CMSC 132: Object-Oriented Programming II

Algorithmic Complexity II

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Overview

- **Critical sections**
- **Comparing complexity**
- **Types of complexity analysis**

Analyzing Algorithms

- **Goal**
  - **Find asymptotic complexity of algorithm**
- **Approach**
  - **Ignore less frequently executed parts of algorithm**
  - **Find critical section of algorithm**
  - **Determine how many times critical section is executed as function of problem size**

Critical Section of Algorithm

- **Heart of algorithm**
- **Dominates overall execution time**
- **Characteristics**
  - **Operation central to functioning of program**
  - **Contained inside deeply nested loops**
  - **Executed as often as any other part of algorithm**
- **Sources**
  - **Loops**
  - **Recursion**

Critical Section Example 1

- Code (for input size n)
  1. A
  2. for (int i = 0; i < n; i++)
  3.   B
  4. C

critical section

- Code execution
  - A $\Rightarrow$ once
  - B $\Rightarrow$ n times
  - C $\Rightarrow$ once
- Time $\Rightarrow 1 + n + 1 = O(n)$

Critical Section Example 2

- Code (for input size n)

1. A
2. for (int i = 0; i < n; i++)
3.   B
4.   for (int j = 0; j < n; j++)
5.   C
6. D

critical section

- Code execution
  - A ⇒ once
  - B ⇒ n times
  - C ⇒ n^2 times
  - D ⇒ once
- Time ⇒ $1 + n + n^2 + 1 = O(n^2)$

Critical Section Example 3

- Code (for input size n)

```
1. A
2. for (int i = 0; i < n; i++)
3.    for (int j = i+1; j < n; j++)
4.       B
```

critical section

- Code execution
  - A ⇒ once
  - B ⇒ ½ n (n-1) times
- Time ⇒ 1 + ½ n^2 = $O(n^2)$

Critical Section Example 4

- **Code (for input size n)**
  1. **A**
  2. **for (int i = 0; i < n; i++)**
  3. **for (int j = 0; j < 10000; j++)**
  4. **B**

critical section

- **Code execution**
  - **A ⇒ once**
  - **B ⇒ 10000 n times**
- **Time ⇒ 1 + 10000 n = O(n)**

Critical Section Example 5

- **Code (for input size n)**

```
for (int i = 0; i < n; i++)
   for (int j = 0; j < n; j++)
      A
for (int i = 0; i < n; i++)
   for (int j = 0; j < n; j++)
      B
```

critical sections

- **Code execution**
  - **A $\Rightarrow$ n^2 times**
  - **B $\Rightarrow$ n^2 times**
- **Time $\Rightarrow$ $n^2 + n^2 = O(n^2)$**

Critical Section Example 6

- **Code (for input size n)**
  1. **i = 1**
  2. **while (i < n) {**
  3. **A**
  4. **i = 2 × i**

     }
  5. **B**

critical section

- **Code execution**
  - **A ⇒ log(n) times**
  - **B ⇒ 1 times**
- **Time ⇒ log(n) + 1 = O(log(n))**

Critical Section Example 7

- Code (for input size n)

```
DoWork (int n)
   if (n == 1)
      A
   else
      DoWork(n/2)
      DoWork(n/2)
```

critical sections

- Code execution
  - A $\Rightarrow$ 1 times
  - DoWork(n/2) $\Rightarrow$ 2 times
- Time(1) $\Rightarrow$ 1   Time(n) = 2 $\times$ Time(n/2) + 1

Recursive Algorithms

- **Definition**
  - **An algorithm that calls itself**
- **Components of a recursive algorithm**

1. **Base cases**
   - **Computation with no recursion**
2. **Recursive cases**
   - **Recursive calls**
   - **Combining recursive results**

Recursive Algorithm Example

- **Code (for input size n)**

```
DoWork (int n)
   if (n == 1)
      A
   else
      DoWork(n/2)
      DoWork(n/2)
```

base case

recursive cases

Comparing Complexity

- **Compare two algorithms**
  - **$f(n)$, $g(n)$**
- **Determine which increases at faster rate**
  - **As problem size n increases**
- **Can compare ratio**

$$\lim_{n \to \infty} \frac{f(n)}{g(n)}$$

  - **If ∞, f() is larger**
  - **If 0, g() is larger**
  - **If constant, then same complexity**

Complexity Comparison Examples

- $\log(n)$ vs. $n^{½}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \rightarrow \lim_{n\to\infty} \frac{\log(n)}{n^{½}} \rightarrow 0$$

- 1.001^{n} vs. n^{1000}

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \rightarrow \lim_{n\to\infty} \frac{1.001^{n}}{n^{1000}} \rightarrow ??$$

Not clear, use L'Hopital's Rule

Additional Complexity Measures

- Upper bound
  - Big-O $\Rightarrow$ O(…)
  - Represents upper bound on # steps
- Lower bound
  - Big-Omega $\Rightarrow$ Ω(…)
  - Represents lower bound on # steps
- Combined bound
  - Big-Theta $\Rightarrow$ Θ(…)
  - Represents combined upper/lower bound on # steps
  - Best possible asymptotic solution

2D Matrix Multiplication Example

- Problem
  - C = A * B
- Lower bound
  - $\Omega(n^2)$ Required to examine 2D matrix
- Upper bounds
  - $O(n^3)$ Basic algorithm
  - $O(n^{2.807})$ Strassen's algorithm (1969)
  - $O(n^{2.376})$ Coppersmith & Winograd (1987)
- Improvements still possible (open problem)
  - Since upper & lower bounds do not match

Additional Complexity Categories

Name	Description
P	Deterministic polynomial time
NP	Nondeterministic polynomial time
PSPACE	Polynomial space
EXPSPACE	Exponential space
Decidable	Can be solved by finite algorithm
Undecidable	Not solvable by finite algorithm

- If a problem has a algorithm that solves it in time X, then the problem is said to be in X
  - e.g., matrix multiplication is in P

Why do we care?

- If a problem can't be solved with in P time, then no algorithm can solve *all* cases exactly in a *reasonable* amount of time
  - But there might be an algorithm that always quickly gives close approximations
  - Or an algorithm that gives quick exact answers for the cases we care about

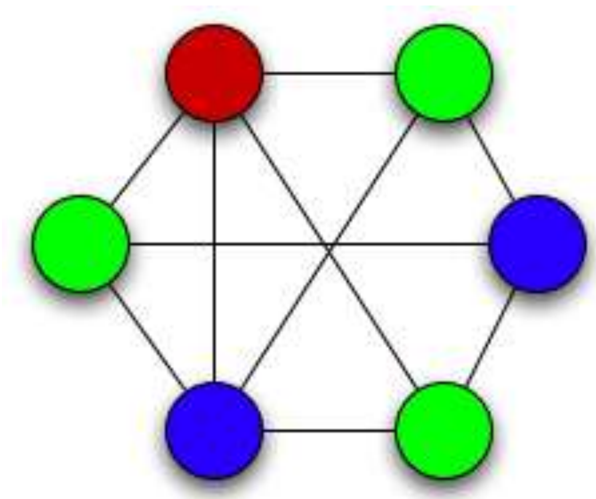
- Issues such as PSPACE vs. EXPSPACE are interesting theoretical issues, and sometime provide practice insight

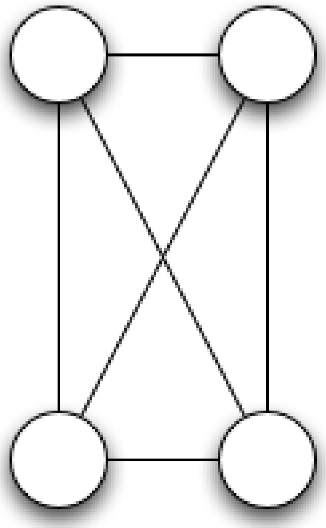
NP Time Algorithms

- **Two ways of thinking about it**
- **First way:**
  - **Given a problem, and a potential answer, can we verify that the answer is correct in polynomial time?**
- **Second way:**
  - **Whenever the algorithm wants, it can clone itself in two**
  - **If either clone finds an answer, the entire system produces an answer**

Graph 3-coloring

- Given a graph (vertices and undirected edges)
- Can you find a way to color each vertex either blue, red, or green
- Such that no two vertices connected by an edge have the same color?





Some graphs are hard

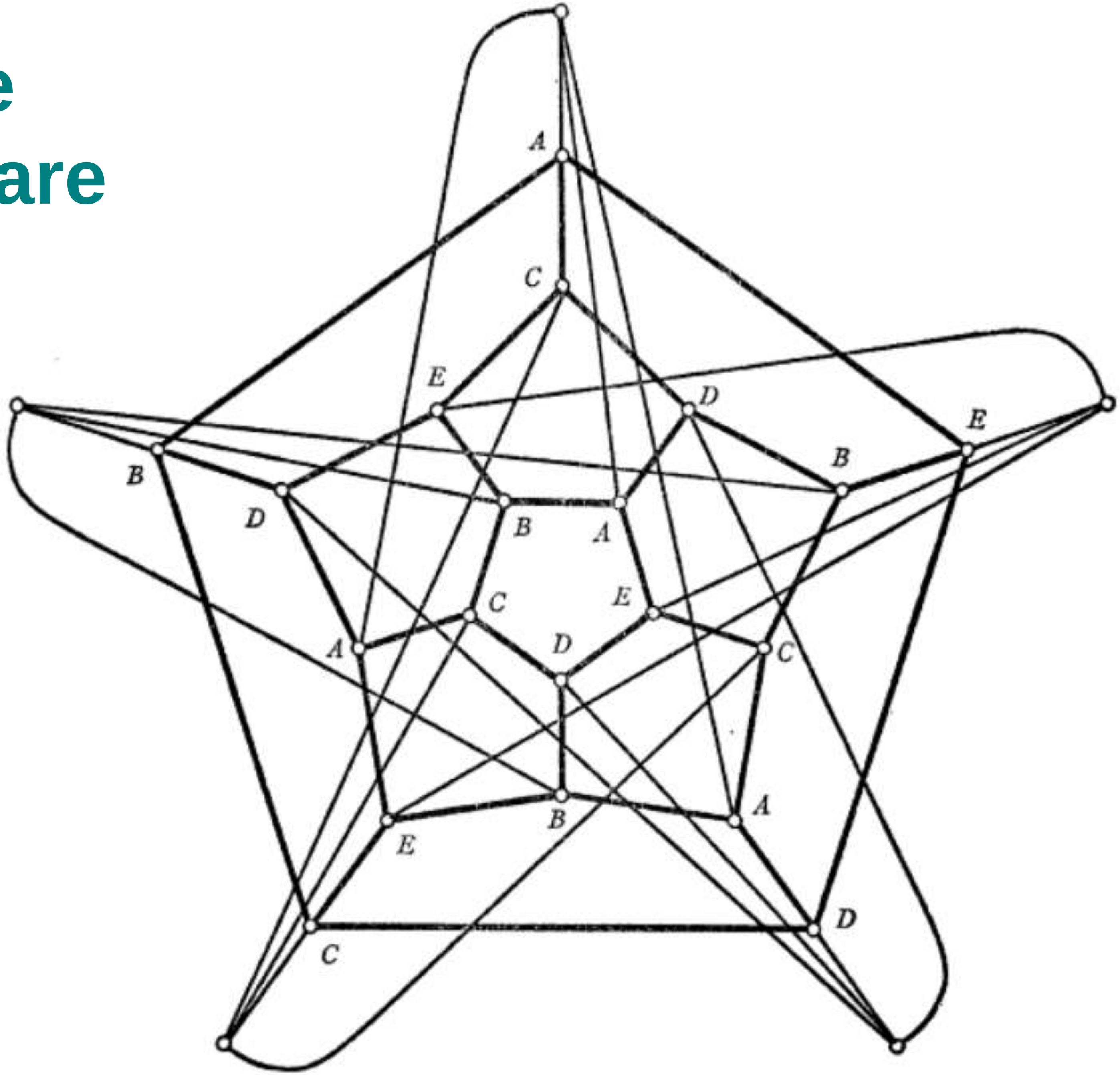


FIG. 1

3 coloring a graph is in NP

- There exist NP algorithms to 3 color a graph
  - Guess a 3 coloring
  - Verify that the coloring is valid
    - Easy to do in O(n) time
- No one knows if there exists a polynomial time algorithm to find a 3 coloring for an arbitrary graph
  - If you come up with one, even one that runs in time $O(n^{100})$, you win one million dollars
  - Seriously

NP Time Algorithm

- Many interesting problems are solvable with an NP algorithm, but not known to be solvable with a P algorithm
  - Boolean satisfiability
  - Traveling salesman problem (TLP)
  - Bin packing
- Key to solving many optimization problems
  - Most efficient trip routes
  - Most efficient schedule for employees
  - Most efficient usage of resources

NP Complete problems

- Some problems are NP-complete
  - They are in NP
  - And if a polynomial time algorithm existed for the program
  - Then every single last problem in NP could be solved in polynomial time
- Almost all problems that are in NP but are not known to be in P are NP-complete
  - But not all

P = NP?

- Are NP problems solvable in polynomial time?
  - Prove P=NP
    - Show polynomial time solution exists for any NP-complete problem
  - Prove P≠NP
    - Show no polynomial-time solution possible for some problem in NP
    - The expected answer
- The most important open problem in CS
  - $1 million prize offered by Clay Math Institute
  - Plus front page, NY Times, job offers galore, instant Ph.D. in Computer Science

Algorithmic Complexity Summary

- **Asymptotic complexity**
  - **Fundamental measure of efficiency**
  - **Independent of implementation & computer platform**
- **Learned how to**
  - **Examine program**
  - **Find critical sections**
  - **Calculate complexity of algorithm**
  - **Compare complexity**