CMSC 250
Discrete Structures

Predicate Calculus


Ambiguity
12
A 13 C
14

Examples

- Everyone loves logic.
- There is a person only a mom could love.
- Every sufficiently large odd number can be written as the summation of three primes.

Predicate Calculus

- Propositional Logic – uses statements
- Predicate Calculus – uses predicates
  - Predicates must be applied to a subject in order to be true or false
- Subject / Predicate
  - John / went to the store.
  - The sky / is blue.
- P(x)
  - Means this predicate represented by P
  - Applied to the object represented by x

Quantification

- $\exists x$ – There exists an x
- $\forall x$ – For all x's

Usually specified from a domain

- $\exists x \in Z$ – There exists an x in the integers
- $\forall x \in R$ – For all x's in the reals
- Domain – set where these subjects come from

Quantifier Example

- $\exists x \mid (\forall y)\ [y \leqslant x]$

- Depends on the domain
  - Finite set (e.g. $\{1,2,3,4,5,6\}$)?
  - Infinite set (e.g. $\Re$ – set of reals)?

Another Example

- $\forall x \in D$, such that $x^2 \geqslant x$
  - $D = \{0,1,2,3,4,5,6\}$
  - $D = R$
  - $D = Z$

Translation

- A student of mine is wearing a blue shirt.
  - Domain: people who are my students S
  - Quantification: There is at least one
  - Predicate: wearing a blue shirt

$\exists x \in$ S such that B(x)

where B(x) represents "wearing a blue shirt"

- All of my students are in class.
  - Domain: people who are my students S
  - Quantification: All of them
  - Predicate: are in class

$\forall x \in$ S such that C(x)

where C(x) represents "being in class"

Negation of Quantified Statements

~ ($\exists$x $\in$ people such that **H(x))**

$\equiv$ $\forall$**x** $\in$ **people** such that **~ H(x)**

~(There is a person who is here.) $\equiv$

For all people, each person is not here.

Same in meaning as "There is no person here."

~ ($\forall$x $\in$ people such that **H(x))**

$\equiv$ $\exists$**x** $\in$ **people** such that **~ H(x)**

~(For all people, each person is here.) $\equiv$

There is at least one person who is not here.

Multiple Predicate Translation

- **A student of mine is wearing a blue shirt.**
  - Domain: all people P
  - Quantification: There is at least one
  - Predicates: "wearing a blue shirt" and "is my student"

$\exists$**x** $\in$ **P such that B(x) ^ S(x)**

B(x) represents "wearing a blue shirt"

S(x) represents "being my student"

- **All of my students are in class.**
  - Domain: all people P
  - Quantification: All of them
  - Predicates: "are in class" and "is my student"

$\forall$**x** $\in$ **P such that S(x)** $\rightarrow$ **C(x)**

C(x) represents "being in class"

S(x) represents "being my student"

Multiple Quantification

- $\exists b \in B\ \exists c \in C,\ S(b,c)$
- $\exists c \in C\ \exists b \in B,\ S(b,c)$

- $\forall b \in B\ \forall c \in C,\ S(b,c)$
- $\forall c \in C\ \forall b \in B,\ S(b,c)$

Where C={all chairs}, B={all bears}, and S(b,c) represents "b sitting in c"

Mixed Multiple Quantification

- $\forall c \in C\ \exists b \in B,\ S(b,c)$
- $\forall b \in B\ \exists c \in C,\ S(b,c)$

- $\exists b \in B\ \forall c \in C,\ S(b,c)$
- $\exists c \in C\ \forall b \in B,\ S(b,c)$

Where C={all chairs}, B={all bears}, and S(b,c) represents "b sitting in c"

Negations of Multiply Quantified Statements

- $\sim(\forall c \in C\ \exists b \in B,\ S(b,c))$
- $\exists c \in C\ \forall b \in B,\ \sim S(b,c)$

Where C={all chairs}, B={all bears}, and S(b,c) represents "b sitting in c"

Other Variations

- Exactly one child attends school
  - $\exists c \in C\ \exists s \in S\ A(c,s) \wedge \sim[\ \exists p \in C\ \exists b \in S,\ \ p \neq c\ \wedge\ A(p,b)]$
  - $\exists c \in C\ \exists s \in S,\ A(c,s) \wedge [\forall p \in C\ \forall b \in S,\ \ p = c \vee \sim A(p,b)]$
- At most 1 child attends school
  - $\forall c,p \in C\ \forall s,b \in S,\ (A(c,s) \wedge A(p,b)) \rightarrow c = p$
- At least 2 children attend school
  - $\exists c,p \in C\ \exists s,b \in S,\ A(c,s) \wedge A(p,b) \wedge p \neq c$

$\exists x \in \mathbf{R}$ such that $x^2=2$

- Which of the following are equivalent ways of expressing the statement?
  a. The square of each real number is 2.
  b. Some real numbers have square 2.
  c. The number x has square 2, for some real number x.
  d. If x is a real number, then $x^2 = 2$.
  e. Some real number has square 2.
  f. There is at least one real number whose square is 2.

∀ integers n, if n^2 is even then n is even

- Which of the following are equivalent ways of expressing the statement?
  a. All integers have even squares and are even.
  b. Given any integer whose square is even, that integer is itself even.
  c. For all integers, there are some whose square is even.
  d. Any integer with an even square is even.
  e. If the square of an integer is even, then that integer is even.
  f. All even integers have even squares.

Formal to Informal

- $\forall$ students S, if S is in CMSC 250, then S has taken CMSC 131.

- If a student is in CMSC 250 , then that student has taken CMSC 131.
- All students in CMSC 250 have taken CMSC 131.
- Every student in CMSC 250 has taken CMSC 131.
- Each student who is in CMSC 250 has taken CMSC 131.
- Given any student in CMSC 250, that student has taken CMSC 131.

Informal to Formal

- All Java programs have at least 5 lines.
  - ∀x, if x is a Java program, then x has at least five lines
- Any valid argument with true premises has a true conclusion.
  - ∀*x*, if *x* is a valid argument with true premises, then *x* has a true condition
  - ∀ arguments *x*, if *x* is valid and *x* has true premises then *x* has a true conclusion
  - ∀ valid arguments *x*, if *x* has true premises then *x* has a true conclusion.
- The sum of two even integers is even.
  - ∀ integers *x* and *y*, if *x* and *y* are even, then $x + y$ is even.
- The product of any two odd integers is odd.
  - ∀ integers *m* and *n*, if *m* and *n* are odd then *mn* is odd.

Degenerate or Vacuous Cases

- Predicates
  - B(s) “student s is wearing blue”
  - I(s,c) “student s is in class c”

- $\forall s\ B(s)$ – all my students are wearing blue
- $\forall s\ \forall c\ I(s,c)$
- $\forall s\ \exists c\ I(s,c)$
- $\exists c\ \forall s\ I(s,c)$

If there are no students …

Variants of Quantified Conditional Statements

- Statement: $\forall x \in D, P(x) \rightarrow Q(x)$

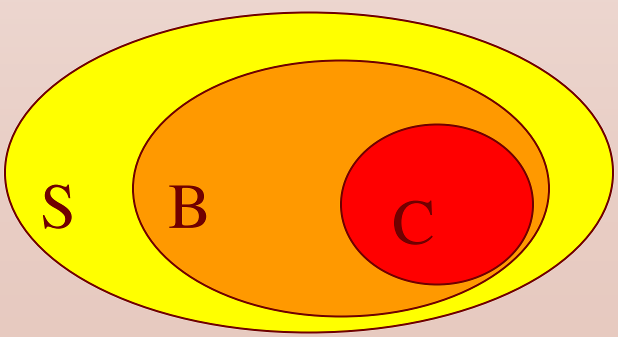
- Contrapositive: $\forall x \in D, \sim Q(x) \rightarrow \sim P(x)$
- Converse: $\forall x \in D, Q(x) \rightarrow P(x)$
- Inverse: $\forall x \in D, \sim P(x) \rightarrow \sim Q(x)$

- Same logical variants apply to existentially quantified conditional statements

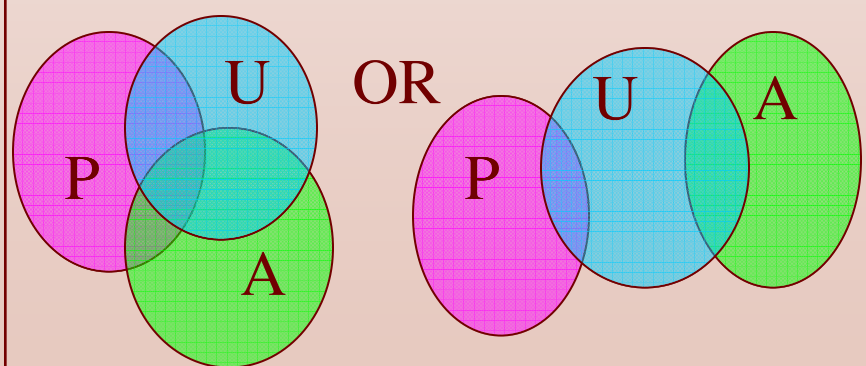
Euler Diagrams

- Circles used to tell "Truth Sets"
  - Where the predicate applied to a object is true
- A dot is used to tell a specific instance
- If "all" then a completely contained circle
- If "some" then an overlapping circle

All college students are brilliant.
All brilliant people are scientists.
∴**All college students are scientists.**



Some poets are unsuccessful.
Some athletes are unsuccessful.
∴**Some poets are athletes.**



Rules of Inference for Quantified Statements

Universal Modus Ponens	**Universal Modus Tollens**
$\forall \mathbf{x} \in D, P(x) \rightarrow Q(x)$ $P(a)$ $a \in D$ $\therefore \mathbf{Q(a)}$	$\forall \mathbf{x} \in D, P(x) \rightarrow Q(x)$ $\sim Q(a)$ $a \in D$ $\therefore \mathbf{\sim P(a)}$
Universal Instantiation	**Existential Generalization**
$\forall x \in D, P(x)$ $a \in D$ $\therefore \mathbf{P(a)}$	$P(c)$ $c \in D$ $\therefore \exists \mathbf{x} \in \mathbf{D, P(x)}$

Rules that DON'T exist or need more clarification

- Existential Modus Ponens - Doesn't exist
- Existential Modus Tollens - Doesn't exist

- Universal Generalization: $P(a) \therefore \forall x \in D, P(x)$
  - only if a is completely arbitrary in the domain
- Existential Instantiation: $\exists x \in D, P(x) \therefore P(a)$
  - only if a is completely arbitrary in the domain

Errors in Deduction

Converse Error

$\forall x \in D,\ P(x) \to Q(x)$

$Q(a)$

$\therefore\ \mathbf{P(a)}$

Called: Asserting the consequence

Inverse Error

$\forall x \in D,\ P(x) \to Q(x)$

$\sim P(a)$

$\therefore\ \mathbf{\sim Q(a)}$

Called: Denying the hypothesis

Direct Proofs by Deduction

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\sim Q(a)$ where $a \in D$
- therefore: $\exists x \in D, \sim P(x)$

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\forall x \in D, R(x) \rightarrow \sim P(x)$
- $P(b)$ where $b \in D$
- therefore: $Q(b) \wedge \sim R(b)$

Direct Proofs by Deduction

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\forall x \in D, \sim P(x) \vee R(x)$
- $P(b)$ where $b \in D$
- therefore: $\exists x \in D, Q(x) \wedge R(x)$

- $\forall x \in D, P(x) \rightarrow Q(x)$
- $\forall y \in D, \sim P(y) \rightarrow R(y)$
- $\exists z \in D, \sim Q(z)$
- therefore: $\exists x \in D, R(x)$

More Translation Examples

- Everybody is older than somebody.
  - $\forall x \in P$, $\exists y \in P$ such that x is older than y

- Somebody is older than everybody.
  - $\exists x \in P$, $\forall y \in P$, such that x is older than y

More Practice in Translation

- There is a person only a mom could love.
- Development:

There is at least one person (only a mom could love).
There is at least one person (if anyone loves him it must be a mom)
There is at least one person (if anyone loves him then that person is a mom)

$$\exists \mathbf{x} \in \mathbf{P}\ \forall \mathbf{s} \in \mathbf{P},\ \mathbf{L(s,x)} \rightarrow \mathbf{M(s)}$$

L(s,x) means "s loves x"
M(s) means "s is a mom"

Another way:

There is at least one person, x,(There's nobody in the world who loves x is not a Mom)

There's a person,x, (it's not the case (there is a person who Loves x and is not a Mom)).

$$\exists \mathbf{x} \in \mathbf{P}\ [\sim(\exists \mathbf{s} \in \mathbf{P},\ \mathbf{L(s,x)}\ \wedge\ \sim\mathbf{M(s)})]$$

More Practice in Translation

- No two people share the same toothbrush.
- Development:

It is not the case (there there are two (or more) people sharing a toothbrush).

~(there are two (or more) people sharing a toothbrush)

~(at least two people share a toothbrush)

$$\sim(\exists s,m \in P\ \exists t \in T,\ K(s,t) \wedge K(m,t) \wedge s \neq m)$$

Another way:

If we look at every pair of people/toothbrush combination, one of the three is false.

$$\forall s,m \in P\ \forall t \in T,\ \sim K(s,t) \vee \sim K(m,t) \vee s=m$$

A(c,s) = "child c attends school s"

- **$\exists c\ \exists s\ A(c,s)$**
  - find one child/school combo which makes it true
  - one child attends some school somewhere
- **$\forall c\ \forall s\ A(c,s)$**
  - must be true for all child/school combos
  - all children must attend all schools
- **$\forall c\ \exists s\ A(c,s)$**
  - for all children select any one school to which that child attends
  - all children attend some school
- **$\forall s\ \exists c\ A(c,s)$**
  - for all schools select any one child to which that school attends
  - all schools have at least one child
- **$\exists s\ \forall c\ A(c,s)$**
  - select any one school and assert that all children attend that one school
  - there is a school that all children attend
- **$\exists c\ \forall s\ A(c,s)$**
  - select any one child and assert that all schools are attended by that one child
  - there is a child who attends all schools

A(c,s) = "child c attends school s"

- Negation of "Every child attends school".
  - **$\sim[\forall c\ \exists s\ A(c,s)]$**
- At least one child did not attend school.
  - **$\sim[\forall c\ \exists s\ A(c,s)]$**
    - It is not the case that all children attend school.
  - **$\exists c \sim[\exists s\ A(c,s)]$**
    - There is one child for whom it is not the case that there exists a school which he/she attends.
  - **$\exists c\ \forall\ s \sim A(c,s)$**
    - There is one child for whom all schools are ones that he/she does not attend.

A(c,s) = "child c attends school s"

- Negation of "At least one child attends school".
  - **$\sim [\exists c\ \exists s\ A(c,s)]$**
- No children attend school.
  - **$\sim[\exists c\ \exists s\ A(c,s)]$**
    - It is not the case that there is a child who attends school.
  - **$\forall c \sim[\exists s\ A(c,s)]$**
    - For all children it is not the case that you can select a school that child attends.
  - **$\forall c\ \forall s \sim A(c,s)$**
    - For all child/school combinations it is not the case that the child attends that school.

True or false? … Why?

- $\forall x \in \boldsymbol{Z}^+, \exists y \in \boldsymbol{Z}^+$ such that $x = y + 1$
- $\forall x \in \boldsymbol{Z}, \exists y \in \boldsymbol{Z}$ such that $x = y + 1$

- $\forall x \in \boldsymbol{R}^+, \exists y \in \boldsymbol{R}^+$ such that $xy = 1$
- $\forall x \in \boldsymbol{R}, \exists y \in \boldsymbol{R}$ such that $xy = 1$

- $\forall x \in \boldsymbol{Z}^+$ and $\forall y \in \boldsymbol{Z}^+, \exists y \in \boldsymbol{Z}^+$ such that $z = x - y$
- $\exists x \in \boldsymbol{R}^+$ such that $\forall y \in \boldsymbol{R}^+, xy < y$

Are these Equivalent?

- $\forall x \in \boldsymbol{D}, (P(x) \wedge Q(x))$, and $(\forall x \in \boldsymbol{D}, P(x)) \wedge (\forall x \in \boldsymbol{D}, Q(x))$
- $\exists x \in \boldsymbol{D}, (P(x) \wedge Q(x))$, and $(\exists x \in \boldsymbol{D}, P(x)) \wedge (\exists x \in \boldsymbol{D}, Q(x))$
- $\forall x \in \boldsymbol{D}, (P(x) \vee Q(x))$, and $(\forall x \in \boldsymbol{D}, P(x)) \vee (\forall x \in \boldsymbol{D}, Q(x))$
- $\exists x \in \boldsymbol{D}, (P(x) \vee Q(x))$, and $(\exists x \in \boldsymbol{D}, P(x)) \vee (\exists x \in \boldsymbol{D}, Q(x))$

Valid Arguments?

- No good car is cheap.
- A Volvo is a good car.

∴ A Volvo is not cheap.

- No good car is cheap.
- A Pinto is cheap.

∴ A Pinto is not good.

- No good car is cheap.
- A Subaru is not cheap.

∴ A Subaru is a good car.

- No good car is cheap.
- A Focus is not a good car.

∴ A Focus is cheap.

$\forall x$, if x is a good car, then x is not cheap.