CMSC 330:

Operational Semantics

Introduction

- We've looked at several formal methods for defining the syntax of a programming language
  - Regular expressions
  - Context-free grammars

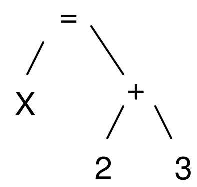
- What about formal methods for defining the semantics of a programming language?
  - I.e., what does a program mean?

Roadmap: Compilation of Program

Grammar:
S→A
A → id = E
E → T+E | T
T → P * T | P
P → id | n | (E)

Program:
X = 2 + 3

Recursive Descent Parsing: AST



Postorder: postfix
X 2 3 + =

Program semantics: What does program mean

?

Compilation:
Push X
Push 2
Push 3
Add top 2
Assign top to second

Is this correct? What is program supposed to do?

Formal Semantics

- Formal semantics of a programming language
  - Mathematical model of all possible computations performed by programs written in that language
- Three main approaches to formal semantics
  - Denotational
  - Operational
  - Axiomatic

Formal Semantics (cont.)

- Denotational semantics
  - Translate parts of language into another language
    - Usually a mathematical function
    - Meaning of program expressed as mathematical object
- Operational semantics
  - Describe effect of parts of language
    - Usually on (a mathematical model of) an abstract machine
    - For lambda calculus, can use syntactic transformations
- Axiomatic semantics
  - Describe each part of language through logical axioms
    - Foundation of many program verification systems

Operational Semantics

- We will briefly look at operational semantics
  - Using a subset of OCaml as an example
- Useful for
  - Specifying the meaning of a program
  - Proving the correctness of an algorithm
    - Through formal verification
    - For cryptographic algorithms, combinatorial circuits, etc…
    - Currently limited to smaller programs

Roadmap: Semantics of a Program

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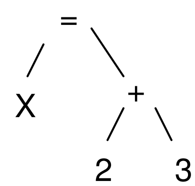

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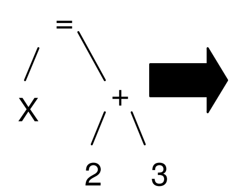
Program Semantics:



value

Compilation:
Push X
Push 2
Push 3
Add top 2
Assign top to second

Evaluation

- We're going to define a relation $E \to v$
  - This means "expression E evaluates to v"
- So we need a formal way of defining programs and of defining things they may evaluate to
- We'll use grammars to describe each of these
  - One to describe **abstract syntax trees** E
  - One to describe OCaml **values** v

OCaml Programs

- E ::= x | n | true | false | [] | if E then E else E
  | fun x = E | E E

  - x stands for any identifier
  - n stands for any integer
  - true and false stand for the two boolean values
  - [] is the empty list
  - Using = in fun instead of -> to avoid some confusion later

Values

- *v ::= n | true | false | [] | v::v*
  - *n* is an integer (*not* a string corresp. to an integer)
    - Same idea for *true*, *false, []*
  - *v1::v2* is the pair with *v1* and *v2*
    - This will be used to build up lists
    - Notice: nothing yet requires *v2* to be a list
  - Important: Be sure to understand the difference between *program text* S and *mathematical objects v*
    - E.g., the text 3 evaluates to the mathematical number *3*
  - To help, we'll use different colors and italics
    - If not present, it's up to you to remember which is which

Grammars for Trees

- We're just using grammars to describe trees

E ::= x | n | true | false | [] | if E then E else E
  | fun x = E | E E

v ::= n | true | false | [] | v::v

Given a program, we know how to convert it to an AST using recursive descent parsing

```
type ast =
    Id of string
  | Num of int
  | Bool of bool
  | Nil
  | If of ast * ast * ast
  | Fun of string * ast
  | App of ast * ast
```

```
type value =
    Val_Num of int
  | Val_Bool of bool
  | Val_Nil
  | Val_Pair of value *
               value
```

Goal: For any AST, we want an operational rule to obtain a value that represents the execution of that AST

Operational Semantics Rules

$$\mathrm{n} \longrightarrow \mathit{n}$$

$$\mathrm{true} \longrightarrow \mathit{true}$$

$$\mathrm{false} \longrightarrow \mathit{false}$$

$$[\,] \longrightarrow \mathit{[\,]}$$

- Each basic entity evaluates to its corresponding value

Operational Semantics Rules (cont.)

- How about built-in functions?

$$(+)\ n\ m \rightarrow n + m$$

- We're applying the + function
  - We put parens around it because it's not in infix notation
    - Will skip this from now on
  - Ignore currying for the moment
    - Pretend we have multi-argument functions
- On the right-hand side, we're computing the mathematical sum; the left-hand side is source code
- But what about + (+ 3 4) 5 ?
  - We need recursion

Rules with Hypotheses

- To evaluate $+\ E_1\ E_2$, we need to evaluate E_1, then evaluate E_2, then add the results
  - This is call-by-value

$$\frac{E_1 \to n \qquad\qquad E_2 \to m}{+\ E_1\ E_2 \to n + m}$$

  - This is a "natural deduction" style rule
  - It says that if the hypotheses above the line hold, then the conclusion below the line holds
    - i.e., if E_1 executes to value n and if E_2 executes to value m, then $+\ E_1\ E_2$ executes to value $n+m$

$$\frac{E_1 \rightarrow n \qquad E_2 \rightarrow m}{+ \; E_1 \; E_2 \rightarrow n + m}$$

Error Cases

- What if E1 and E2 aren't integers?
  - E.g., what if we write + false true ?
  - It can be parsed, but we can't execute it
- Previous rule does not cover such a case
  - Because we wrote n, m in the hypothesis
    - So they must be integers
- Convention
  - If there is no rule to cover a case
    - Then the expression is erroneous
  - A program that evaluates to an erroneous expression
    - Produces a run-time error in practice

Trees of Semantic Rules

- When we apply rules to an expression, we actually get a tree
  - Corresponds to the recursive evaluation procedure
    - For example: + (+ 3 4) 5

$$\dfrac{\dfrac{3 \to 3 \qquad 4 \to 4}{(+\ 3\ 4) \to 7} \qquad 5 \to 5}{+\ (\ +\ 3\ 4)\ 5 \to 12}$$

Rules for If

$$\frac{E_1 \to \mathit{true} \qquad E_2 \to v}{\text{if } E_1 \text{ then } E_2 \text{ else } E_3 \to v}$$

$$\frac{E_1 \to \mathit{false} \qquad E_3 \to v}{\text{if } E_1 \text{ then } E_2 \text{ else } E_3 \to v}$$

- Examples
  - if false then 3 else 4 $\to$ *4*
  - if true then 3 else 4 $\to$ *3*
- Notice that only one branch is evaluated

Rule for ::

$$\frac{E_1 \rightarrow v_1 \qquad\qquad E_2 \rightarrow v_2}{::\ E_1\ E_2 \rightarrow v_1 :: v_2}$$

- So :: allocates a pair in memory
- Are there any conditions on E_1 and E_2?
  - No! We will allow E_2 to be anything
  - OCaml's type system will disallow non-lists

Rules for Identifiers

- Let's assume for now that the only identifiers are parameter names
  - Example: (fun x = + x 3) 4
  - When we see x in the body, we need to look it up
  - So we need to keep some sort of environment
    - This will be a map from identifiers to values

Semantics with Environments

- Extend rules to the form $A; E \rightarrow v$
  - Means in environment A, program text E evaluates to v
- Notation
  - We write • for the empty environment (may be omitted)
  - We write $A(x)$ for the value that x maps to in A
  - We write $A, x{:}v$ for the same environment as A, except x is now v
    - x might or might not have mapped to anything in A
  - We write A, A' for the environment with the bindings of A' added to and overriding the bindings of A

Rules for Identifiers and Application

$$\frac{}{A;\, x \rightarrow A(x)}$$

no hypothesis means
"in all cases"

$$\frac{A;\, E_2 \rightarrow v \qquad A,\, x{:}v;\, E_1 \rightarrow v'}{A;\, (\,(\text{fun } x = E_1)\;\; E_2) \rightarrow v'}$$

- To evaluate a user-defined function applied to an argument:
  - Evaluate the argument (call-by-value)
  - Evaluate the function body in an environment in which the formal parameter is bound to the actual argument
  - Return the result

Example: (fun x = + x 3) 4 = ?

$$\frac{\bullet;\ 4 \longrightarrow \mathit{4} \qquad \dfrac{\bullet,\ x{:}\mathit{4};\ x \longrightarrow \mathit{4} \qquad \bullet,\ x{:}\mathit{4};\ 3 \longrightarrow \mathit{3}}{\bullet,\ x{:}\mathit{4};\ +\ x\ 3 \longrightarrow \mathit{7}}}{\bullet;\ (\text{fun } x = +\ x\ 3)\ \ 4 \longrightarrow \mathit{7}}$$

Nested Functions

- This works for cases of nested functions
  - ...as long as they are fully applied
- But what about the true higher-order cases?
  - Passing functions as arguments, and returning functions as results
  - We need closures to handle this case
  - ...and a closure was just a function and an environment
  - We already have notation around for writing both parts

Closures

- Formally, we add closures *(A, λx.E)* to values
  - A is the environment in which the closure was created
  - x is the parameter name
  - E is the source code for the body
- *λx* is a binding of x in E
- *v ::= n | true | false | [] | v::v | (A, λx.E)*

Revised Rule for Lambda

$$\frac{}{A;\ \text{fun } x = E \ \rightarrow\ (A, \lambda x.E)}$$

- To evaluate a function definition, create a closure when the function is created
  - Notice that we don't look inside the function body

Revised Rule for Application

$$\frac{A;\ E_1 \rightarrow (A',\ \lambda x.E) \qquad A;\ E_2 \rightarrow v \qquad A,\ A',\ x{:}v;\ E \rightarrow v'}{A;\ (E_1\ \ E_2) \rightarrow v'}$$

- To apply something to an argument:
  - Evaluate it to produce a closure
  - Evaluate the argument (call-by-value)
  - Evaluate the body of the closure, in
    - The current environment, extended with the closure's environment, extended with the binding for the parameter

Example

$$\dfrac{\bullet;\ (\text{fun } x = (\text{fun } y = +\ x\ y)) \to \mathit{(\bullet,\ \lambda x.(fun\ y = +\ x\ y))} \qquad \bullet;\ 3 \to \mathit{3} \qquad x{:}\mathit{3};\ (\text{fun } y = +\ x\ y) \to \mathit{(}x{:}\mathit{3,\ \lambda y.(+\ x\ y))}}{\bullet;\ (\text{fun } x = (\text{fun } y = +\ x\ y))\ \ 3 \to \mathit{(}x{:}\mathit{3,\ \lambda y.(+\ x\ y))}}$$

Let <previous> = (fun x = (fun y = + x y)) 3

Example (cont.)

$$\dfrac{\bullet;\ \text{<previous>} \longrightarrow\ (x{:}3,\ \lambda y.(+\ x\ y)) \qquad \bullet;\ 4 \longrightarrow\ 4 \qquad x{:}3,\ y{:}4;\ (+\ x\ y) \longrightarrow\ 7}{\bullet;\ (\ \text{<previous>}\ 4\ ) \longrightarrow 7}$$

Why Did We Do This?

- Operational semantics are useful for
  - Describing languages
    - Not just OCaml! It's pretty hard to describe a big language like C or Java, but we can at least describe the core components of the language
  - Giving a precise specification of how they work
    - Look in any language standard – they tend to be vague in many places and leave things undefined
  - Reasoning about programs
    - We can actually prove that programs do something or don't do something, because we have a precise definition of how they work