CMSC 132: Object-Oriented Programming II

Algorithmic Complexity II

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Overview

- Critical sections
- Comparing complexity
- Types of complexity analysis

Analyzing Algorithms

- **Goal**
  - **Find asymptotic complexity of algorithm**
- **Approach**
  - **Ignore less frequently executed parts of algorithm**
  - **Find critical section of algorithm**
  - **Determine how many times critical section is executed as function of problem size**

Critical Section of Algorithm

- Heart of algorithm
- Dominates overall execution time
- Characteristics
  - Operation central to functioning of program
  - Contained inside deeply nested loops
  - Executed as often as any other part of algorithm
- Sources
  - Loops
  - Recursion

Critical Section Example 1

- Code (for input size n)

1. A
2. for (int i = 0; i < n; i++)
3.   B
4. C

critical section

- Code execution
  - A $\Rightarrow$ once
  - B $\Rightarrow$ n times
  - C $\Rightarrow$ once
- Time $\Rightarrow 1 + n + 1 = O(n)$

Critical Section Example 2

- Code (for input size n)

```
1. A
2. for (int i = 0; i < n; i++)
3.    B
4.    for (int j = 0; j < n; j++)
5.       C
6. D
```

critical section

- Code execution
  - A ⇒ once
  - B ⇒ n times
  - C ⇒ n^2 times
  - D ⇒ once
- Time ⇒ $1 + n + n^2 + 1 = O(n^2)$

Critical Section Example 3

- Code (for input size n)
  1. A
  2. for (int i = 0; i < n; i++)
  3. for (int j = i+1; j < n; j++)
  4. B

critical section

- Code execution
  - A $\Rightarrow$ once
  - B $\Rightarrow$ $\frac{1}{2}\, n\, (n-1)$ times
- Time $\Rightarrow$ $1 + \frac{1}{2} n^2 = O(n^2)$

Critical Section Example 4

- Code (for input size n)
  1. A
  2. for (int i = 0; i < n; i++)
  3.   for (int j = 0; j < 10000; j++)
  4.   B

critical section

- Code execution
  - A ⇒ once
  - B ⇒ 10000 n times
- Time ⇒ 1 + 10000 n = O(n)

Critical Section Example 5

- **Code (for input size n)**

```
for (int i = 0; i < n; i++)
   for (int j = 0; j < n; j++)
      A
for (int i = 0; i < n; i++)
   for (int j = 0; j < n; j++)
      B
```

critical sections

- **Code execution**
  - **A $\Rightarrow$ n^2 times**
  - **B $\Rightarrow$ n^2 times**
- **Time $\Rightarrow$ $n^2 + n^2 = O(n^2)$**

Critical Section Example 6

- Code (for input size n)
  1. i = 1
  2. while (i < n) {
  3.   A
  4.   i = 2 × i

     }
  5. B

critical section

- Code execution
  - A ⇒ log(n) times
  - B ⇒ 1 times
- Time ⇒ log(n) + 1 = O(log(n))

Critical Section Example 7

- Code (for input size n)

```
DoWork (int n)
   if (n == 1)
      A
   else
      DoWork(n/2)
      DoWork(n/2)
```

critical sections

- Code execution
  - A ⇒ 1 times
  - DoWork(n/2) ⇒ 2 times
- Time(1) ⇒ 1   Time(n) = 2 × Time(n/2) + 1

Recursive Algorithms

- **Definition**
  - **An algorithm that calls itself**
- **Components of a recursive algorithm**

1. **Base cases**
   - **Computation with no recursion**
2. **Recursive cases**
   - **Recursive calls**
   - **Combining recursive results**

Recursive Algorithm Example

- **Code (for input size n)**

```
DoWork (int n)
  if (n == 1)
    A
  else
    DoWork(n/2)
    DoWork(n/2)
```

base case

recursive cases

Asymptotic Complexity Categories

Complexity	Name	Example
$O(1)$	Constant	Array access
$O(\log(n))$	Logarithmic	Binary search
$O(n)$	Linear	Largest element
$O(n \log(n))$	N log N	Optimal sort
$O(n^2)$	Quadratic	2D Matrix addition
$O(n^3)$	Cubic	2D Matrix multiply
$O(n^k)$	Polynomial	Linear programming
$O(k^n)$	Exponential	Integer programming
$O(n!)$	Factorial	Brute-force search TSP
$O(n^n)$	N to the N	

From smallest to largest

For size n, constant $k > 1$

Comparing Complexity

- Compare two algorithms
  - f(n), g(n)
- Determine which increases at faster rate
  - As problem size n increases
- Can compare ratio

$$\lim_{n \to \infty} \frac{f(n)}{g(n)}$$

  - If ∞, f() is larger
  - If 0, g() is larger
  - If constant, then same complexity

Complexity Comparison Examples

- $\log(n)$ vs. $n^{½}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \rightarrow \lim_{n\to\infty} \frac{\log(n)}{n^{½}} \rightarrow 0$$

- 1.001^{n} vs. n^{1000}

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \rightarrow \lim_{n\to\infty} \frac{1.001^{n}}{n^{1000}} \rightarrow ??$$

Not clear, use L'Hopital's Rule

Additional Complexity Measures

- Upper bound
  - Big-O $\Rightarrow$ $O(\ldots)$
  - Represents upper bound on # steps
- Lower bound
  - Big-Omega $\Rightarrow$ $\Omega(\ldots)$
  - Represents lower bound on # steps
- Combined bound
  - Big-Theta $\Rightarrow$ $\Theta(\ldots)$
  - Represents combined upper/lower bound on # steps
  - Best possible asymptotic solution

2D Matrix Multiplication Example

- Problem
  - C = A * B

- Lower bound
  - $\Omega(n^2)$ Required to examine 2D matrix
- Upper bounds
  - $O(n^3)$ Basic algorithm
  - $O(n^{2.807})$ Strassen's algorithm (1969)
  - $O(n^{2.376})$ Coppersmith & Winograd (1987)
- Improvements still possible (open problem)
  - Since upper & lower bounds do not match

Additional Complexity Categories

- **Name** **Description**
  - P Deterministic polynomial time
  - NP Nondeterministic polynomial time
  - PSPACE Polynomial space
  - EXPSPACE Exponential space
  - Decidable Can be solved by finite algorithm
  - Undecidable Not solvable by finite algorithm

- If a problem has a algorithm that solves it in time X, then the problem is said to be in X
  - e.g., matrix multiplication is in P

Why do we care?

- If a problem can't be solved with in P time, then no algorithm can solve *all* cases exactly in a *reasonable* amount of time
  - But there might be an algorithm that always quickly gives close approximations
  - Or an algorithm that gives quick exact answers for the cases we care about

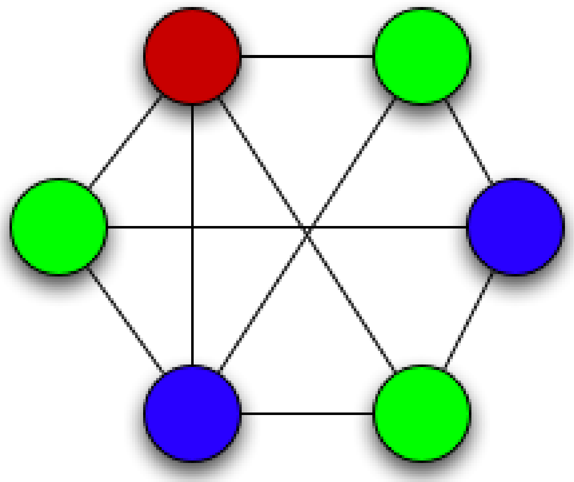
- Issues such as PSPACE vs. EXPSPACE are interesting theoretical issues, and sometime provide practice insight

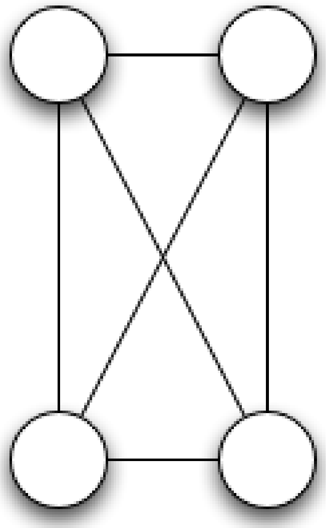
NP Time Algorithms

- Two ways of thinking about it
- First way:
  - Given a problem, and a potential answer, can we verify that the answer is correct in polynomial time?
- Second way:
  - Whenever the algorithm wants, it can clone itself in two
  - If either clone finds an answer, the entire system produces an answer

Graph 3-coloring

- Given a graph (vertices and undirected edges)
- Can you find a way to color each vertex either blue, red, or green
- Such that no two vertices connected by an edge have the same color?





Some graphs are hard

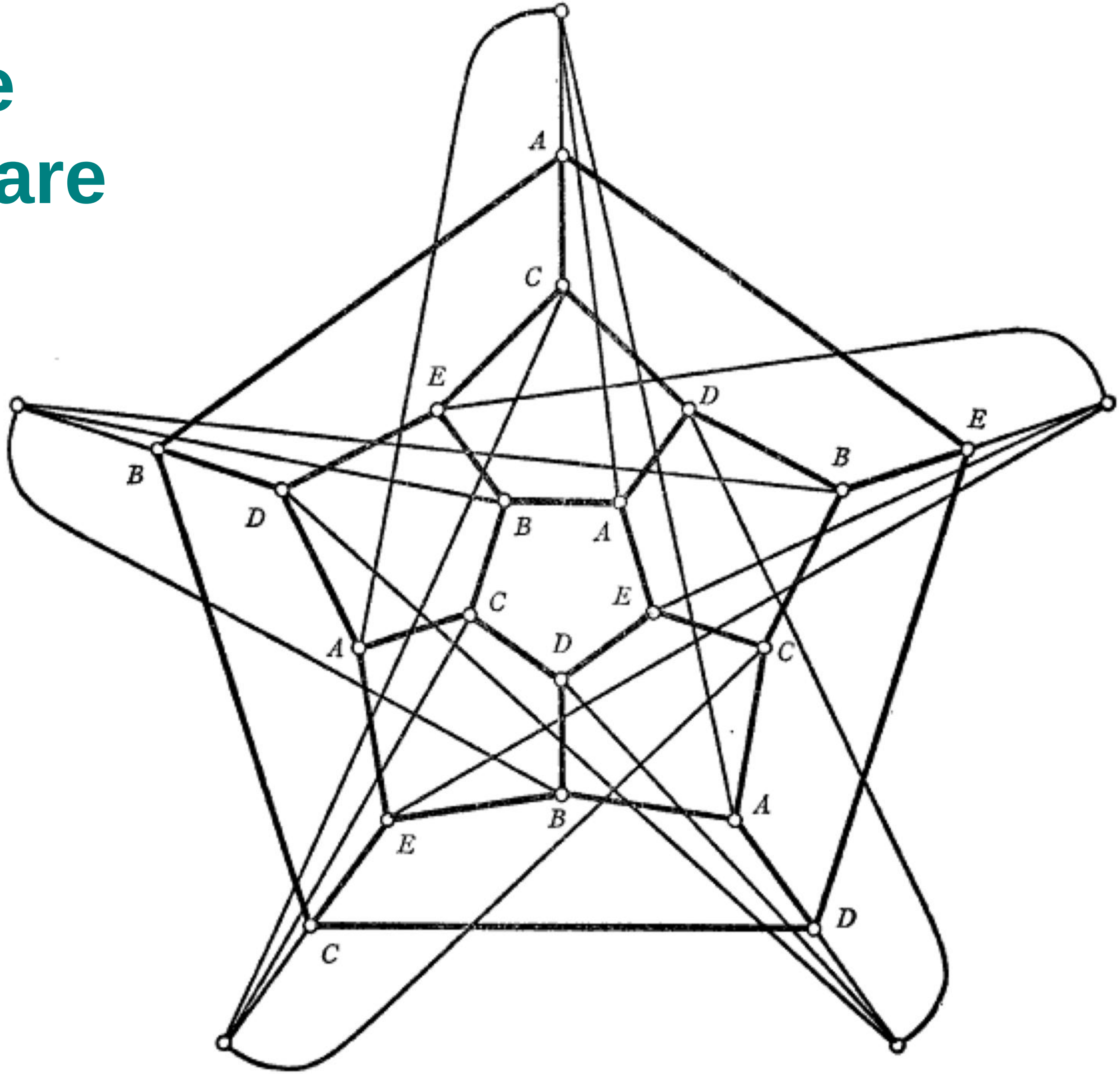


FIG. 1

3 coloring a graph is in NP

- **There exist NP algorithms to 3 color a graph**
  - **Guess a 3 coloring**
  - **Verify that the coloring is valid**
    - **Easy to do in O(n) time**

- **No one knows if there exists a polynomial time algorithm to find a 3 coloring for an arbitrary graph**
  - **If you come up with one, even one that runs in time $O(n^{100})$, you win one million dollars**
  - **Seriously**

NP Time Algorithm

- **Many interesting problems are solvable with an NP algorithm, but not known to be solvable with a P algorithm**
  - **Boolean satisfiability**
  - **Traveling salesman problem (TLP)**
  - **Bin packing**
- **Key to solving many optimization problems**
  - **Most efficient trip routes**
  - **Most efficient schedule for employees**
  - **Most efficient usage of resources**

NP Complete problems

- Some problems are NP-complete
  - They are in NP
  - And if a polynomial time algorithm existed for the program
  - Then every single last problem in NP could be solved in polynomial time
- Almost all problems that are in NP but are not known to be in P are NP-complete
  - But not all

P = NP?

- **Are NP problems solvable in polynomial time?**
  - **Prove P=NP**
    - **Show polynomial time solution exists for any NP-complete problem**
  - **Prove P≠NP**
    - **Show no polynomial-time solution possible for some problem in NP**
    - **The expected answer**
- **The most important open problem in CS**
  - **$1 million prize offered by Clay Math Institute**
  - **Plus front page, NY Times, job offers galore, instant Ph.D. in Computer Science**

Algorithmic Complexity Summary

- Asymptotic complexity
  - Fundamental measure of efficiency
  - Independent of implementation & computer platform
- Learned how to
  - Examine program
  - Find critical sections
  - Calculate complexity of algorithm
  - Compare complexity