CMSC 250
Discrete Structures

Functions

Terminology

- Domain: set which holds the values to which we apply the function
- Co-domain: set which holds the possible values (results) of the function
- Range: set of actual values received when applying the function to the values of the domain

Function

- A "total" function is a relationship between elements of the domain and elements of the co-domain where each and every element of the domain relates to one and only one value in the co-domain
- A "partial" function does not need to map every element of the domain.
- f: X $\rightarrow$Y
  - f is the function name
  - X is the domain
  - Y is the co-domain
  - The range of f is: $\{y \in Y \mid \exists y \in Y$ such that $f(x)=y\}$
  - $x \in X$ $y \in Y$ f sends x to y
  - $f(x) = y$ f of x; value of f at x; image of x under f

Formal Definitions

- Range of f = $\{y \in Y \mid \exists x \in X, f(x) = y\}$
  - where X is the domain and Y is the co-domain
- Inverse image of y = $\{x \in X \mid f(x) = y\}$
  - the set of things that map to y
- Arrow Diagrams
  - Determining if they are functions using an arrow diagram

Examples

- $f(x) = \sqrt{x}$
- $f(a/b) = a + b$; $f:\mathbf{Q} \rightarrow \mathbf{Z}$
- $f(n) = n^2$, for $n \in \mathbf{Z}^+$; $f:\mathbf{Z}^+ \rightarrow \mathbf{Z}^+$
  - $f(3)$
  - $f(-1)$
  - $f(f(4))$
- $f(n) = 3$
- $f((x,y,z)) = (x \wedge y) \vee z$; $f: B \times B \times B \rightarrow B$
  - $f((T,F,T))$
  - $f((F,F,T))$

Terminology of Functions

- Equality of Functions
  $\forall f,g \in \{\text{functions}\}, f=g \leftrightarrow \forall x \in X, f(x)=g(x)$
- F <u>is a One-to-One</u> (or Injective) Function iff
  $\forall x_1,x_2 \in X \quad F(x_1) = F(x_2) \rightarrow x_1=x_2$
  $\forall x_1,x_2 \in X \quad x_1 \neq x_2 \rightarrow F(x_1) \neq F(x_2)$
- F <u>is NOT a One-to-One</u> Function iff
  $\exists\, x_1,x_2 \in X, (F(x_1) = F(x_2)) \wedge (x_1 \neq x_2)$
- F <u>is an Onto</u> (or Surjective) Function iff
  $\forall y \in Y \quad \exists x \in X, \; F(x) = y$
- F <u>is NOT an Onto</u> Function iff
  $\exists y \in Y\ \forall x \in X, \; F(x) \neq y$

Proving Functions one-to-one and onto

$f: R \rightarrow R \qquad f(x)=3x-4$

- Prove or give a counter example that f is one-to-one:
  - Use def:

$$\forall x_1, x_2 \in R, f(x_1) = f(x_2) \rightarrow x_1 = x_2$$

- Prove or give a counter example that f is onto
  - Use def:

$$\forall y \in R \exists x \in R, f(x) = y$$

Examples

- $f(x) = 3x + 9$; $f:\mathbf{Z}\to\mathbf{Z}$
  - $f:\mathbf{Q}\to\mathbf{Z}$
  - $f:\mathbf{R}\to\mathbf{R}$

- $f(x) = 5x - 3$; $f:\mathbf{Z}\to\mathbf{Z}$

One-to-One Correspondence (Bijection)

- F:X $\rightarrow$Y is bijective $\leftrightarrow$

  F:X $\rightarrow$Y is one-to-one & onto
- F:X $\rightarrow$Y is bijective $\leftrightarrow$

  It has an inverse function

$$\exists F^{-1} : Y \rightarrow X$$

$$\forall x \in X, F(x) = y \rightarrow F^{-1}(y) = x$$

$$\forall y \in Y, F^{-1}(y) = x \rightarrow F(x) = y$$

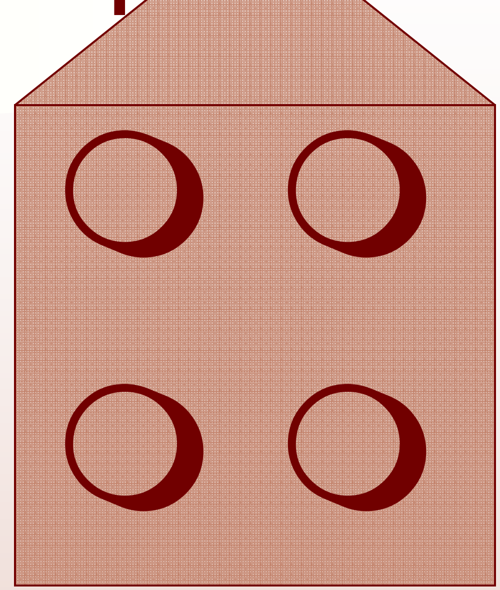
Proving Something is a Bijection

- $F: Z \rightarrow Z \quad F(x)=x+1$

- Prove it is one-to-one
- Prove it is onto
- Then it is a bijection

- So it has an inverse function
  - find F^{-1}

Pigeonhole Principle



- Basic Form

  A function from one finite set to a smaller finite set cannot be one-to-one; there must be at least two elements in the domain that have the same image in the co-domain.

Pigeonhole Principle

- If you have n items (pigeons) and m slots (pigeon holes) where $n > m$, then at least one slot contains more than one item.
- If the co-domain is smaller than the domain, than the function cannot be one-to-one.
- $\forall f: X \rightarrow Y$ where $n(X)$ and $n(Y)$ are finite

  $n(X) > n(Y) \rightarrow \exists a,b \in X$ where $a \neq b$ s.t. $f(a)=f(b)$
- $\forall f: X \rightarrow Y$ where $n(X)$ and $n(Y)$ are finite

  $\forall a,b \in X$ where $a \neq b$ s.t. $f(a) \neq f(b) \rightarrow n(X) > n(Y)$

Examples

- Can these be one-to-one:
  - f: birthdays $\rightarrow$ months
  - f: 13 specific birthdays $\rightarrow$ months
  - f: 11 specific birthdays $\rightarrow$ months
    - Is it certain to be?
  - f: birthdays in this class $\rightarrow$ possible birthdays
    - Can this be?
    - Is it?
    - What are chances no two have same birthday?
    - What are chances two do have same birthday?

$$\frac{P(366, \#\text{ of students})}{366^{\#\text{ of students}}}$$

$$1 - \frac{P(366, \#\text{ of students})}{366^{\#\text{ of students}}}$$

Examples

- Using this class as the domain,
  - Must two people share a birth month?
  - Must two people share a birthday?

- A = {1,2,3,4,5,6,7,8}
  - If I select 5 integers at random from this set, must two of the numbers sum to 9?
  - If I select 4 integers?

Examples

- S = {6 black socks, 5 blue socks, 8 red socks}
- How many do you have to pick from S before knowing for certain (without looking) that you have a matching pair?
- Draw arrow diagram …

Another Example

- You have an urn containing:
  - 7 red balls
  - 5 yellow balls
  - 9 green balls
- What if you remove one ball at a time at random without putting them back in, how many do you need to remove to ensure you have removed one from each color?
- What if you replace the ball after you remove it?
- What if you don't replace them, but only care that you have removed at least two colors?

Other (more useful) Forms of the Pigeonhole Principle

- Generalized Pigeonhole Principle
  - For any function f from a finite set X to a finite set Y and for any positive integer k, if $n(X) > k*n(Y)$, then there is some $y \in Y$ such that y is the image of at least k+1 distinct elements of X.

- Contrapositive Form of Generalized Pigeonhole Principle
  - For any function f from a finite set X to a finite set Y and for any positive integer k, if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most $k*n(Y)$ elements.

Examples

- Using Generalized Form:
  - Assume 50 people in the room, how many must share the same birth month?
  - $n(A)=5$ $n(B)=3$ $F:P(A)\rightarrow P(B)$

    How many elements of P(A) must map to a single element of P(B)?
- Using Contrapositive of the Generalized Form:
  - $G:X\rightarrow Y$ Where Y is the set of 2 digit integers that do not have distinct digits. Assuming no more than 5 elements of X can map to a single element of Y, how big can X be?

Another Example

- You have 10 cars each holds up to 4 people.

- Can you fit 40 people?
- Can you fit 41 people?
- If you have 30 people does some car have to have 4 people in it?
  - What is the largest number that at least one car will be required to have?
  - Think of as sets P and C, and function $f: P \rightarrow C$ and an answer $k \in Z$
  - $f(p) = c$ (person p goes in car c)
  - $n(P) = 30$, $n(C) = 10$, goal: $n(P) \leq k \cdot n(C)$
  - $30 \leq k \cdot 10 \Rightarrow 3 \leq k$ (at least one car must have 3 people)
- What about 31 people
  - $31 \leq k \cdot 10 \Rightarrow 3.1 \leq k$ (at least one car must have 4 people)

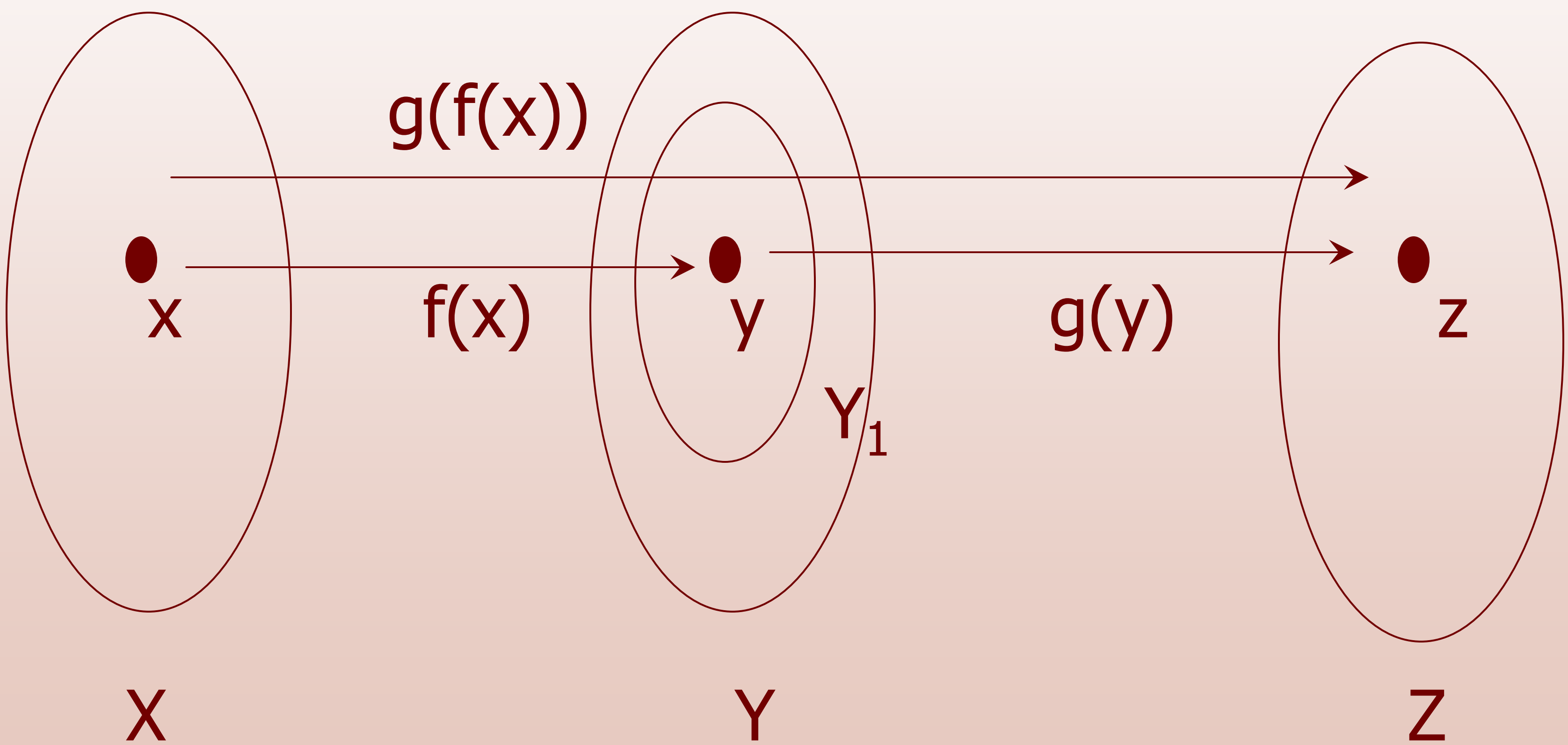
More Examples

- If you have 85 people, how many must have the same last initial?

- There are 5 buses that can each carry up to 25 students. There are 100 students to carry. Show that at least 3 buses must have at least 16 students each.

Composition of Functions

- $f: X \to Y_1$ and $g: Y \to Z$ where $Y_1 \subseteq Y$
  - $g \circ f: X \to Z$ where $\forall x \in X$, $g(f(x)) = g \circ f(x)$



Composition on Finite Sets

- Example
  - X = {1,2,3}
  - Y1 = {a,b,c,d}
  - Y={a,b,c,d,e}
  - Z = {x,y,z}

f(1)=c	g(a)=y	$(g \circ f)(1)=g(f(1))=z$
f(2)=b	g(b)=y	$(g \circ f)(2)=g(f(2))=y$
f(3)=a	g(c)=z	$(g \circ f)(3)=g(f(3))=y$
	g(d)=x	
	g(e)=x	

Composition for Infinite Sets

- $f: Z \rightarrow Z \quad f(n) = n+1$
- $g: Z \rightarrow Z \quad g(n) = n^2$

- $(g \circ f)(n) = g(f(n)) = g(n+1) = (n+1)^2$
- $(f \circ g)(n) = f(g(n)) = f(n^2) = n^2+1$

- Note: $g \circ f \neq f \circ g$

Identity Function

- i_X the identity function for the domain X

  $i_X : X \rightarrow X \qquad \forall x \in X, i_X(x) = x$

- i_Y the identity function for the domain Y

  $i_Y : Y \rightarrow Y \qquad \forall y \in Y, i_Y(y) = y$

- Composition with the identity functions

Composition of a function with its inverse function

- $f \circ f^{-1} = i_Y$
- $f^{-1} \circ f = i_X$

- Composing a function with its inverse returns you to the starting place.

  (Note: $f: X \to Y$ and $f^{-1}: Y \to X$)

Function Properties w/ Composition

- If $f{:}X \to Y$ and $g{:}Y \to Z$ are both one-to-one, then $(g \circ f){:}X \to Z$ is one-to-one
- If $f{:}X \to Y$ and $g{:}Y \to Z$ are both onto, then $g \circ f{:}X \to Z$ is onto when $Y = Y_1$
- Can a function $f{:}X \to Y$ be one-to-one if $n(X) > n(Y)$?

Cardinality

- Comparing the "sizes" of sets
- $\forall A,B \in \{sets\}$, A and B have the same cardinality $\leftrightarrow$ there is a one-to-one correspondence from A to B
  - Card(A) = Card(B)

    $\leftrightarrow \exists f \in \{functions\}, f{:}A \rightarrow B \wedge f$ is a bijection
- As a relation (which we will talk more about next)
  - <u>Reflexive</u> – A has the same cardinality as A
  - <u>Symmetric</u> – If A has the same cardinality as B, B has the same cardinality as A
  - <u>Transitive</u> – If A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C

Sets of Integers

- $\mathbf{Z}^+ = \{1,2,3,4,\ldots\}$
  - Infinite set classified as "Countably Infinite"
- $\mathbf{Z}^{\geq 0}$
  - Infinite set classified as "Countably Infinite"
- $\mathbf{Z}$
  - $f:\mathbf{Z}^+ \rightarrow \mathbf{Z}$, $f(n) = \lfloor {}^{n}/_{2} \rfloor (-1)^{n+1}$
  - Prove that f is one-to-one and onto (left to reader)
  - Infinite set classified as "Countably Infinite"
- $\mathbf{Z}^{even}$
  - Infinite set classified as "Countably Infinite"

- $\text{Card.}(\mathbf{Z}^+) = \text{Card.}(\mathbf{Z}^{\geq 0}) = \text{Card.}(\mathbf{Z}) = \text{Card.}(\mathbf{Z}^{even})$

Real Numbers

- We'll take just a part of this infinite set
- Reals between 0 and 1 (non-inclusive)
  - $X = \{x \in R | \quad 0<x<1\}$
- All elements of X can be written as
  - $0.a_1a_2a_3\ldots\ a_n\ldots$

Cantor's Proof

- Assume the set $X = \{x \in R | 0 < x < 1\}$ is countable
- Then the elements in the set can be listed

  $0.a_{11}a_{12}a_{13}a_{14}\ldots a_{1n}\ldots$

  $0.a_{21}a_{22}a_{23}a_{24}\ldots a_{2n}\ldots$

  $0.a_{31}a_{32}a_{33}a_{34}\ldots a_{3n}\ldots$

  … … … …
- Select the digits on the diagonal
- build a number

$$d_n = \begin{cases} 1 & if\ a_{nn} \neq 1 \\ 2 & if\ a_{nn} = 1 \end{cases}$$

- d differs in the n^{th} position from the n^{th} in the list

Cardinality and Subsets

- Since any subset of a countably infinite set is countably infinite and the subset of the reals is uncountable, the set of all reals is also uncountable

- All Reals
  - The set of all reals can't be countably infinite
  - So it is uncountably infinite
  - Card.($\{x \in R | 0<x<1\}$) = Card.(R)

Positive Rationals $\mathbf{Q}^+$

- Card.(Q^+) =?= Card.(Z)
  - Yes – see book for proof
- Card.(Q^+) =?= Card.(R)
  - No, because it is the same as Z

log function properties (from back cover of textbook)

definition of log: $\log_b x = y \leftrightarrow b^y = x$

$\log_b b = 1$	$\log_b x = \dfrac{\log_n x}{\log_n b}$
$\log_b x^a = a * \log_b x$	

$$\log_b (x * y) = \log_b x + \log_b y$$

$$\log_b (\frac{x}{y}) = \log_b x - \log_b y$$

$$\log_b (x) = \log_b y \rightarrow x = y$$