CMSC 250
Discrete Structures

Counting

Counting Elements in a List

- How many integers in the list from 1 to 10?

- How many integers in the list from m to n? (assuming $m \leq n$)
  - $n - m + 1$
  - Can you prove this?

Prove: # elements in list

- Base case (List of size 1, x=y)
  - $y - x + 1 = y - y + 1$ (by substitution) $= 1$
- IH (generic x, y=k [where $x \leq k$])
  - Assume size of list x to k, is $k - x + 1$
- IS
  - Show size of list x to k + 1, is $(k + 1) - x + 1$
  - Prove
    - Split into two lists …

How many in list divisible by something

- How many positive three digit integers are there?
  - (this means only the ones that require 3 digits)
  - 999 – 100 + 1 = 900
- How many three digit integers are divisible by 5?
  - think about the definition of divisible by

    $x \mid y \leftrightarrow \exists\, k \in Z,\ y = kx$ and then count the k's that work

    100, 101, 102, 103, 104, 105, 106,… 994, 995, 996, 997, 998, 999

    20*5   21*5   …   199*5
  - count the integers between 20 and 199
  - 199 – 20 + 1 = 180

Probability

likelihood of a specific event

- Sample Space = set of all possible outcomes
- Random = outcome of trial is unknown
- Event = subset of sample space
- Equal Probability Formula:
  - Given a finite sample space S where all outcomes are equally likely
  - Select an event E from the sample space S
  - The probability of event E from sample space S:

$$P(E) = \frac{n(E)}{n(S)}$$

Flipping Two Coins

- Sample Space = {(H,H), (H,T), (T,H), (T,T)}

- What do the following events represent?
  - Probability of no heads
  - Probability of at least one head
  - Probability of same sides on the two coins
- Compute the above probabilities

- If flip two coins 100 times will you get 25-50-25?
  - Probability & actual outcomes often differ

Standard Playing Cards

- Values: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- Suits: D(♦), H(♥), S(♠), C(♣)

- Probability of drawing the Ace of Spades
- Probability of drawing a Spade
- Probability of drawing a face card
- Probability of drawing a red face card

Rolling Two Six-Sided Dice

- Sample Space
  - Ordered pairs

    {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),
    (2,1),(2,2),(2,3),(2,4),(2,5),(2,6),
    …
    (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}
  - Simplified

    {11,12,13,14,15,16,
    21,22,23,24,25,26,
    …
    61,62,63,64,65,66}

- Set representing rolling a 10?
- The probability of rolling a 10?

- Set representing rolling a pair?
- Probability of rolling a pair?

Multi-level Probability

- If I toss a coin once – the probability of Head = ½
- If I toss that coin 5 times
  - The probability of all heads

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2^5}$$

  - The probability of exactly 4 heads

$$\frac{5}{2^5}$$

Combination Examples

- Breakfast
  - Main
  - Fruit

- Red, White, Blue marbles
  - How can you arrange 10 of them
  - __ __ __ __ __ __ __ __ __ __
  - $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^{10}$

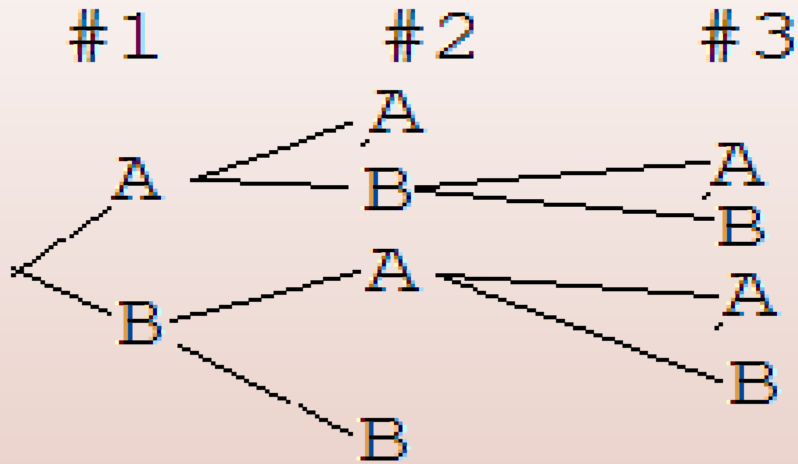
Multiplication Rule

- 1^{st} step can be performed n_1 ways
- 2^{nd} step can be performed n_2 ways
- …
- K^{th} step can be performed n_k ways
- Operation can be performed $n_1 * n_2 * \ldots * n_k$ ways

- Cartesian product n(A)=3, n(B)=2, n(C)=4
  - n(AxBxC) = 24
  - n(AxB) = 6, n((AxB)xC) = 24

Tournament Play

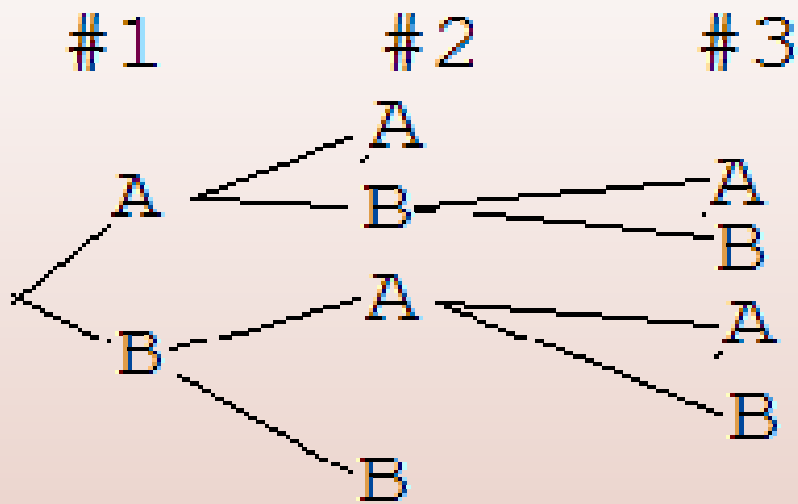
- Team A and Team B in "Best of 3" Tournament
- Where they each have an equal likelihood of winning each game



  – Do leaves add up to 1?
  – Do we have to play 3 games?
  – Do A and B have an equal chance of winning?

What if A wins 2 of every 3 games?

- Each line for A must have a 2/3
- Each line for B must have a 1/3



- How likely is A to win the tournament?
- How likely is B to win the tournament?

Permutation Example

- How many ways to take a picture?
  - With 1 person?
  - With 2 people?
  - With 3 people?
  - With 4 people?
  - With 5 people?

- Number of ways to arrange n objects
  - $n!$
  - $10^n < n!$

Permutation Example

- How many ways to rewrite CAT?
- $3! = 6$
- cat, cta, act, atc, tca, tac

Multiplication Rule

- If there are n steps to a decision, each step having c(k) choices, the total number of choices is:

$$\prod_{i=1}^{n} c(i)$$

Examples

- Choose a PIN from a {0,1,2,3,4,A,B,C} keyboard with PIN length 3

  $= 8 \cdot 8 \cdot 8 = 8^3 = 512$
- Choose a PIN of the form #L# where
  - $\# \in \{0,1,2,\ldots,9\}$
  - $L \in \{0,1,2,\ldots,9\}$

  $= 10 \cdot 26 \cdot 10$
- Choose a 3-value PIN of digits and letters

  $= 36 \cdot 36 \cdot 36$
  - Without repeats

  $= 36 \cdot 35 \cdot 34$

Permutation Example

- Hexadecimal numbers are made using digits: 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F with subscript 16.

- How many 5-digit long begin with a digit 3 through B, and end with a digit 5 through F?
  - $9 \cdot 16 \cdot 16 \cdot 16 \cdot 11 = 405{,}504$

- How many 6-digit long begin with one of 4 through D, and end with a digit 2 through E?
  - $10 \cdot 16 \cdot 16 \cdot 16 \cdot 16 \cdot 13 = 8{,}519{,}680$

Probabilities with PINs

- Number of 4 digit PINs of (0,1,2,.)
  - With repetition allowed = $4 \cdot 4 \cdot 4 \cdot 4 = 256$
  - With no repetition allowed = $4 \cdot 3 \cdot 2 \cdot 1 = 24$
- What is the probability that your 4 digit PIN has no repeated characters?
- What is the probability that your 4 digit PIN does have repeated characters?

- Probability of the complement of an event

$$P(E') = P(E^c) = 1 - P(E)$$

Permutation Example

- Consider strings of length n over the set $\{a,b,c,d\}$.
- How many strings contain at least one pair of adjacent characters that are the same?
  - Total number of length n is 4^n
  - No two adjacent = $4 \cdot 3 \cdot 3 \cdot 3 = 4 \cdot 3^{n-1}$
  - So $4^n - 4 \cdot 3^{n-1}$
- What is the probability that a random string of length 10 contains at least one pair of adjacent characters that are the same?

$$= \frac{4^{10} - 4 \cdot 3^9}{4^{10}} = 1 - \left(\frac{3}{4}\right)^9 \cong 92.5\%$$

Difference Rule Formally

- If A is a finite set and $B \subseteq A$, then

$$n(A - B) = n(A) - n(B)$$

- One application:

  probability of the complement of an event

$$P(E') = P(E^c) = 1 - P(E)$$

PINs with less specified length Addition Rule

- Assume it can be a 2, 3 or 4 length PIN
- Partition the problem
  - Number of 2 length PINs w/rep allowed: $4 \cdot 4 = 16$
  - Number of 3 length PINs w/rep allowed: $4 \cdot 4 \cdot 4 = 64$
  - Number of 4 length PINs w/rep allowed: $4 \cdot 4 \cdot 4 \cdot 4 = 256$

- Number PINs if allowing length of 2, 3 or 4 = 336

Addition Rule Formally

- If $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k = A$
  and $A_1, A_2, A_3, \ldots, A_k$ are pairwise disjoint
  (i.e. if these subsets form a partition of A)

- $n(A) = n(A_1) + n(A_2) + n(A_3) + \ldots + n(A_k)$

Another example for Multiplication Rule and Addition Rule

- How many 3 digit integers are divisible by 5?
  - How many end in a 0? $9 \cdot 10 \cdot 1 = 90$
  - How many end in a 5? $9 \cdot 10 \cdot 1 = 90$
  - These form a partition with the set of numbers divisible by 5 so
    - $90 + 90 = 180$

Where Multiplication Rule Doesn't Work

- People = {Angel, Bob, Carol, Dan}
- Need to be appointed as
  - President, vice-president, and treasurer
  - Nobody can hold more than one office
  - Angel doesn't want to be president
  - Only Bob and Dan want to be vice-president
- Draw tree …

Where Multiplication Rule Doesn't Work

- P1 picks a card and doesn't look at it
- P2 picks a card

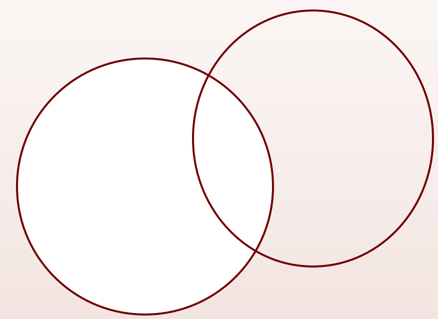
- What are the chances P2 picked a red face card?

- Draw tree …

Inclusion/Exclusion Rule

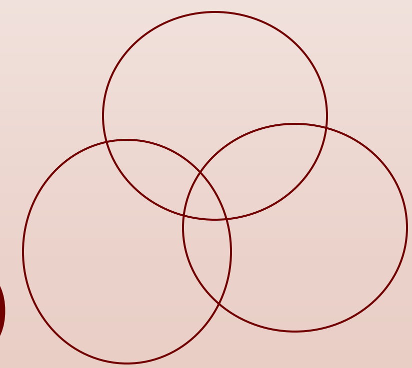
- If there are two sets:

  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$



- If there are three sets:

  $$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$



Inclusion/Exclusion Example

- How many integers from 1 to 1000 are multiples of 3, 5, or 8?
- Sets
  - A = 3
  - B = 5
  - C = 5
- n(A B C) =

  $n(A) + n(B) + n(C)$

  $- n(A \cap B) - n(A \cap C) - n(B \cap C)$

  $- n(A \cap B \cap C)$

Permutation Example

- How many ways to rewrite COMPUTER?
- 8! = 40320

computer, computre, compuetr, compuert, compurte, compuret, comptuer, compture, compteur, compteru, comptrue, comptreu, compeutr, compeurt, competur, competru, comperut, compertu, comprute, compruet, comprtue, comprteu, compreut, compretu, comupter, comuptre, comupetr, comupert, comuprte, comupret, comutper, comutpre, comutepr, comuterp, comutrpe, comutrep, comueptr, comueprt, comuetpr, comuetrp, comuerpt, comuertp, comurpte, comurpet, comurtpe, comurtep, comurept, comuretp, comtpuer, comtpure, comtpeur, comtperu, comtprue, comtpreu, comtuper, comtupre, comtuepr, comtuerp, comturpe, comturep, comtepur, comtepru, comteupr, comteurp, comterpu, comterup, comtrpue, comtrpeu, comtrupe, comtruep, comtrepu, comtreup, comeputr, comepurt, comeptur, comeptru, comeprut, comeprtu, comeuptr, comeuprt, comeutpr, comeutrp, comeurpt, comeurtp, cometpur, cometpru, cometupr, meturp, cometrpu, cometrup, comerput, comerptu, comerupt, comerutp, comertpu, comertup, comrpute, comrpuet, comrptue, comrpteu, comrpeut, comrpetu, comrupte, comrupet, comrutpe, comrutep, comruept, comruetp, comrtpue, comrtpeu, comrtupe, comrtuep, comrtepu, comrteup, comreput, comreptu, comreupt, comreutp, comretpu, comretup, copmuter, copmutre, copmuetr, copmuert, copmurte, copmuret, copmtuer, copmture, copmteur, copmteru, copmtrue, copmtreu, copmeutr, copmeurt, copmetur, copmetru, copmerut, copmertu, copmrute, copmruet, copmrtue, copmrteu, copmreut, copmretu, copumter, copumtre, copumetr, copumert, copumrte, copumret, coputmer, coputmre, …

Permutation Example

- How many to rewrite CO-M-P-U-T-E-R?
- 7! = 5040
- COMPUTER, COMPUTRE, COMPUETR, COMPUERT, COMPURTE, COMPURET, COMPTUER, COMPTURE, COMPTEUR, COMPTERU, COMPTRUE, COMPTREU, COMPEUTR, COMPEURT, COMPETUR, …

Factorial

- How to write a factorial method?

```
long factorial(int number)
{
    if (number <= 1)
        return 1;
    else return (number * factorial(--number));
}
```

Computing Combinations

```
List combinations(String str)
{   List combinations = new List();
    if (str.length() == 1) combinations.add(str);
    else
    {       char[] strChars = str.toCharArray();
            for (int i = 0; i < strChars.length; i++)
            {       // get the sub part of the string to get more combinations
                    // sub part is original string minus the current ith character
                    char[] subStrChars = new char[strChars.length - 1];

                    // get subcombos
                    List subCombos = combinations(new String(subStrChars));

                    // add current char to beginning of other strings
                    for (int j = 0; j < subCombos.size(); j++)
                            combinations.add(new String(strChars[i] + subCombos.get(j).toString()));
            }
    }
    return combinations;
}
```

Permutations

- Different ways of arranging objects
  - In a line or circle
  - Without duplication/ all items distinguishable
  - Note: <u>order is taken into account</u>
- Number of linear permutations of N objects = N!
  - N possible for 1^{st} position * (N-1) for 2^{nd} * …* (1) for last
- Number of circular permutations of N objects = (N-1)!
  - Fix one person,
  - Then (N-1) possible for next position * (N-2) for 2^{nd} * …* (1) for last

Theater Seating

- 6 people sit in a row with exactly 6 seats.
- How many ways can they be seated together in the row?
- $6! = 720$

- One is a doctor who must sit on the aisle.
- How many ways can they be seated together in the row?
  - Two aisles = $2 \cdot 5! = 240$
  - One aisle = $5! = 120$

- 6 people are 3 married couples. Each wants to sit together with the husband on the left.
- How many ways can they be seated together in the row?
- $3! = 6$

Permutations

- What if just want 5-letter words from COMPUTER?

$$= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$$

$$= \frac{8!}{(8-5)!}$$

r-Permutations

- If there are n things in the set, and you want to line-up only r of them.

$$P(n,r) = {}_rP_n = \frac{n!}{(n-r)!}$$

- Example:
  - Class = {Alice, Bob, Carol, Dan}
  - Select a president and a vice president to represent the class

Combinations

- Different ways of selecting objects
  - Counting subsets
  - Without duplication / all items distinguishable
  - Note: <u>order is not taken into account</u>

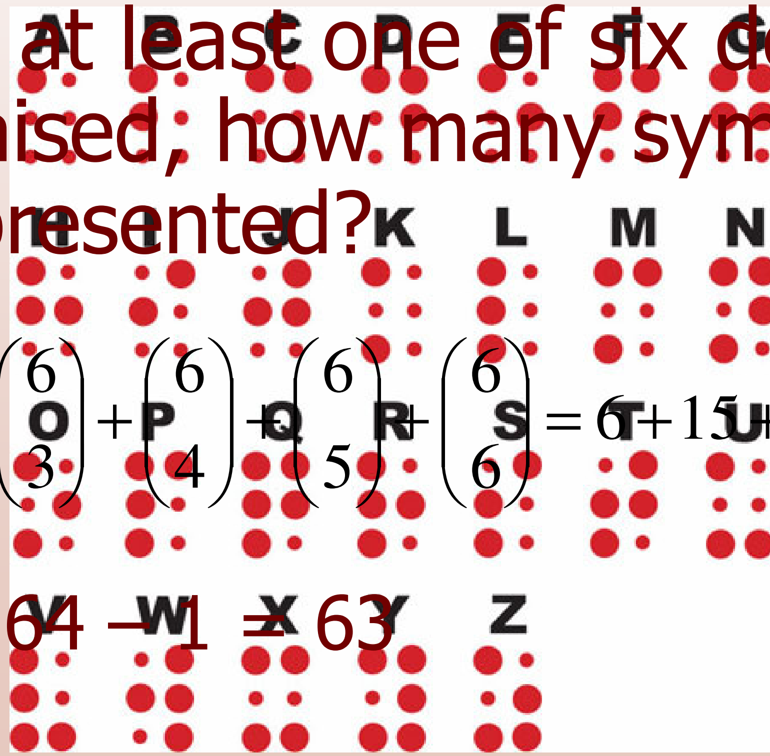
$$\forall n, r \in Z^{\geq 0} \, where \; n \geq r,$$

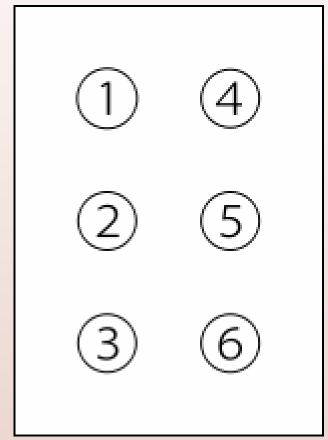
$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$$

- Examples:
  - Class = {Alice, Bob, Carol, Dan}
    - Select two class representatives
    - Select three class representatives

Combinations Example

- Braille code is represented by rectangular arrangement of six dots, each of which is raised or flat.
- Given that at least one of six dots must be raised, how many symbols can be represented?





$$= \binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6} = 6+15+20+15+6+1=63$$

$= 2^6 - 1 = 64 - 1 = 63$

Harder Examples
selecting “class representatives”

- Class = {Alice, Bob, Carol, Dan, Erin, Fred}

- Select 2 – no restrictions
- Select 2 – assuming Alice and Bob must stay together
- Select 3 – no restrictions
- Select 3 – assuming Alice and Bob must stay together
- Select 3 – assuming Alice and Bob refuse to serve together

Different Types of Members

- {Alice, Bob, Carol, Dan, Erin, Fred, George, Harry}
  - pink names are girls and blue names are boys

- 8 people in the set: 3 girls & 5 boys
  - Make a 5 member team of 2 girls and 3 boys
  - Make a 5 member team that has only one girl
  - Make a 5 member team that has no girls
  - Make a 5 member team that has at least one girl

Student Council
(Counting Subsets of a Set)

- A student council consists of:
  - 3 Freshmen
  - 4 Sophomores
  - 4 Juniors
  - 5 Seniors
- How many committees of eight members of the council contain at least one member from each class?

Student Council

(Counting Subsets of a Set)

- Total number of committees is: $\binom{16}{8} = \frac{16!}{8! \cdot 8!} = 12{,}870$
- Answer is total – number of committees with no members from at least one class
- Let A_1, A_2, A_3, A_4 be sets without freshmen, sophomores, juniors, seniors
- $A_1 \cup A_2 \cup A_3 \cup A_4$ is the set with no members from at least one class
- $N(A_1 \cup A_2 \cup A_3 \cup A_4) =$

$$N(A_1) + N(A_2) + N(A_3) + N(A_4) -$$
$$N(A_1 \cap A_2) - N(A_1 \cap A_3) - N(A_1 \cap A_4) -$$
$$N(A_2 \cap A_3) - N(A_2 \cap A_4) - N(A_3 \cap A_4) +$$
$$N(A_i \cap A_j \cap A_k) -$$
$$N(A_1 \cap A_2 \cap A_3 \cap A_4)$$

Combinations with Repetition

- {a,b,c,d}
- How many 3-combinations can I make without repetition?
  - [a,b,c] [a,b,d] [a,c,d] [b,c,d]
- How many 3-combinations can I make with unlimited repetition allowed?
  - [a,a,a] [a,a,b] [a,a,c] [a,a,d]
  - [a,b,b] [a,b,c] [a,b,d]
  - [a,c,c] [a,c,d] [a,d,d]
  - [b,b,b] [b,b,c] [b,b,d]
  - [b,c,c] [b,c,d] [b,d,d]
  - [c,c,c] [c,c,d] [c,d,d] [d,d,d]

Combinations with Repetition

- Is there an easy way to count these?
- Reform problem to putting 3 marks in 4 categories

a	b	c	d	
x		x	x	[a,c,d]
xx		x		[a,a,c]
x			xx	[a,d,d]

- We can also represent this as:
  - X||X|X [a,c,d]
  - XX||X| [a,a,c]
  - X|||XX [a,d,d]
- Easy to solve in this form …
  - How many words can you make from XXX|||

$$\binom{6}{3} = \frac{6!}{3! \cdot 3!} = 20$$

r-Combinations w/ Repetitions

- A large pile of coins consists of pennies, nickels, dimes and quarters (at least 30 each)

- How many different collections of 30 coins can be chosen? $$=\binom{30+4-1}{30}=\binom{33}{30}=\frac{33!}{3!\cdot 30!}=5456$$

- If the different kinds of collections of 30 coins are equally likely, what is the probability that a combination contains at least four coins of each type? $$=\frac{\binom{(30-4\cdot 4)+4-1}{30-4\cdot 4}}{5456}=\frac{\binom{17}{14}}{5456}=\frac{\left(\frac{17!}{3!\cdot 14!}\right)}{5456}=\frac{680}{5456}\cong 12.5\%$$

Permutation w/ Repeated Elements

- What about NEEDLES
- $\mathrm{NE_1E_2DLE_3S} = 7!$
  - $\mathrm{NE_1E_2DLE_3S}$, $\mathrm{NE_1E_3DLE_2S}$,
  - $\mathrm{NE_2E_1DLE_3S}$, $\mathrm{NE_2E_3DLE_1S}$
  - $\mathrm{NE_3E_1DLE_2S}$, $\mathrm{NE_3E_2DLE_1S}$

$$= \frac{7!}{6} = \frac{7!}{3!}$$

Permutations w/ Repeated Elements

- What about MISSISSIPPI?
- $MI_1S_1S_2I_2S_3S_4I_3P_1P_2I_4$
- 11!

- M (1), I (4), S (4), P(2)

$$= \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!}$$

Permutations but of indistinguishable items

- Assume you have a set of 15 beads
  - 3 Red
  - 2 Black
  - 4 Orange
  - 6 Green
- Select positions of R's, then B's, then O's then G's

$$\binom{15}{3}\cdot\binom{12}{2}\cdot\binom{10}{4}\cdot\binom{6}{6} = \frac{15!}{3!\cdot 2!\cdot 4!\cdot 6!}$$

Combinations with Repitition

- In general
  - n – categories
  - r – items to choose from
  - Repetition allowed and order doesn't count

$$\underbrace{\mathbf{x\ x\ x}}_{\mathbf{r}} \quad \underbrace{\mathbf{|\ |\ |}}_{\mathbf{n\ -\ 1}}$$

$$\binom{r+n-1}{r}$$

Combinations w/ Repetition Example

- A person giving a party wants to set out 15 assorted cans of soft drinks,
- The store he shops at sells five different types of soft drinks
  - How many different selections of cans of 15 soft drinks can he make? $\binom{15+5-1}{15}$
  - If root beer is one of the types, how many selections include at least six cans of root beer? $\binom{9+5-1}{9}$
  - If each selection is equally likely, what is the probability a selection contains at least six cans of root beer?

Combinations w/ Repetition Example

- There are 6 kinds of computers
- How many ways to choose 25?
  - x x x x x x x x x x x x x x x x x x x x x x x x x | | | | |
  - r – choose 25
  - n – 6 categories
- Assume one type is an iPAQ, how many ways can you choose if exactly 10 must be iPAQs?
  - Restate as 15 computers, 5 types
- What if at least 10 must be iPAQs?
  - Restate as 15 computers, 6 types
- What are the chances that if we select 25 at random, that exactly 10 will be iPAQs?

Probability with Combinations

- Assume there are 32 people in the class
- And that 7 will be chosen to get extra homework
- What is the probability that you get extra homework

- Number of ways to select the "lucky 7"
- Number of ways to select if "I get HW"

- P(I get HW)

Properties of r-Permutations and proofs of those properties

- $P(n,1) = n$
- $P(n,2) = n^2 - n$
- $P(n,2) + P(n,1) = n^2$

- $P(n,n) = n!$
- $P(n,n-1) = n!$

Properties of Combinations
(Can you prove this?)

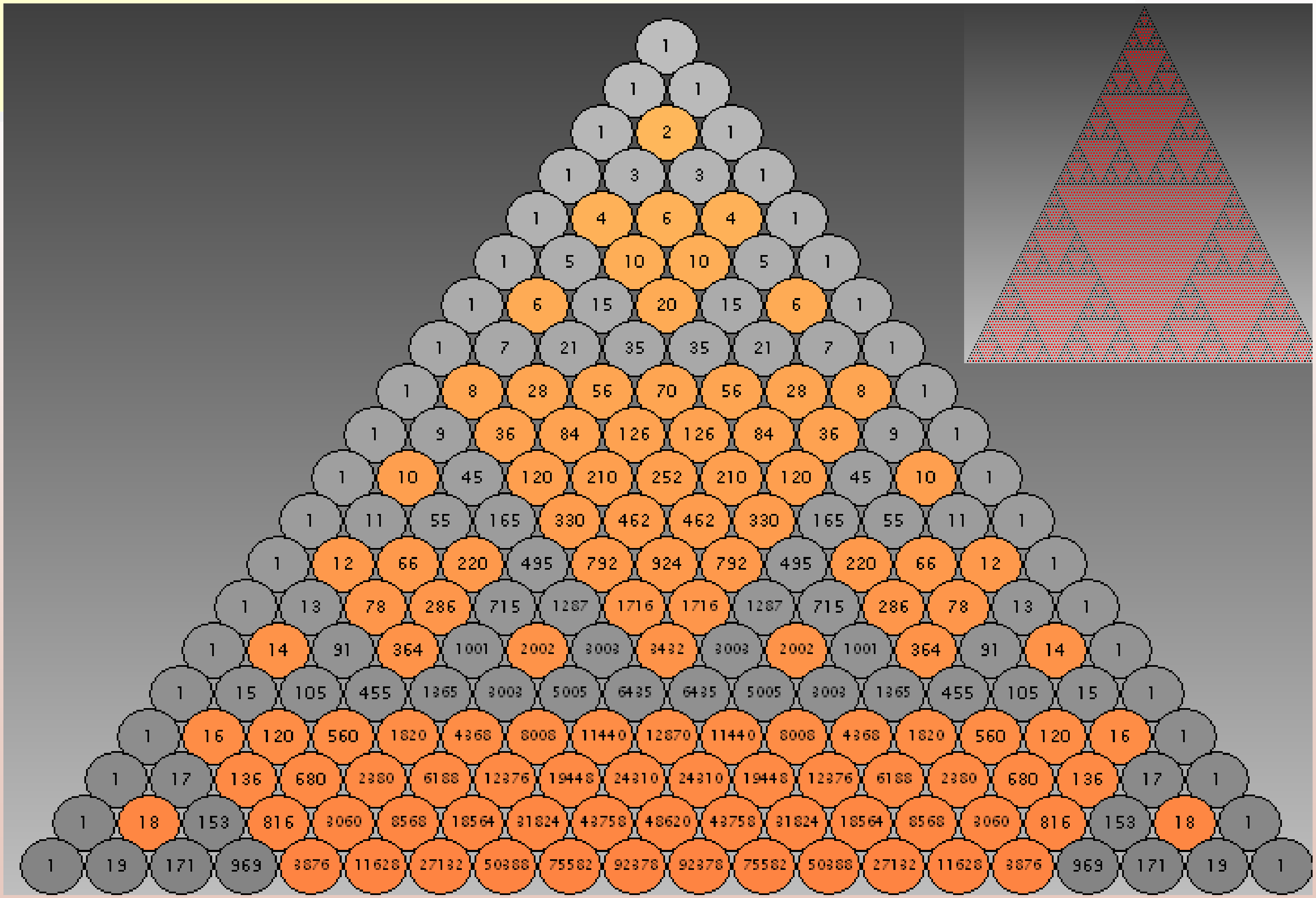
$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

$$\binom{n}{1} = n \qquad \binom{n}{n-1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2} \qquad \binom{n}{n-2} = \frac{n(n-1)}{2}$$

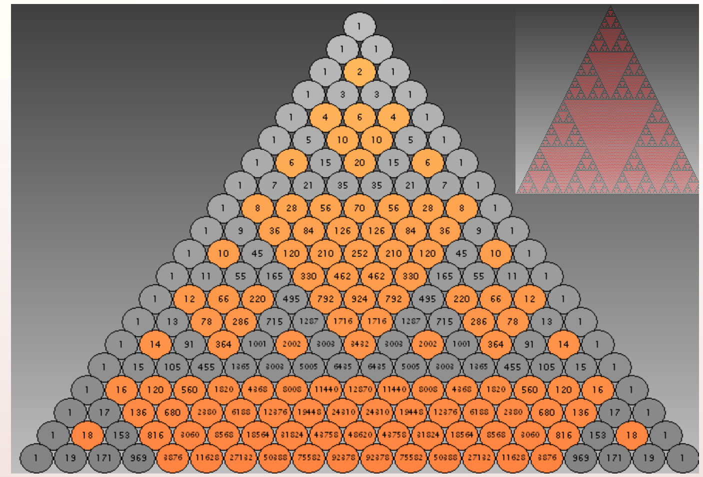
$$\binom{n}{r} = \binom{n}{n-r}$$

Pascal’s Triangle



Binomial Theorem

- $(x+y)^2$
- $(x+y)^3$
- …
- $(x+y)^n$



- Given any real numbers a and b and any nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Conditional Probability

- Probability of B given that A is known to have happened

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- If P(B) = P(B|A),

  then event B is Independent of event A