CMSC 330: Organization of Programming Languages

Lambda Calculus

Programming Language Features

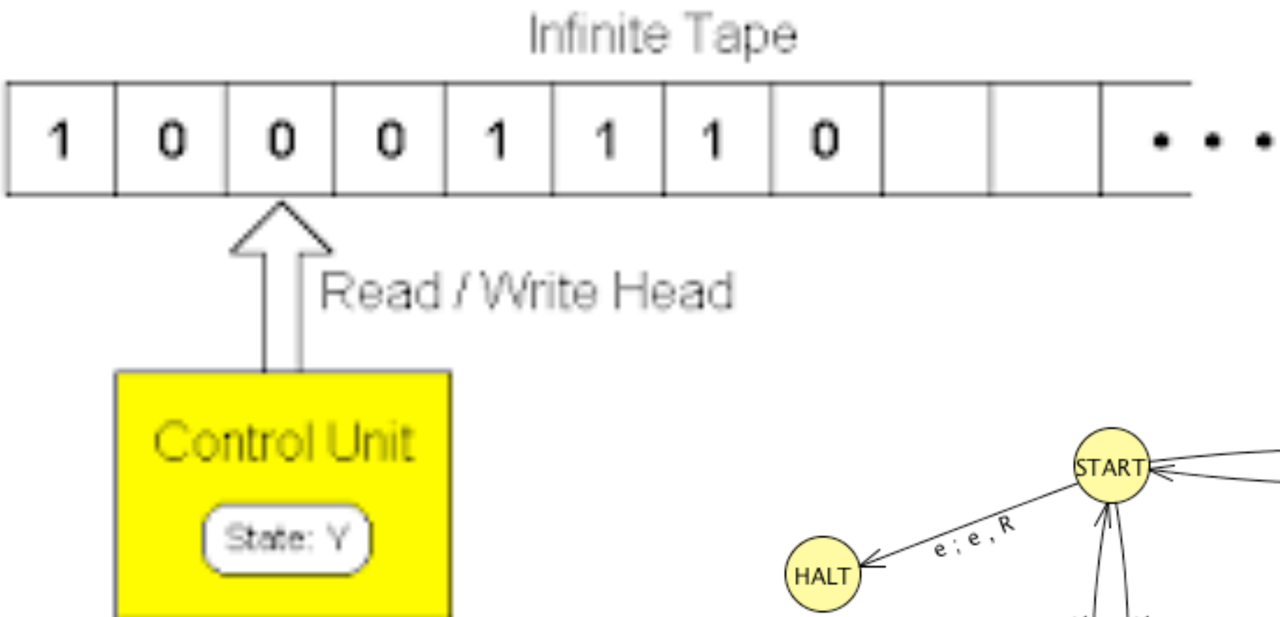
- Many features exist simply for convenience
  - Multi-argument functions foo (a, b, c)
    - Use currying or tuples
  - Loops while (a < b) …
    - Use recursion
  - Side effects a := 1
    - Use functional programming

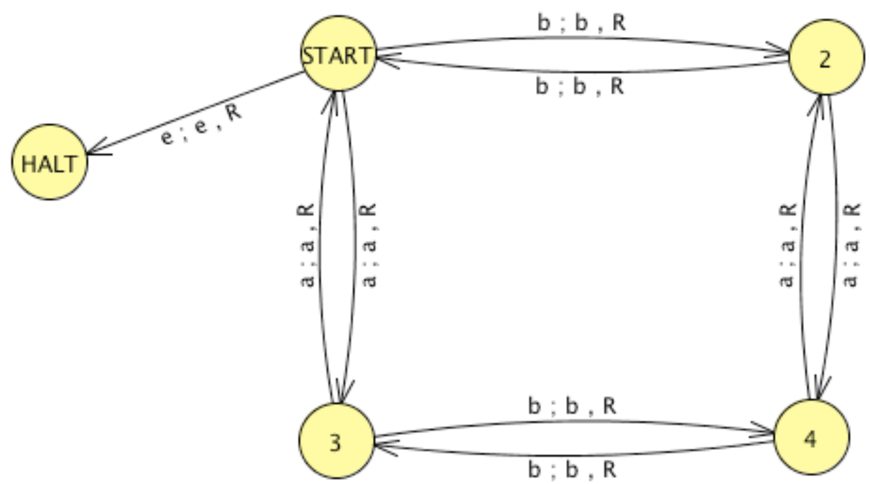
- So what language features are really needed?

Turing Completeness

- Computational system that can
  - Simulate a Turing machine
  - Compute every Turing-computable function
- A programming language is Turing complete if
  - It can map every Turing machine to a program
  - A program can be written to emulate a Turing machine
  - It is a superset of a known Turing-complete language
- Most powerful programming language possible
  - Since Turing machine is most powerful automaton

Turing Machine





Programming Language Theory

- Come up with a "core" language
  - That's as small as possible
  - But still Turing complete
- Helps illustrate important
  - Language features
  - Algorithms
- One solution
  - Lambda calculus

Mini C

You only have:

- If statement
- Plus 1
- Minus 1
- functions

Sum n = 1+2+3+4+5…n in Mini C

```
int add1(int n){return n+1;}
int sub1(int n){return n-1;}
int add(int a,int b){
    if(b == 0) return a;
    else return add( add1(a),sub1(b));
}
int sum(int n){
    if(n == 1) return 1;
    else return add(n, sum(sub1(n)));
}
int main(){
    printf("%d\n",sum(5));
}
```

Lambda Calculus (λ-calculus)

- Proposed in 1930s by
  - Alonzo Church
    (born in Washington DC!)

- Formal system
  - Designed to investigate functions & recursion
  - For exploration of foundations of mathematics
- Now used as
  - Tool for investigating computability
  - Basis of functional programming languages
    - Lisp, Scheme, ML, OCaml, Haskell…

Lambda Expressions

- A lambda calculus expression is defined as

$$
\begin{array}{lll}
e ::= & x & \text{variable} \\
 & \mid\ \lambda x.e & \text{function} \\
 & \mid\ e\ e & \text{function application}
\end{array}
$$

  - Note that this is CFG is ambiguous, but that's not a problem for defining the terms in the language – we are not using it for parsing (i.e., different parse trees = different expressions)

- $\lambda x.e$ is like **`(fun x -> e)`** in OCaml
- That's it! Nothing but higher-order functions

Three Conveniences

- "Syntactic sugar" for local declarations
  - let x = e1 in e2 is short for (λx.e2) e1
- Scope of λ extends as far right as possible
  - Subject to scope delimited by parentheses
  - λx. λy.x y is same as λx.(λy.(x y))
- Function application is left-associative
  - x y z is (x y) z
  - Same rule as OCaml

OCaml implementation

```
type id = string
```

- e ::= x
  | λx.e
  | e e

```
type exp = Var of id
  | Lam of id * exp
  | App of exp * exp
```

y	`Var "y"`
λx.x	`Lam ("x", Var "x")`
λx.λy.x y	`Lam ("x", (Lam ("y", App (Var "x", Var "y")`
(λx.λy.x y) λx.x x	`App (Lam ("x", Lam ("y", App (Var "x", Var "y"))), Lam ("x", App (Var "x", Var "x")))`

Lambda Calculus Semantics

- All we've got are functions
  - So all we can do is call them
- To evaluate (λx.e1) e2
  - Evaluate e1 with x replaced by e2
- This application is called beta-reduction
  - (λx.e1) e2 → e1[x:=e2]
    - e1[x:=e2] is e1 with occurrences of x replaced by e2
    - This operation is called *substitution*
      - **Replace** formals with actuals
      - Instead of using environment to map formals to actuals
  - We allow reductions to occur *anywhere* in a term
    - Order reductions are applied does not affect final value!

Beta Reduction Example

- $(\lambda x.\lambda z.x\ z)\ y$
  - $\rightarrow (\lambda x.(\lambda z.(x\ z)))\ y$ // since λ extends to right
  - $\rightarrow (\lambda x.(\lambda z.(x\ z)))\ y$ // apply $(\lambda x.e1)\ e2 \rightarrow e1[x:=e2]$
    // where $e1 = \lambda z.(x\ z)$, $e2 = y$
  - $\rightarrow \lambda z.(y\ z)$ // final result

Parameters
- Formal
- Actual

- Equivalent OCaml code
  - (fun x -> (fun z -> (x z))) y → fun z -> (y z)

Lambda Calculus Examples

- $(\lambda x.x)\ z \rightarrow z$
- $(\lambda x.y)\ z \rightarrow\ y$
- $(\lambda x.x\ y)\ z \rightarrow\ \ z\ y$
  - A function that applies its argument to y

Lambda Calculus Examples (cont.)

- $(\lambda x.x\ y)\ (\lambda z.z) \rightarrow (\lambda z.z)\ y \rightarrow y$
- $(\lambda x.\lambda y.x\ y)\ z \rightarrow \lambda y.z\ y$
  - A curried function of two arguments
  - Applies its first argument to its second
- $(\lambda x.\lambda y.x\ y)\ (\lambda z.zz)\ x \rightarrow (\lambda y.(\lambda z.zz)y)x \rightarrow (\lambda z.zz)x \rightarrow xx$

Static Scoping & Alpha Conversion

- Lambda calculus uses static scoping
- Consider the following
  - $(\lambda x.x\ (\lambda x.x))\ z \rightarrow ?$
    - The rightmost "x" refers to the second binding
  - This is a function that
    - Takes its argument and applies it to the identity function
- This function is "the same" as $(\lambda x.x\ (\lambda y.y))$
  - Renaming bound variables consistently is allowed
    - This is called alpha-renaming or alpha conversion
  - Ex. $\lambda x.x = \lambda y.y = \lambda z.z \qquad \lambda y.\lambda x.y = \lambda z.\lambda x.z$

Defining Substitution

- Use recursion on structure of terms
  - x[x:=e] = e // Replace x by e
  - y[x:=e] = y // y is different than x, so no effect
  - (e1 e2)[x:=e] = (e1[x:=e]) (e2[x:=e])
    // Substitute both parts of application
  - (λx.e')[x:=e] = λx.e'
    - In λx.e', the x is a parameter, and thus a local variable that is different from other x's. Implements static scoping.
    - So the substitution has no effect in this case, since the x being substituted for is different from the parameter x that is in e'
  - (λy.e')[x:=e] = ?
    - The parameter y does not share the same name as x, the variable being substituted for
    - Is λy.(e' [x:=e]) correct? No…

Variable capture

- How about the following?
  - (λx.λy.x y) y → ?
  - When we replace y inside, we don't want it to be captured by the inner binding of y, as this violates static scoping
  - I.e., (λx.λy.x y) y ≠ λy.y y
- Solution
  - (λx.λy.x y) is "the same" as (λx.λz.x z)
    - Due to alpha conversion
  - So alpha-convert (λx.λy.x y) y to (λx.λz.x z) y first
    - Now (λx.λz.x z) y → λz.y z

Completing the Definition of Substitution

- Recall: we need to define (λy.e')[x:=e]
  - We want to avoid capturing (free) occurrences of y in e
  - Solution: alpha-conversion!
    - Change y to a variable w that does not appear in e' or e
      (Such a w is called fresh)
    - Replace all occurrences of y in e' by w.
    - Then replace all occurrences of x in e' by e!
- Formally:
  (λy.e')[x:=e] = λw.((e' [y:=w]) [x:=e]) (w is fresh)

Beta-Reduction, Again

- Whenever we do a step of beta reduction
  - (λx.e1) e2 → e1[x:=e2]
  - We must alpha-convert variables as necessary
  - Usually performed implicitly (w/o showing conversion)

- Examples
  - (λx.λy.x y) y = (λx.λz.x z) y → λz.y z // y → z
  - (λx.x (λx.x)) z = (λy.y (λx.x)) z → z (λx.x) // x → y

OCaml Implementation: Substitution

```
(* substitute e for y in m *)
let rec subst m y e =
  match m with
      Var x ->
        if y = x then e (* substitute *)
        else m          (* don't subst *)
    | App (e1,e2) ->
        App (subst e1 y e, subst e2 y e)
    | Lam (x,e0) -> …
```

OCaml Impl: Substitution (cont'd)

```
(* substitute e for y in m *)
let rec subst m y e = match m with …
    | Lam (x,e0) ->
      if y = x then m
      else if not (List.mem x (fvs e)) then
        Lam (x, subst e0 y e)
      else
        let z = newvar() in (* fresh *)
        let e0' = subst e0 x (Var z) in
        Lam (z,subst e0' y e)
```

Shadowing blocks substitution

Safe: no capture possible

Might capture; need to α-convert

OCaml Impl: Reduction

```
let rec reduce e =
  match e with
      App (Lam (x,e), e2) -> subst e x e2
    | App (e1,e2) ->
      let e1' = reduce e1 in
      if e1' != e1 then App(e1',e2)
      else App (e1,reduce e2)
    | Lam (x,e) -> Lam (x, reduce e)
    | _ -> e
```

Straight β rule

Reduce lhs of app

Reduce rhs of app

Reduce function body

nothing to do

Encodings

- The lambda calculus is Turing complete
- Means we can encode any computation we want
  - If we're sufficiently clever...
- Examples
  - Booleans
  - Pairs
  - Natural numbers & arithmetic
  - Looping

Booleans

- Church's encoding of mathematical logic
  - true = λx.λy.x
  - false = λx.λy.y
  - if a then b else c
    - Defined to be the λ expression: a b c
- Examples
  - if true then b else c = (λx.λy.x) b c → (λy.b) c → b
  - if false then b else c = (λx.λy.y) b c → (λy.y) c → c

Booleans (cont.)

- Other Boolean operations
  - not = λx.((x false) true)
    - not x = if x then false else true
    - not true → (λx.(x false) true) true → ((true false) true) → false
  - and = λx.λy.((x y) false)
    - and x y = if x then y else false
  - or = λx.λy.((x true) y)
    - or x y = if x then true else y
- Given these operations
  - Can build up a logical inference system

Pairs

- Encoding of a pair a, b
  - (a,b) = λx.if x then a else b
  - fst = λf.f true
  - snd = λf.f false
- Examples
  - fst (a,b) = (λf.f true) (λx.if x then a else b) →
    (λx.if x then a else b) true →
    if true then a else b → a
  - snd (a,b) = (λf.f false) (λx.if x then a else b) →
    (λx.if x then a else b) false →
    if false then a else b → b

Natural Numbers (Church* Numerals)

- Encoding of non-negative integers
  - 0 = λf.λy.y
  - 1 = λf.λy.f y
  - 2 = λf.λy.f (f y)
  - 3 = λf.λy.f (f (f y))

    i.e., n = λf.λy.<apply f n times to y>
  - Formally: n+1 = λf.λy.f (n f y)

*(Alonzo Church, of course)

Operations On Church Numerals

- Successor
  - succ = λz.λf.λy.f (z f y)
  - 0 = λf.λy.y
  - 1 = λf.λy.f y
- Example
  - succ 0 =

    (λz.λf.λy.f (z f y)) (λf.λy.y) →

    λf.λy.f ((λf.λy.y) f y) →

    λf.λy.f ((λy.y) y) →

    λf.λy.f y

    = 1

Since (λx.y) z → y

Operations On Church Numerals (cont.)

- IsZero?
  - iszero = λz.z (λy.false) true

    This is equivalent to λz.((z (λy.false)) true)

- Example
  - iszero 0 =

    (λz.z (λy.false) true) (λf.λy.y) →

    (λf.λy.y) (λy.false) true →

    (λy.y) true →

    Since (λx.y) z → y

    true
  - 0 = λf.λy.y

Arithmetic Using Church Numerals

- If M and N are numbers (as λ expressions)
  - Can also encode various arithmetic operations
- Addition
  - $M + N = \lambda f.\lambda y.(M\ f)((N\ f)\ y)$

    Equivalently: $+ = \lambda M.\lambda N.\lambda f.\lambda y.(M\ f)((N\ f)\ y)$
    - In prefix notation (+ M N)
- Multiplication
  - $M * N = \lambda f.(M\ (N\ f))$

    Equivalently: $* = \lambda M.\lambda N.\lambda f.\lambda y.(M\ (N\ f))\ y$
    - In prefix notation (* M N)

Arithmetic (cont.)

- $1 = \lambda f.\lambda y.f\ y$
- $2 = \lambda f.\lambda y.f\ (f\ y)$

- Prove 1+1 = 2
  - $1+1 = \lambda x.\lambda y.(1\ x)((1\ x)\ y) =$
  - $\lambda x.\lambda y.((\lambda f.\lambda y.f\ y)\ x)((1\ x)\ y) \rightarrow$
  - $\lambda x.\lambda y.(\lambda y.x\ y)((1\ x)\ y) \rightarrow$
  - $\lambda x.\lambda y.x\ ((1\ x)\ y) \rightarrow$
  - $\lambda x.\lambda y.x\ (((\lambda f.\lambda y.f\ y)\ x)\ y) \rightarrow$
  - $\lambda x.\lambda y.x\ ((\lambda y.x\ y)\ y) \rightarrow$
  - $\lambda x.\lambda y.x\ (x\ y) = 2$

- With these definitions
  - Can build a theory of arithmetic

Looping & Recursion

- Define $D = \lambda x.x\ x$, then
  - $D\ D = (\lambda x.x\ x)\ (\lambda x.x\ x) \rightarrow (\lambda x.x\ x)\ (\lambda x.x\ x) = D\ D$
- So $D\ D$ is an infinite loop
  - In general, self application is how we get looping

The Fixpoint Combinator

Y = λf.(λx.f (x x)) (λx.f (x x))

- Then

  Y F =

  (λf.(λx.f (x x)) (λx.f (x x))) F →

  (λx.F (x x)) (λx.F (x x)) →

  F ((λx.F (x x)) (λx.F (x x)))

  = F (**Y** F)

- **Y** F is a *fixed point* (aka "fixpoint") of F
- Thus **Y** F = F (**Y** F) = F (F (**Y** F)) = ...
  - We can use **Y** to achieve recursion for F

Example

fact = λf. λn.if n = 0 then 1 else n * (f (n-1))

- The second argument to fact is the integer
- The first argument is the function to call in the body
  - We'll use Y to make this recursively call fact

(**Y** fact) 1 = (fact (**Y** fact)) 1

→ if 1 = 0 then 1 else 1 * ((**Y** fact) 0)

→ 1 * ((Y fact) 0)

→ 1 * (fact (Y fact) 0)

→ 1 * (if 0 = 0 then 1 else 0 * ((**Y** fact) (-1))

→ 1 * 1 → 1

Discussion

- Lambda calculus is Turing-complete
  - Most powerful language possible
  - Can represent pretty much anything in "real" language
    - Using clever encodings
- But programs would be
  - Pretty slow (10000 + 1 → thousands of function calls)
  - Pretty large (10000 + 1 → hundreds of lines of code)
  - Pretty hard to understand (recognize 10000 vs. 9999)
- In practice
  - We use richer, more expressive languages
  - That include built-in primitives

The Need For Types

- Consider the untyped lambda calculus
  - false = λx.λy.y
  - 0 = λx.λy.y
- Since everything is encoded as a function...
  - We can easily misuse terms…
    - false 0 → λy.y
    - if 0 then ...

  …because everything evaluates to some function
- The same thing happens in assembly language
  - Everything is a machine word (a bunch of bits)
  - All operations take machine words to machine words

Simply-Typed Lambda Calculus (STLC)

- e ::= n | x | λx:t.e | e e
  - Added integers n as primitives
    - Need at least two distinct types (integer & function)…
    - …to have type errors
  - Functions now include the type of their argument

Simply-Typed Lambda Calculus (cont.)

- $t ::= \text{int} \mid t \rightarrow t$
  - int is the type of integers
  - $t1 \rightarrow t2$ is the type of a function
    - That takes arguments of type t1 and returns result of type t2
  - t1 is the domain and t2 is the range
  - Notice this is a recursive definition
    - So we can give types to higher-order functions

Types are limiting

- STLC will reject some terms as ill-typed, even if they will not produce a run-time error
  - Cannot type check Y in STLC
    - Or in Ocaml, for that matter!
- Surprising theorem: All (well typed) simply-typed lambda calculus terms are strongly normalizing
  - They will terminate
  - Proof is not by straightforward induction
    - Applications “increase” term size

Summary

- Lambda calculus shows issues with
  - Scoping
  - Higher-order functions
  - Types
- Useful for understanding how languages work