CMSC 250
Discrete Structures

Exam 2 Review

Summations

- What is next in the series … $4, \frac{9}{4}, \frac{16}{9}, \frac{25}{16}, \_\_\_\_\_$
- General formula for a series $a_k = \frac{(k+1)^2}{k^2}, \quad k \geq 1$
- Identical series $a_k = \frac{k}{k+1}, \ k \geq 1 \quad b_i = \frac{i-1}{i}, \ i \geq 2$
- Summation and product notation $\sum_{k=1}^{6} 2^k \qquad \prod_{k=1}^{5} 2k$
- Properties (splitting/merging, distribution)
- Change of variables $\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}$
- Applications (indexing, loops, algorithms)

Properties

- Merging and Splitting

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

$$\sum_{k=m}^{n} a_k = \sum_{k=m}^{i} a_k + \sum_{k=i+1}^{n} a_k$$

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$$

$$\prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{i} a_k\right) \cdot \left(\prod_{k=i+1}^{n} a_k\right)$$

- Distribution

$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k)$$

Using the Properties

$$a_k = k+1; \quad b_k = k-1$$

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$$

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right)$$

Mathematical Induction

- Definition
  - Used to verify a property of a sequence
  - Formal definition (next slide)
- What proofs must have
- We proved:
  - General summation/product
  - Inequalities
  - Strong induction

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\forall r \in R^{>1}, \forall a \in R, \forall n \in Z^{\geq 0}, \sum_{j=0}^{n} ar^{j} = \frac{ar^{n+1} - a}{r-1}$$

- Misc
  - Recurrence relations
  - Quotient remainder theorem
  - Correctness of algorithms (Loop Invariant Theorem)

Inductive Proof

- Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer.
- Suppose the following two statements are true.
  - $P(a)$ is true.
  - For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.
- Then the statement for all integers $n \geq a$, $P(n)$ is true.

Inductive Proofs Must Have

- Base Case (value)
  - Prove base case is true
- Inductive Hypothesis (value)
  - State what will be assumed in this proof
- Inductive Step (value)
  - Show
    - State what will be proven in the next section
  - Proof
    - Prove what is stated in the show portion
    - Must use the Inductive Hypothesis sometime

Prove this statement:

$$\forall n \in Z^{\geq 3}, 2n + 1 \leq 2^n$$

Base Case (n=3): $LHS: 2(3)+1 = 6+1 = 7$

$RHS: 2^3 = 8 \qquad LHS \leq RHS$

Inductive Hypothesis (n=k): $2k + 1 \leq 2^k$

Inductive Step (n=k+1):

Show: $2(k+1)+1 \leq 2^{k+1}$

Proof:

Prove this statement:

$$\forall n \in Z^{\geq 3}, 2n+1 \leq 2^n$$

Inductive Step (n=k+1):

Show: $2(k+1)+1 \leq 2^{k+1}$

Proof: $2k+2+1 \leq 2^{k+1} \Rightarrow 2k+1+2 \leq 2^{k+1} \Rightarrow 2k+1+2 \leq 2^k \cdot 2$

$IH \Rightarrow 2(k+1) \leq 2^k \Rightarrow (\text{Multiply by two}) \Rightarrow 4k+2 \leq 2^k \cdot 2$

New goal: $2k+1+2 < 4k+2$

$$2k+1 < 4k$$

$$1 < 2k$$

$$\frac{1}{2} \leq k$$

Which is true since k⩾3.

So $2k+1+2 \leq 4k+2 \leq 2^k \cdot 2$ and $2(k+1)+1 \leq 2^{k+1}$

Another Example

- For all integers $n \geq 1$, $3 \,\Big|\, \left(2^{2n} - 1\right)$

Sets

- Set
  - Notation – $\in$ versus $\subseteq$
  - Definitions – Subset, proper subset, partitions/disjoint sets
  - Operations ($\cup$, $\cap$, $-$, $'$, $\times$)
  - Properties and inference rules
  - Venn diagrams
  - Empty set properties
- Proofs
  - Element argument, set equality
  - Propositional logic / predicate calculus
  - Inference rules
  - Counterexample
  - Types – generic particular, induction, contra's, CW
- Russell's Paradox (The Barber's Puzzle) & Halting Problem

Set Operations

Formal Definitions and Venn Diagrams

Union:

$$A \bigcup B = \{x \in U \mid x \in A \vee x \in B\}$$

Intersection:

$$A \bigcap B = \{x \in U \mid x \in A \wedge x \in B\}$$

Complement:

$$A^c = A' = \{x \in U \mid x \notin A\}$$

Difference:

$$A - B = \{x \in U \mid x \in A \wedge x \notin B\}$$

$$A - B = A \bigcap B'$$

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in (X \cup Y) \Leftrightarrow x \in X$ or $x \in Y$
2. $x \in (X \cap Y) \Leftrightarrow x \in X$ and $x \in Y$
3. $x \in (X - Y) \Leftrightarrow x \in X$ and $x \notin Y$
4. $x \in X^c \Leftrightarrow x \notin X$
5. $(x, y) \in (X \times Y) \Leftrightarrow x \in X$ and $y \in Y$

Properties of Sets (Theorems 5.2.1 & 5.2.2)

- Inclusion $A \bigcap B \subseteq A$ $\qquad$ $A \bigcap B \subseteq B$

  $A \subseteq A \bigcup B$ $\qquad$ $B \subseteq A \bigcup B$

- Transitivity

  $$A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$$

- DeMorgan's for Complement

  $(A \bigcup B)' = A' \bigcap B'$ $\qquad$ $(A \bigcap B)' = A' \bigcup B'$

- Distribution of union and intersection

  $$A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$$
  $$A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$$

Prove A=C

$A=\{n \in Z \mid \exists p \in Z, n = 2p\}$

$C=\{m \in Z \mid \exists q \in Z, m = 2q-2\}$

Does A=D

$A=\{x \in Z \mid \exists p \in Z, x = 2p\}$

$D=\{y \in Z \mid \exists q \in Z, y = 3q+1\}$

Easy to disprove universal statements!

HW10, Problem 2

- Did yesterday in class …

Counting

- Counting elements in a list
  - How many in list are divisible by x
- Probability – likelihood of an event

$$P(E) = \frac{n(E)}{n(S)}$$

- Permutations – with and without repetition

$$P(n,r) = {}_rP_n = \frac{n!}{(n-r)!}$$

- Multiplication rule $\prod_{i=1}^{n} c(i)$
  - Tournament play
  - Rearranging letters in words
  - Where it doesn't work
- Difference rule – If A is a finite set and $B \subseteq A$, then $n(A - B) = n(A) - n(B)$
- Addition rule – If $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k = A$ and $A_1, A_2, A_3, \ldots, A_k$ are pairwise disjoint, then $n(A) = n(A_1) + n(A_2) + n(A_3) + \ldots + n(A_k)$
- Inclusion/exclusion rule
- Combinations – with and without repetition, categories

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$$

$$\binom{r+n-1}{r}$$

- Binomial theorem (Pascal's Triangle)

Prove: # elements in list = $n - m + 1$

- Base case (List of size 1, $x=y$)
  - $y - x + 1 = y - y + 1$ (by substitution) $= 1$
- IH (generic x, $y=k$ [where $x \leq k$])
  - Assume size of list x to k, is $k - x + 1$
- IS
  - Show size of list x to $k + 1$, is $(k + 1) - x + 1$
  - Prove
    - Split into two lists …

How many in list divisible by something

- How many positive three digit integers are there?
  - (this means only the ones that require 3 digits)
  - 999 – 100 + 1 = 900
- How many three digit integers are divisible by 5?
  - think about the definition of divisible by

    $x \mid y \leftrightarrow \exists\, k \in Z,\ y = kx$ and then count the k's that work

100, 101, 102, 103, 104, 105, 106,… 994, 995, 996, 997, 998, 999

20*5 21*5 … 199*5

  - count the integers between 20 and 199
  - 199 – 20 + 1 = 180

Multiplication Rule

- 1^{st} step can be performed n_1 ways
- 2^{nd} step can be performed n_2 ways
- …
- K^{th} step can be performed n_k ways

- Operation can be performed $n_1 * n_2 * \ldots * n_k$ ways

- Cartesian product   n(A)=3, n(B)=2, n(C)=4
  - n(AxBxC) = 24
  - n(AxB) = 6, n((AxB)xC) = 24

Multiplication Rule

- If there are n steps to a decision, each step having c(k) choices, the total number of choices is:

$$\prod_{i=1}^{n} c(i)$$

Permutation Example

- How many ways to take a picture?
  - With 1 person?
  - With 2 people?
  - With 3 people?
  - With 4 people?
  - With 5 people?

- Number of ways to arrange n objects
  - n!
  - $10^n < n!$

Difference Rule Formally

- If A is a finite set and $B \subseteq A$, then

$$n(A - B) = n(A) - n(B)$$

- One application:

  probability of the complement of an event

$$P(E') = P(E^c) = 1 - P(E)$$

Addition Rule Formally

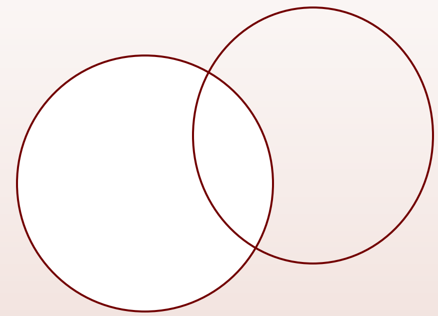
- If $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k = A$
  and $A_1, A_2, A_3, \ldots, A_k$ are pairwise disjoint
  (i.e. if these subsets form a partition of A)

- $n(A) = n(A_1) + n(A_2) + n(A_3) + \ldots + n(A_k)$

Inclusion/Exclusion Rule

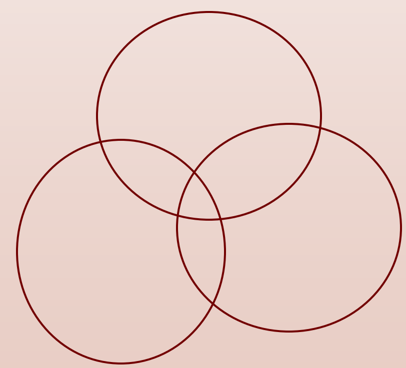
- If there are two sets:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$



- If there are three sets:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C)$$
$$- n(A \cap B) - n(A \cap C) - n(B \cap C)$$
$$+ n(A \cap B \cap C)$$



r-Permutations

- If there are n things in the set, and you want to line-up only r of them.

$$P(n,r) = {}_rP_n = \frac{n!}{(n-r)!}$$

- Example:
  - Class = {Alice, Bob, Carol, Dan}
  - Select a president and a vice president to represent the class

Permutations

- What if just want 5-letter words from COMPUTER?

$= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$

$$= \frac{8!}{(8-5)!}$$

Permutation w/ Repeated Elements

- What about NEEDLES
- $NE_1E_2DLE_3S = 7!$
  - $NE_1E_2DLE_3S$, $NE_1E_3DLE_2S$,
  - $NE_2E_1DLE_3S$, $NE_2E_3DLE_1S$
  - $NE_3E_1DLE_2S$, $NE_3E_2DLE_1S$

$$= \frac{7!}{6} = \frac{7!}{3!}$$

Combinations

- Different ways of selecting objects
  - Counting subsets
  - Without duplication / all items distinguishable
  - Note: <u>order is not taken into account</u>

$$\forall n, r \in Z^{\geq 0} where\ n \geq r,$$

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$$

- Examples:
  - Class = {Alice, Bob, Carol, Dan}
    - Select two class representatives
    - Select three class representatives

Combinations with Repitition

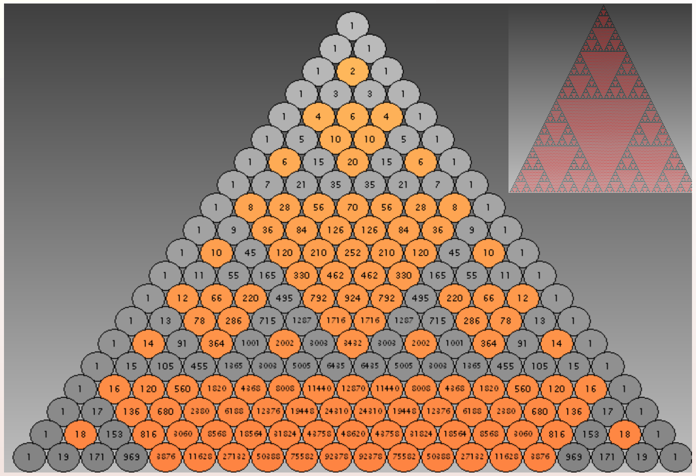
- In general
  - n – categories
  - r – items to choose from
  - Repetition allowed and order doesn't count

$$\underbrace{\mathbf{x\,x\,x}}_{\mathbf{r}}\ \underbrace{\mathbf{|\,|\,|}}_{\mathbf{n\,-\,1}}$$

$$\binom{r+n-1}{r}$$

Binomial Theorem

- $(x+y)^2$
- $(x+y)^3$
- …
- $(x+y)^n$



- Given any real numbers a and b and any nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$