CMSC 250
Discrete Structures

Summation:

Sequences and Mathematical Induction

What is Next?

- 2, 4, 6, 8, 10, …
- 1, 4, 9, 16, 25, …
- 2, 4, 8, 16, 32, …
- 0, 1, 1, 2, 3, 5, …

Sequences

- 2,4,6,8,… for $i \geqslant 1$ $a_i = 2i$
  - infinite sequence with infinite distinct values
- For $i \geqslant 1$ $b_i = (-1)^i$
  - infinite sequence with finite distinct values
- For $1 <= i <= 6$ $c_i = i+5$
  - finite sequence (with finite distinct values)

Identical series?

$$a_k = \frac{k}{k+1},\ \ k \geq 1 \qquad\qquad b_i = \frac{i-1}{i},\ \ i \geq 2$$

Finding the Explicit Formula

- Figure the formula of this sequence

$$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$$

$$a_k = \frac{(-1)^{k+1}}{k^2},\ k \ge 1$$

$$a_k = \frac{(-1)^k}{(k+1)^2},\ k \ge 0$$

- Different sequences with same initial values

$$k >= 0$$

$$a_k = 2k + 1$$

$$b_k = (k-1)^3 + k + 2$$

What is the Formula?

- 2, 4, 6, 8, 10, … $a_k = 2k, k \geq 1$
- 1, 4, 9, 16, 25, … $a_k = k^2, \; k \geq 1$
- 2, 4, 8, 16, 32, … $a_k = 2^k, \; k \geq 1$
- 0, 1, 1, 2, 3, 5, … $\begin{cases} a_0 = 0; a_1 = 1 \\ a_k = a_{k-2} + a_{k-1}, \; k \geq 2 \end{cases}$

Summation & Product Notation

- Sum of Items Specified

$$\sum_{k=1}^{6} 2^k = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$$

- Product of Items Specified

$$\prod_{k=1}^{5} 2k = 2(1)\cdot 2(2)\cdot 2(3)\cdot 2(4)\cdot 2(5)$$

Variable ending point

- n as the index of the final term

$$\sum_{k=0}^{n} \frac{k+1}{n+k}$$

- for n = 2
- for n = 3

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

Telescoping Series

$$\sum_{k=1}^{n} \left(\frac{k}{k+1} - \frac{k+1}{k+2} \right)$$

$$\prod_{i=1}^{n} \left(\frac{i}{i+1} \right)$$

Factorial

- n! = n · (n-1) · (n-2) · … · 2 · 1

- Definition

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

Properties

- Merging and Splitting

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

$$\sum_{k=m}^{n} a_k = \sum_{k=m}^{i} a_k + \sum_{k=i+1}^{n} a_k$$

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$$

$$\prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{i} a_k\right) \cdot \left(\prod_{k=i+1}^{n} a_k\right)$$

- Distribution

$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k)$$

Using the Properties

$$a_k = k + 1; \quad b_k = k - 1$$

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$$

$$\left(\prod_{k=m}^{n} a_k \right) \cdot \left(\prod_{k=m}^{n} b_k \right)$$

Change of Variables (1 of 2)

$$\sum_{j=2}^{4}(j-1)^2 = \sum_{k=1}^{3}k^2$$

Change of Variables (2 of 2)

$$summation: \sum_{k=0}^{6} \frac{1}{k+1} \qquad change\ of\ variable: j = k+1$$

- Calculate new lower and upper limits
  - When k = 0, j = k + 1 = 0 + 1 = 1.
  - When k = 6, j = k + 1 = 6 + 1 = 7.
- Calculate new general term
  - Since j = k + 1, then k = j – 1.
  - Hence $\frac{1}{k+1} = \frac{1}{(j-1)+1} = \frac{1}{j}$

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}$$

Applications

- Indexing arrays using loops
  - When to start and end
  - …
- Algorithms
  - Convert from base 10 to base 2
  - …