CMSC 132: Object-Oriented Programming II

Shortest Paths
Quiz 1

One advantage of adjacency list representation over adjacency matrix representation of a graph is that in adjacency list representation, space is saved for sparse graphs.

A. True
B. False
One advantage of adjacency list representation over adjacency matrix representation of a graph is that in adjacency list representation, space is saved for sparse graphs.

A. True
B. False
Quiz 2

Traversal of a graph is different from tree because

A. There can be a loop in graph so we must maintain a visited flag for every vertex
B. DFS of a graph uses stack, but inorder traversal of a tree is recursive
C. BFS of a graph uses queue, but a time efficient BFS of a tree is recursive.
D. All of the above
Quiz 2

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B. DFS of a graph uses stack, but inorder traversal of a tree is recursive
C. BFS of a graph uses queue, but a time efficient BFS of a tree is recursive.
D. All of the above
Quiz 3

One possible order of Breadth First Search on the following graph

A. MNOPQR
B. NQMPOR
C. QMNPOR
D. QMNPRO
Quiz 3

One possible order of Breadth First Search on the following graph

A. MNOPQR
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Quiz 4

Given two vertices in a graph 1 and 6, which of the two traversals (BFS and DFS) can be used to find if there is path from 1 to 6?

A. Only BFS
B. Only DFS
C. Both BFS and DFS
D. Neither BFS nor DFS
Given two vertices in a graph 1 and 6, which of the two traversals (BFS and DFS) can be used to find if there is path from 1 to 6?

A. Only BFS
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C. Both BFS and DFS
D. Neither BFS nor DFS
Quiz 5

Consider the DAG with $V = \{1, 2, 3, 4, 5, 6\}$, shown below. Which of the following is NOT a topological ordering?

A. 1 2 3 4 5 6
B. 1 3 2 4 5 6
C. 1 3 2 4 6 5
D. 3 2 4 1 6 5
Quiz 5

Consider the DAG with vertices V = {1, 2, 3, 4, 5, 6}, shown below. Which of the following is NOT a topological ordering?

A. 1 2 3 4 5 6  
B. 1 3 2 4 5 6  
C. 1 3 2 4 6 5  
D. 3 2 4 1 6 5
Shortest Paths
Shortest paths

Given an edge-weighted digraph, find the shortest path from $s$ to $t$. 

edge-weighted digraph

- $4 \rightarrow 5$: 0.35
- $5 \rightarrow 4$: 0.35
- $4 \rightarrow 7$: 0.37
- $5 \rightarrow 7$: 0.28
- $7 \rightarrow 5$: 0.28
- $5 \rightarrow 1$: 0.32
- $0 \rightarrow 4$: 0.38
- $0 \rightarrow 2$: 0.26
- $7 \rightarrow 3$: 0.39
- $1 \rightarrow 3$: 0.29
- $2 \rightarrow 7$: 0.34
- $6 \rightarrow 2$: 0.40
- $3 \rightarrow 6$: 0.52
- $6 \rightarrow 0$: 0.58
- $6 \rightarrow 4$: 0.93

shortest path from 0 to 6

- 0 $\rightarrow$ 2: 0.26
- 2 $\rightarrow$ 7: 0.34
- 7 $\rightarrow$ 3: 0.39
- 3 $\rightarrow$ 6: 0.52
Shortest path variants

- Which vertices?
  - Single source: from one vertex $s$ to every other vertex.
  - Source-sink: from one vertex $s$ to another $t$.
  - All pairs: between all pairs of vertices.

- Restrictions on edge weights?
  - Nonnegative weights.
  - Euclidean weights.
  - Arbitrary weights.

- Cycles?
  - No directed cycles.
  - No "negative cycles."

- Simplifying assumption: Shortest paths from $s$ to each vertex $v$ exist.
Weighted directed edge

public class DirectedEdge

    DirectedEdge(int v, int w, double weight)  
    int from()  
    int to()  
    double weight()  
    String toString()

Idiom for processing an edge e: int v = e.from(), w = e.to();
Weighted directed edge implementation

```java
public class DirectedEdge{
    private final int v, w;
    private final double weight;

    public DirectedEdge(int v, int w, double weight){
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int from() { return v; }
    public int to() { return w; }
    public double weight() { return weight; }
}
```

![Directed graph diagram]
public class EdgeWeightedDigraph

EdgeWeightedDigraph(int V)

void addEdge(DirectedEdge e)

Iterable<DirectedEdge> adj(int v)

int V()

int E()

Iterable<DirectedEdge> edges()

String toString()

Conventions. Allow self-loops and parallel edges.
Edge-weighted digraph: adjacency-lists representation
public class EdgeWeightedDigraph{
    private final int V;
    private final Bag<DirectedEdge>[] adj;

    public EdgeWeightedDigraph(int V){
        this.V = V;
        adj = (Bag<DirectedEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<DirectedEdge>();
    }

    public void addEdge(DirectedEdge e){
        int v = e.from();
        adj[v].add(e);
    }

    public Iterable<DirectedEdge> adj(int v){
        return adj[v];
    }
}
Single-source shortest paths

What is the shortest distance and path from A to H?
Single-source shortest paths

- Data structures: Represent the Shortest Path with two vertex-indexed arrays:
  - distTo[v] is length of shortest path from s to v.
  - edgeTo[v] is last edge on shortest path from s to v.

```java
public double distTo(int v){
    return distTo[v];
}
```

```java
public Iterable<DirectedEdge> pathTo(int v){
    Stack<DirectedEdge> path = new Stack<DirectedEdge>();
    DirectedEdge e = edgeTo[v];

    while (e != null){
        path.push(e);
        e = edgeTo[e.from()];
    }
    return path;
}
```
Edge relaxation

- Relax edge $e = v \rightarrow w$.
  - $\text{distTo}[v]$ is length of shortest known path from $s$ to $v$.
  - $\text{distTo}[w]$ is length of shortest known path from $s$ to $w$.
  - $\text{edgeTo}[w]$ is last edge on shortest known path from $s$ to $w$.
  - If $e = v \rightarrow w$ gives shorter path to $w$ through $v$, update both $\text{distTo}[w]$ and $\text{edgeTo}[w]$.

$v \rightarrow w$ successfully relaxes
Edge relaxation

- Relax edge $e = v \rightarrow w$. 
  - $\text{distTo}[v]$ is length of shortest known path from $s$ to $v$. 
  - $\text{distTo}[w]$ is length of shortest known path from $s$ to $w$. 
  - $\text{edgeTo}[w]$ is last edge on shortest known path from $s$ to $w$. 
  - If $e = v \rightarrow w$ gives shorter path to $w$ through $v$, update both $\text{distTo}[w]$ and $\text{edgeTo}[w]$

```java
private void relax(DirectedEdge e) {
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight()) {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
    }
}
```
Generic shortest-paths algorithm

Generic algorithm (to compute SPT from $s$)

Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.
Repeat until optimality conditions are satisfied:
Relax any edge.

Efficient implementations: How to choose which edge to relax?
• Dijkstra's algorithm (nonnegative weights).
• Topological sort algorithm (no directed cycles).
• Bellman-Ford algorithm (no negative cycles).
**Dijkstra's algorithm**

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges pointing from that vertex.
Dijkstra's algorithm Demo

Pick vertex in List with minimum distance.

![Graph Diagram]

<table>
<thead>
<tr>
<th>V</th>
<th>distTo[]</th>
<th>edgeTo</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>B</td>
<td>∞</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>∞</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>∞</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>∞</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>∞</td>
<td></td>
</tr>
</tbody>
</table>
Update A’s neighbors

![Graph with nodes A, B, C, D, E, F, G with distances and edges labeled.]

<table>
<thead>
<tr>
<th>V</th>
<th>distTo[]</th>
<th>edgeTo</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>∞</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>E</td>
<td>∞</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>∞</td>
<td></td>
</tr>
</tbody>
</table>
Update D’s neighbors

![Graph with node values and connections]

<table>
<thead>
<tr>
<th>V</th>
<th>distTo[]</th>
<th>edgeTo</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>D</td>
</tr>
<tr>
<td>F</td>
<td>9</td>
<td>D</td>
</tr>
<tr>
<td>G</td>
<td>5</td>
<td>D</td>
</tr>
</tbody>
</table>
Update B’s neighbors

No Update
Update E’s neighbors

No Update
Update C’s neighbors

```
V | distTo[] | edgeTo
---|----------|-------
A | 0        | --    
B | 2        | A     
C | 3        | D     
D | 1        | A     
E | 3        | D     
F | 8        | C     
G | 5        | D     
```
Update G’s neighbors

<table>
<thead>
<tr>
<th>V</th>
<th>distTo[]</th>
<th>edgeTo</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>D</td>
</tr>
<tr>
<td>F</td>
<td>6</td>
<td>G</td>
</tr>
<tr>
<td>G</td>
<td>5</td>
<td>D</td>
</tr>
</tbody>
</table>
Update F’s neighbors

No Update
Dijkstra's algorithm Demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges pointing from that vertex.
Dijkstra's algorithm

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges pointing from that vertex.
Dijkstra's algorithm Implementation

```java
public class DijkstraSP{
    private DirectedEdge[] edgeTo;
    private double[] distTo;
    private IndexMinPQ<Double> pq;

    public DijkstraSP(EdgeWeightedDigraph G, int s) {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];
        pq = new IndexMinPQ<Double>(G.V());
        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;
        pq.insert(s, 0.0);
        while (!pq.isEmpty()){
            int v = pq.delMin();
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }
}
```
Shortest Path Demo
Shortest Path Demo
Shortest Path Demo
Acyclic shortest paths

- Consider vertices in topological order. Relax all edges pointing from that vertex.
Acyclic shortest paths

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.
Longest paths in edge-weighted DAGs

- Formulate as a shortest paths problem in edge-weighted DAGs.
  - Negate all weights.
  - Find shortest paths.
  - Negate weights in result
- Key point. Topological sort algorithm works even with negative weights.
Longest paths in edge-weighted DAGs

- Parallel job scheduling.
  - Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

<table>
<thead>
<tr>
<th>job</th>
<th>duration</th>
<th>must complete before</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>41.0</td>
<td>1 7 9</td>
</tr>
<tr>
<td>1</td>
<td>51.0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>50.0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>36.0</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>38.0</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>45.0</td>
<td>3 8</td>
</tr>
<tr>
<td>6</td>
<td>21.0</td>
<td>3 8</td>
</tr>
<tr>
<td>7</td>
<td>32.0</td>
<td>3 8</td>
</tr>
<tr>
<td>8</td>
<td>32.0</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>29.0</td>
<td>4 6</td>
</tr>
</tbody>
</table>

Parallel job scheduling solution
Critical path method

• To solve a parallel job-scheduling problem, create edge-weighted DAG:
  • Source and sink vertices.
  • Two vertices (begin and end) for each job.
  • Three edges for each job.
    ➢ Begin to end (weighted by duration)
    ➢ Source to begin (0 weight)
    ➢ End to sink (0 weight)
• One edge for each precedence constraint (0 weight).
Critical path method

Use longest path from the source to schedule each job.
Quiz 1

There are multiple shortest paths between vertices $S$ and $T$. Which one will be reported by Dijkstra’s shortest path algorithm?

A. SDT
B. SBDT
C. SACDT
D. SACET
Quiz 1

There are multiple shortest paths between vertices S and T. Which one will be reported by Dijkstra’s shortest path algorithm?

A. SDT  
B. SBDT  
C. SACDT  
D. SACET
Quiz 2

In an unweighted, undirected connected graph, the shortest path from a node S to every other node is computed most efficiently, in terms of time complexity by

A. Dijkstra’s algorithm starting from S.
B. Performing a DFS starting from S.
C. Performing a BFS starting from S.
D. None of the above
Quiz 2

In an unweighted, undirected connected graph, the shortest path from a node S to every other node is computed most efficiently, in terms of time complexity by

A. Dijkstra’s algorithm starting from S.
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C. Performing a BFS starting from S.
D. None of the above