Structural Induction

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CMSC250 @ UMCP

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Outline

1. Recursively defined structures

2. Proofs
   - Binary Trees
   - Sets

Recursively defined structures

Recursively defined structures

- Many structures in Computer Science are *recursively defined*, i.e *parts of them exhibit the same characteristics and have the same properties as the whole!*
- They are also "well-ordered", in the sense that they exhibit a "well-founded partial order", like the order $\leq$ of $\mathbb{Z}$ or $\subseteq$ for sets.

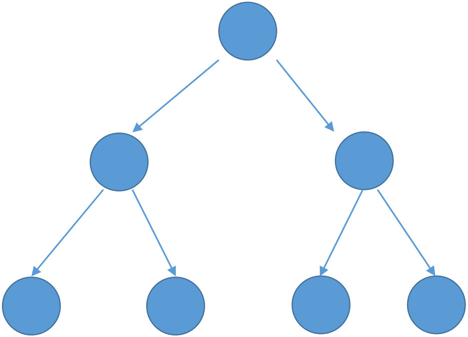
Structural induction as a proof methodology

- **Structural induction** is a proof methodology similar to mathematical induction, only instead of working in the domain of positive integers ($\mathbb{N}$) it works in the domain of such **recursively defined structures**!
- It is terrifically useful for proving *properties* of such structures.
- Its structure is sometimes "looser" than that of mathematical induction.

Proofs

Binary Trees

Pictorially



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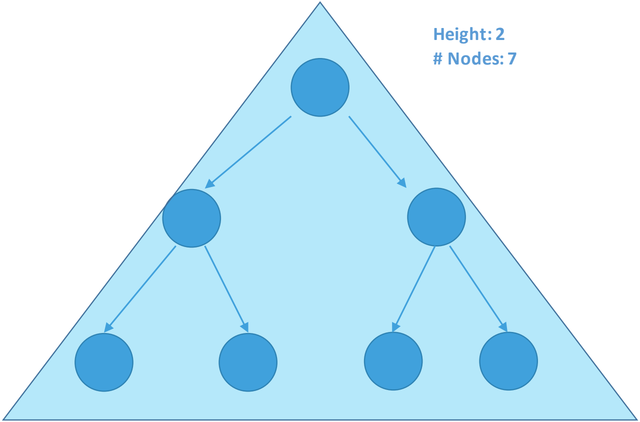

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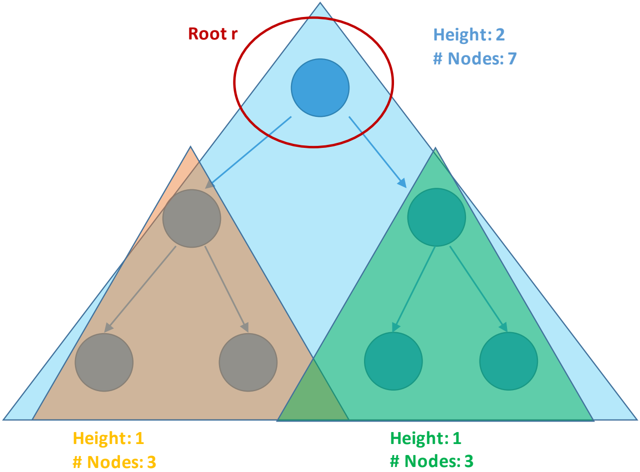

A recursive definition and statement on binary trees

Definition (Non-empty binary tree)

A **non-empty** binary tree T is either:

- **Base case:** A root node r with *no pointers*, or
- **Recursive (or inductive) step:** A root node r pointing to 2 *non-empty binary trees* T_L and T_R

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Claim: $|V| = |E| + 1$

The number of vertices ($|V|$) of a *non-empty binary tree* T is the number of its edges ($|E|$) plus one.

First structurally inductive proof

Proof (via structural induction on non-empty binary trees).

Let T be a non-empty binary tree and P the proposition we want to hold..

1. **Inductive Base**: If T consists of a single root node r (base case for a non-empty binary tree), then $|V| = 1$ and $|E| = 0$, so $P(r)$ holds.
2. **Inductive Hypothesis**: In the recursive part of the definition for a non-empty binary tree, T may consist of a root node r pointing to 1 or 2 non-empty binary trees T_L and T_R. Without loss of generality, we can assume that both T_L and T_R are defined, and we assume $P(T_L)$ and $P(T_R)$.
3. **Inductive Step**: We prove now that $P(T)$ must hold. Denote by V_L, E_L, V_R, E_R the vertex and edge sets of the left and right subtrees respectively. We obtain:

$$\begin{aligned}
|V| &= |V_L| + |V_R| + 1 && \textit{(By definition of non-empty binary trees)} \\
&= (|E_L| + 1) + (|E_R| + 1) + 1 && \textit{(By the Inductive Hypothesis)} \\
&= (|E_L| + |E_R| + 2) + 1 && \textit{(By grouping terms)} \\
&= |E| + 1 && \textit{(By definition of non-empty binary trees)}
\end{aligned}$$

So $P(T)$ holds. □

Here's one for you!

Definition (Height of a non-empty binary tree)

The height $h(T)$ of a non-empty binary tree T is defined as follows:

- **(Base case:)** If T is a single root node r, $h(r) = 0$.
- **(Recursive step:)** If T is a root node connected to two "sub-trees" T_L and T_R, $h(T) = max\{h(T_R), h(T_L)\} + 1$

Theorem ($m(T)$ as a function of $h(T)$)

A non-empty binary tree T of height $h(T)$ has ***at most*** $2^{h(T)+1} - 1$ *nodes.*

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    - Use the Inductive Hypothesis on the T_i!

Sets

Recursive definitions of sets

- Sets can be defined **recursively**!
- Our goal is to find a “flat” definition of them (a *“closed-form” description*), much in the same way we did with recursive sequences and strong induction.
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Practice!

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Our first structurally inductive proof on sets

A proposition on a recursively defined set

Let S be a set defined as follows:

- **Base case**: $4 \in S$
- **Recursive / Inductive step**: If $x \in S$, then $x^2 \in S$.

Then, prove that $\forall x \in S$, x is even.

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Proof

Proof (via structural induction on S).

Let P be the proposition we want to prove. We proceed inductively:

- **Inductive base**: In the base case of the definition of S, we have that $4 \in S$. Since 4 is an even number, $P(\{4\})$ holds.
- **Inductive hypothesis**: The inductive definition of S successively builds sets S' from previous "versions" of S. We assume that $\exists S' : |S'| \geq 1$ and $P(S')$.
- **Inductive step**: We will prove that $P(S' \cup \{x^2 | x \in S'\})$ holds. Let x_0 be an arbitrarily selected element of S'. From the inductive hypothesis, we know that x_0 is even. From a known theorem, we know that x_0^2 is even. But this means that $P(\{x_0\})$, and since x_0 was arbitrarily selected within S', $P(S' \cup \{x^2 | x \in S'\})$ holds.

□

A proof on the Cartesian Plane

An inequality proof

Let S be the subset of $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ defined as follows:

- **Base case**: $(0, 0) \in S$
- **Recursive step**:
  $(a, b) \in S \Rightarrow (((a, b+1) \in S) \wedge ((a+1, b+1) \in S) \wedge (a+2, b+1) \in S)$.
  Prove that $\forall (a, b) \in S, a \leq 2b$.

- *Suggestion:* To make sure you understand the exercise, first list 5 elements in S.

Cartesian plane exercise, proof

In class!

Set equality and structural induction

The set of positive multiples of 3

Let the set S be such that:

- **(Base case:)** $3 \in S$
- **(Recursive step:)** $(x \in S) \wedge (y \in S) \Rightarrow (x + y) \in S$

Then, $S = \{3n, \ \forall n \in \mathbb{N}^*\}$.

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Contradiction | Cases | Mathematical induction | Structural Induction

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Proof

Proof (via weak induction on n and structural induction on S!)

Let $A = \{3n,\ \forall n \in \mathbb{N}^*\}$. To prove $S = A$, we need to prove that $S \subseteq A$ and $A \subseteq S$.

- First, we prove that $A \subseteq S$. The proof is via weak induction on n. Let r be a generic particular for $\mathbb{N}^*$ and $P(n)$ be the statement we want to prove.[a] We proceed inductively:
  1. **Inductive base**: For $r = 1$, $3r = 3 \cdot 1 \in S$. Therefore, $P(1)$ holds.
  2. **Inductive hypothesis**: We assume that for some value of $r \geq 1$, $P(r)$ holds, i.e $3r \in S$
  3. **Inductive step**: We want to prove $P(r+1)$, that is, $3(r+1) \in S$. We know that $3r \in S$ and $3 \in S$ by the inductive base and hypothesis. By the definition of S, this means that $3r + 3 \in S \overset{(Algebra)}{\iff} 3(r+1) \in S$. So, $P(r+1)$ holds.

  Since $r \in \mathbb{N}^*$ was arbitrarily chosen, the result holds for all $n \in \mathbb{N}^*$.

[a] We can do this, since A is parameterized by n.

Proof (continued)

Proof: $S \subseteq A$ part.

- We now prove that $S \subseteq A$.
  1. **Inductive base**: By the base case of the definition of S, we have that $3 \in S$. Since $3 = 3 \cdot 1$ and $1 \in \mathbb{N}^*$, we have that $3 \in A$. So $P(3)$.
  2. **Inductive hypothesis**: Assume that $x, y \in S$ are also contained by A, i.e $x, y \in A$. So $P(x)$, $P(y)$.
  3. **Inductive step**: We must show that $P(x+y)$, i.e $x + y \in A$, because if we do show this, we will have covered the recursive step of A's definition. From the inductive hypothesis, we have that $x, y \in A \Rightarrow \exists i, j \in \mathbb{N}^* : x = 3i,\ y = 3j$. Therefore, $x + y = 3\underbrace{(i+j)}_{z \in \mathbb{N}} \Rightarrow x + y \in A$. Therefore, $P(x+y)$.

□

Recursively defined languages!

Definition

A recursively defined language Let $\Sigma = \{a, b\}$ be an alphabet. We define the language $\mathcal{L}$ as follows:

- **(Base case:)** $\epsilon \in \mathcal{L}$
- **(Recursive step:)** If $x \in \mathcal{L}$, $axa \in \mathcal{L}$ and $bxb \in \mathcal{L}$.[a]

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[a] $\sigma_1\sigma_2$ is the **concatenation** of σ_1 and σ_2. Concatenation can be applied to n strings, $n = 1, 2, 3, \ldots$. σ^n is the concatenation of σ $n-$ many times.

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Claim

$\forall\sigma \in \mathcal{L}, |\sigma|$ is even.[a]

[a] $|\sigma|$ is the number of characters in σ.

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    - Reminds us of weak mathematical induction?
3. In the inductive step, use the recursive part of the definition of S to prove that the new set constructed (call it S') satisfies the proposition (so $P(S')$).
    - The inductive hypothesis will undoubtedly be used.