Properties
  - Reflexive, symmetric, transitive
  - Complement, inverse relations
  - Closures (reflexive, symmetric, transitive)
  - More: antisymmetric, …
- Equivalence relations
  - Relation induced by a partition
  - Reflexive, symmetric, and transitive
  - Equivalence classes
- Partial/total order relations

Graphs & Trees

- Definitions/terminology
  - Vertices, edges
  - Simple, complete, bipartite, sub, connected
  - Degree
- Circuits – Euler and Hamiltonian
- Matrix representation
- Trees
  - Graph that is circuit-free and connected
  - Terminology – circuit-free, trivial tree, forest
  - n vertices and n – 1 edges
  - Rooted (level, height, children, etc), binary, spanning (minimum)
- TSP, Graph Isomorphism are NP
- Graph isomorphic invariants

Preparation for Final

- Review
  - Lecture notes
  - Homework assignments
  - Quizzes
  - Exams
  - Book
- Best wishes to all!!!