# CMSC 132: Object-Oriented Programming II

## Sorting

# What Is Sorting?

- To arrange a collection of items in some specified order.
  - Numerical order
  - Lexicographical order
- ***Input:*** sequence $<a_1, a_2, \ldots, a_n>$ of numbers.
- ***Output:*** permutation $<a'_1, a'_2, \ldots, a'_n>$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$ .
- Example
  - Start → 1 23 2 56 9 8 10 100
  - End → 1 2 8 9 10 23 56 100

# Why Sort?

- A classic problem in computer science.
  - Data requested in sorted order
    - e.g., list students in increasing GPA order
- Searching
  - To find an element in an array of a million elements
    - Linear search:  average **500,000** comparisons
    - Binary search: worst case **20** comparisons
  - Database, Phone book
- Eliminating duplicate copies in a collection of records
- Finding a missing element, Max, Min

# Sorting Algorithms

- **Selection Sort**
- **Insertion Sort**
- **Bubble Sort**
- **Shell Sort**
- $T(n)=O(n^2)$ **Quadratic growth**
- In clock time

| | | | |
|---|---|---|---|
| **10,000** | **3 sec** | **20,000** | **17 sec** |
| 50,000 | 77 sec | 100,000 | 5 min |

- Double input -> 4X time
  - Feasible for small inputs, quickly unmanagable
- Halve input -> 1/4 time
  - Hmm... can recursion save the day?
  - If have two sorted halves, how to produce sorted full result?

# Divide and Conquer

1. Base case: the problem is small enough, solve directly
2. Divide the problem into two or more similar and smaller subproblems
3. Recursively solve the subproblems
4. Combine solutions to the subproblems

# Merge Sort

- Divide and conquer algorithm
- Worst case: O(nlogn)
- Stable
  - maintain the relative order of records with equal values
  - Input: 12, 5, 8, 13, 8, 27
  - Stable: 5, 8, 8, 12, 13, 27
  - Not Stable:5, 8, 8, 12, 13, 27

# Stable Sort Example

$(Dave, A)$
$(Alice, B)$
$(Ken, A)$
$(Eric, B)$
$(Carol, A)$

Sort by name →

$(Alice, B)$
$(Carol, A)$
$(Dave, A)$
$(Eric, B)$
$(Ken, A)$

Now, sort by section

$(Carol, A)$
$(Dave, A)$
$(Ken, A)$
$\boxed{\begin{matrix}(Eric, B)\\(Alice, B)\end{matrix}}$

Not Stable

$(Carol, A)$
$(Dave, A)$
$(Ken, A)$
$\boxed{\begin{matrix}(Alice, B)\\(Eric, B)\end{matrix}}$

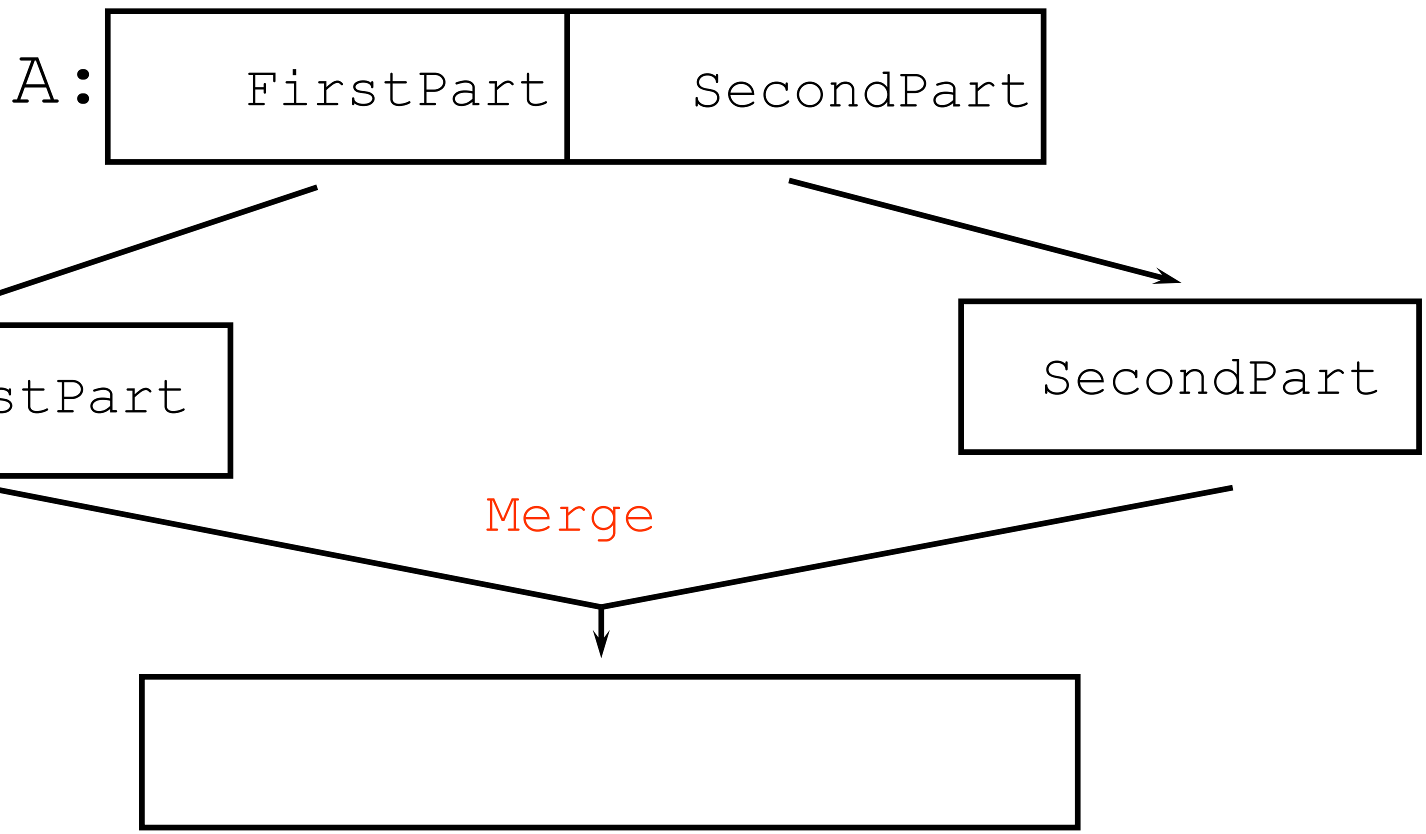
Stable

# Merge Sort: Idea

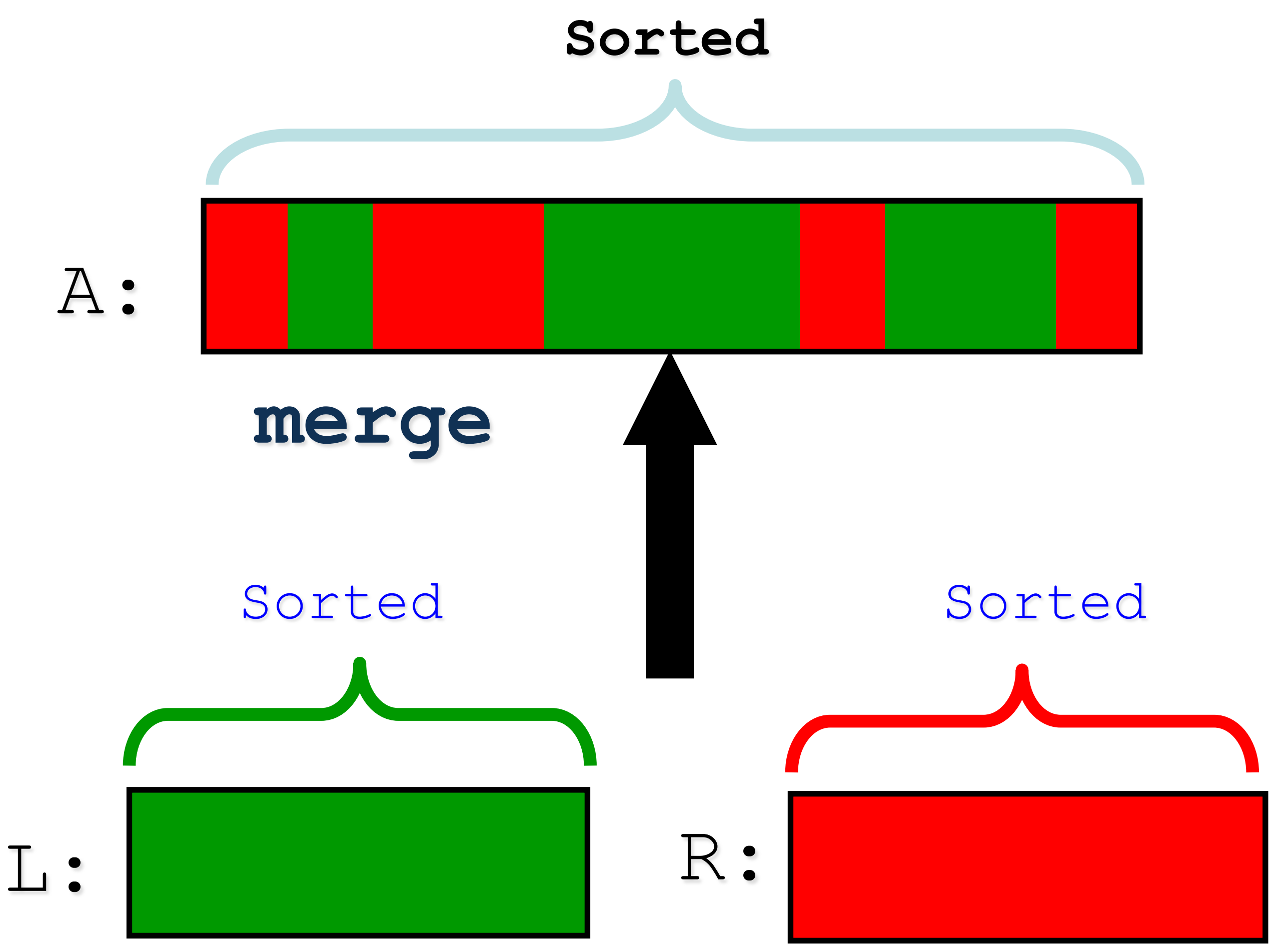
Divide into two halves

Recursively sort

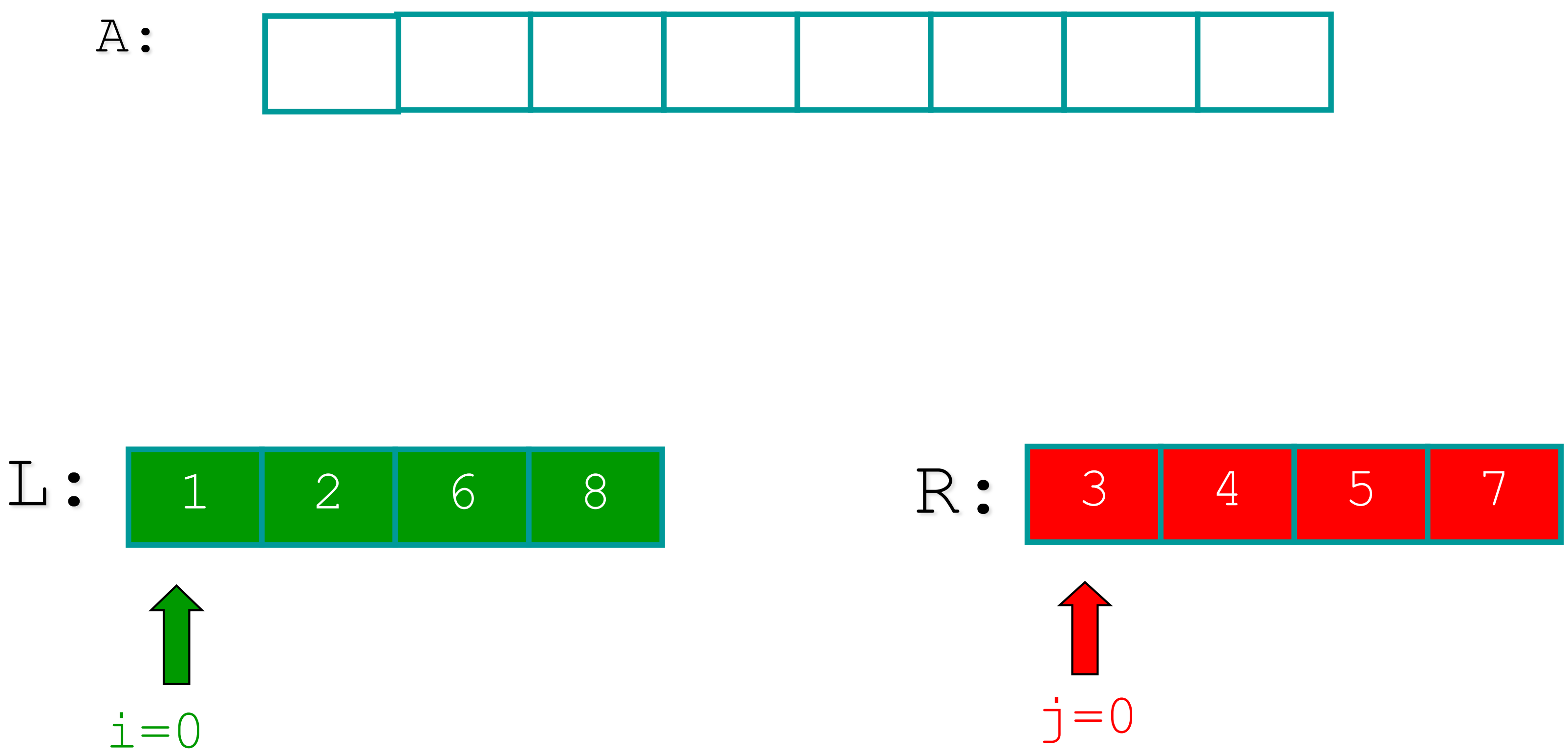
A is sorted!



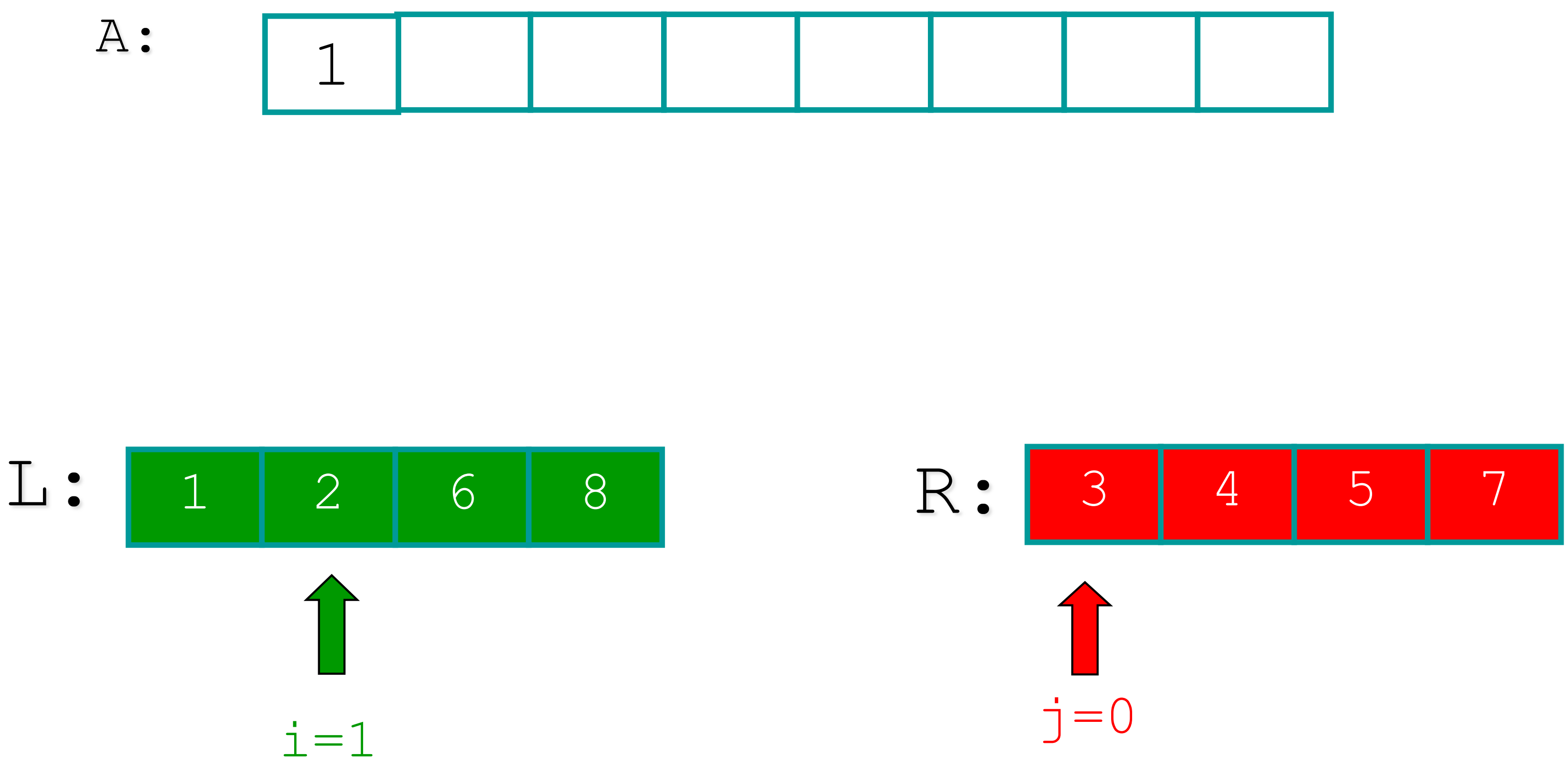
# Merge-Sort: Merge



# Merge Example



# Merge Example



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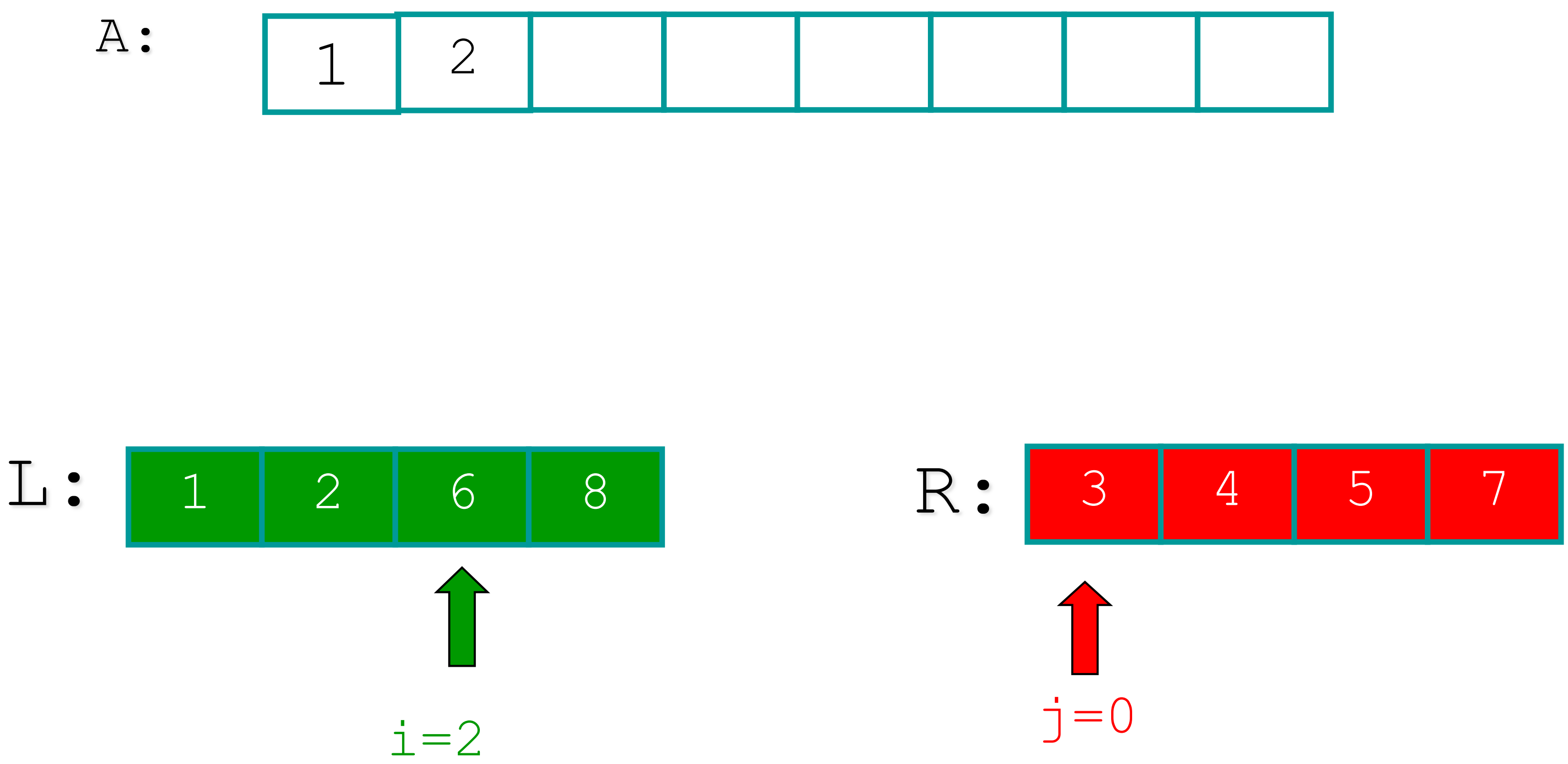
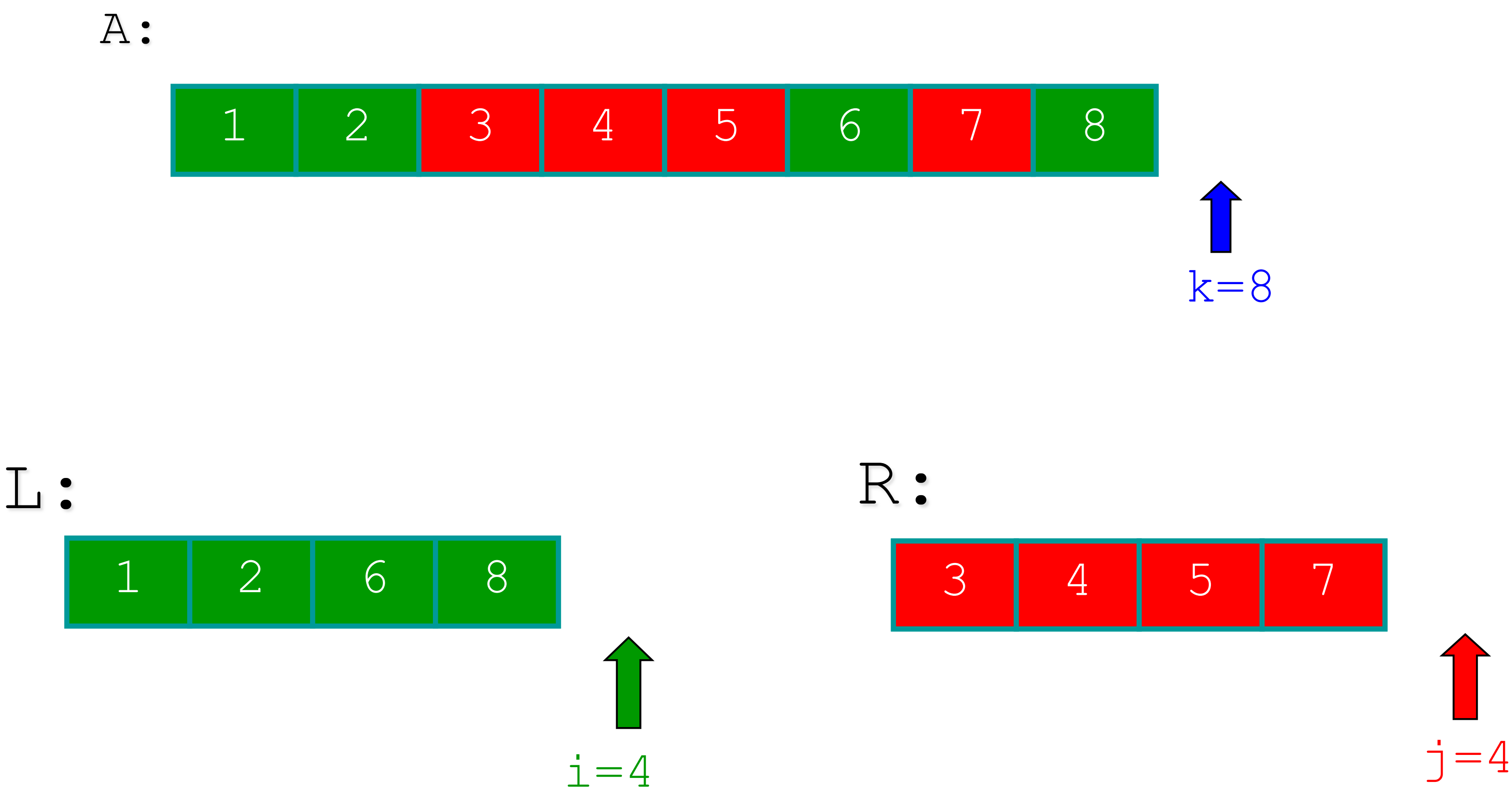


# Merge Example cont.



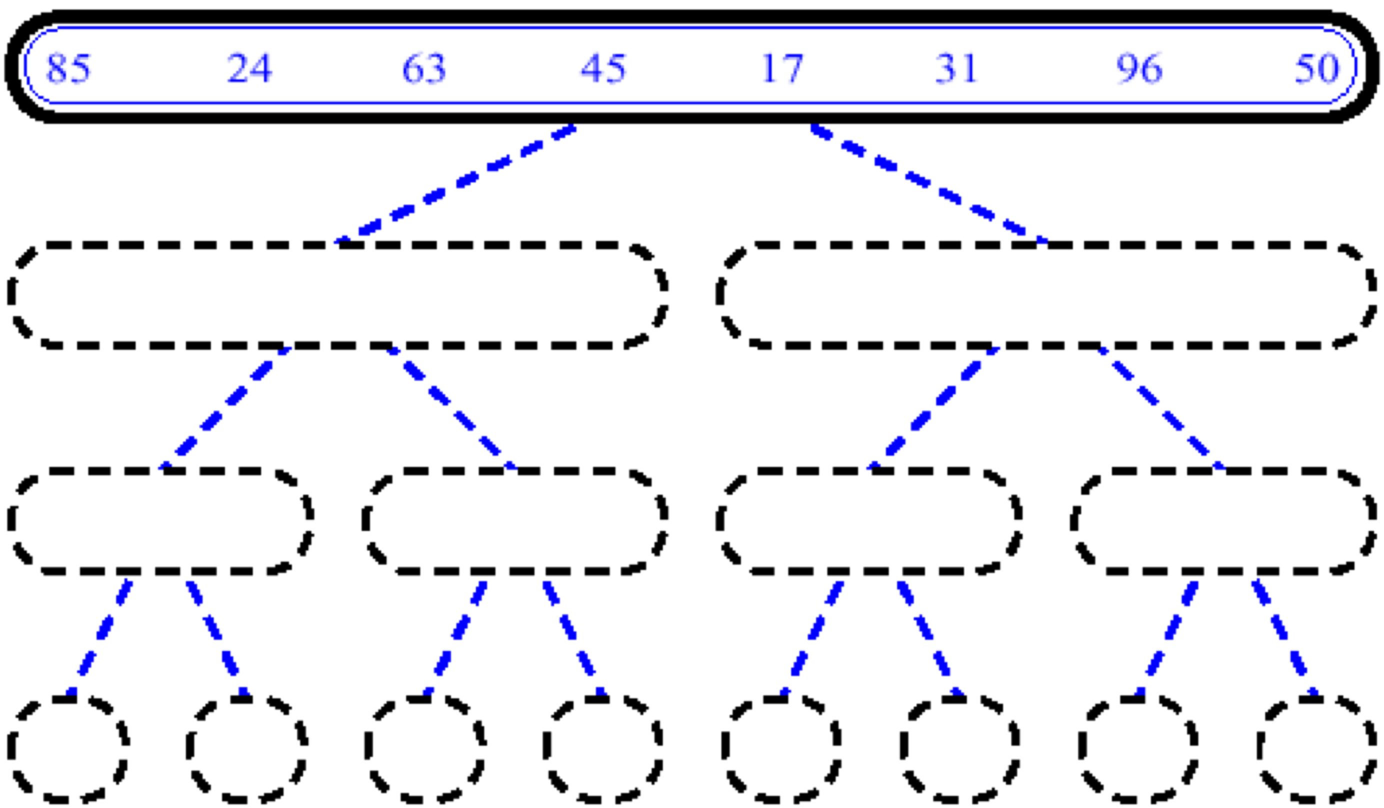
# Merge sort algorithm

**MERGE-SORT** ***A*[1 . . *n*]**

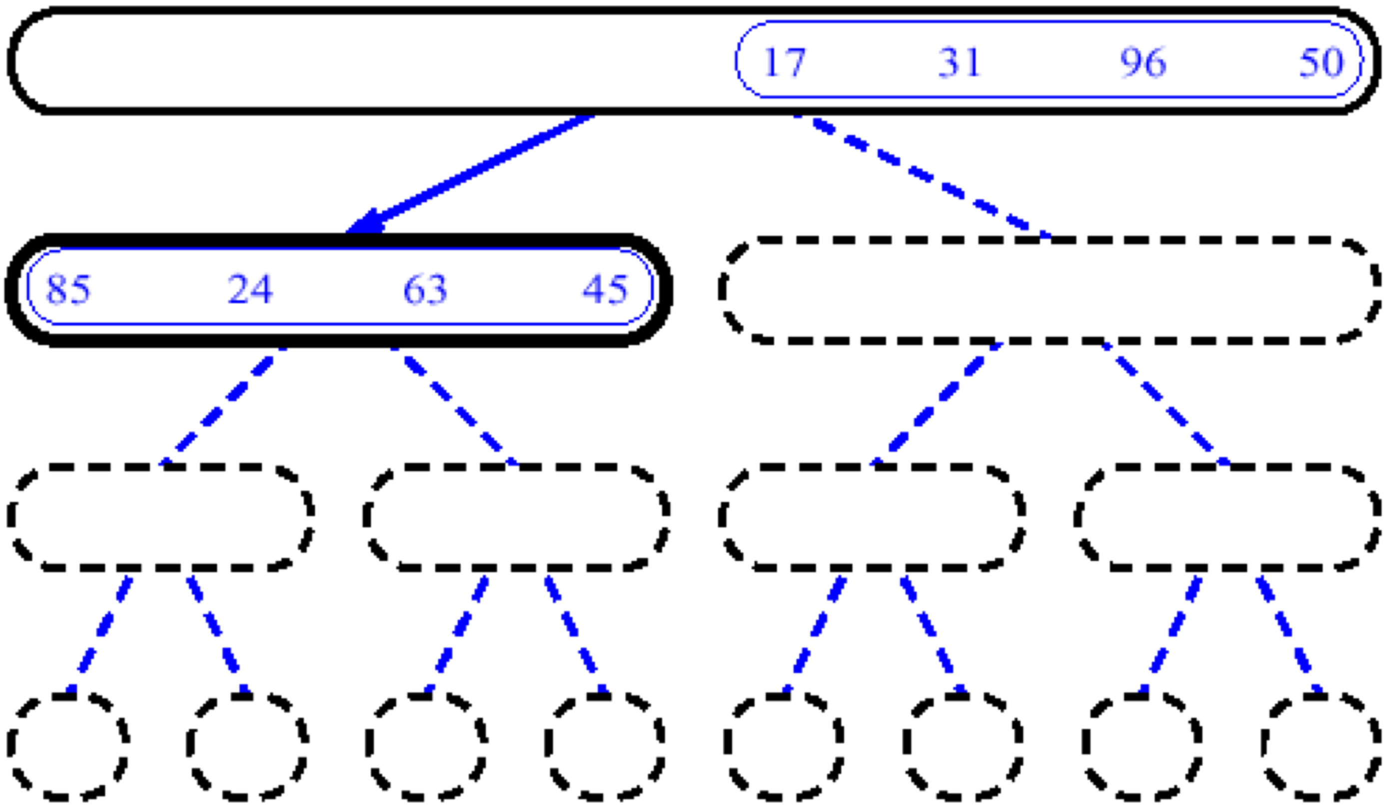
1. If $n = 1$, done.
2. Recursively sort $A[\,1\,..\lceil n/2\rceil\,]$ and $A[\,\lceil n/2\rceil+1\,..\,n\,]$.
3. "*Merge*" the 2 sorted lists.

***Key subroutine:*** **MERGE**

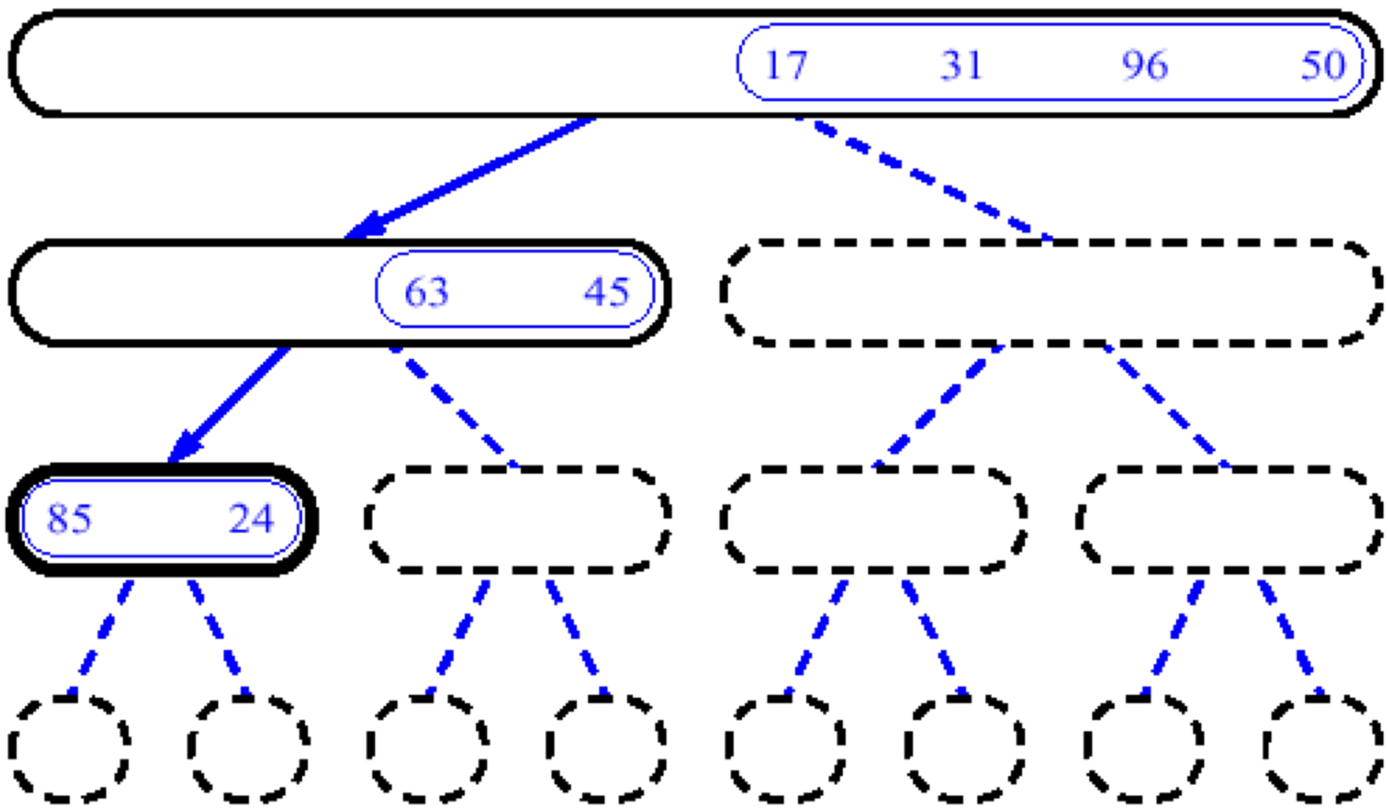
# Merge sort (Example)



# Merge sort (Example)



# Merge sort (Example)



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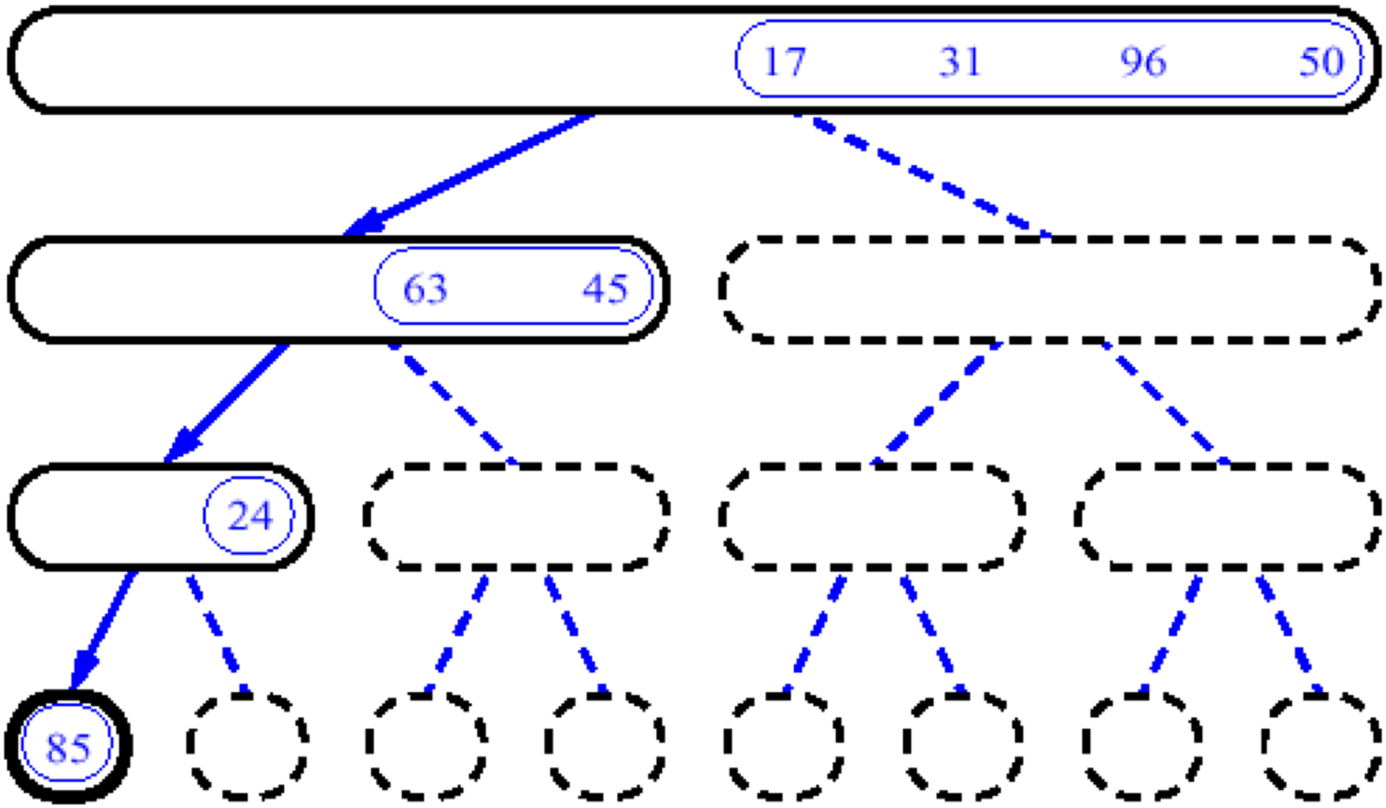

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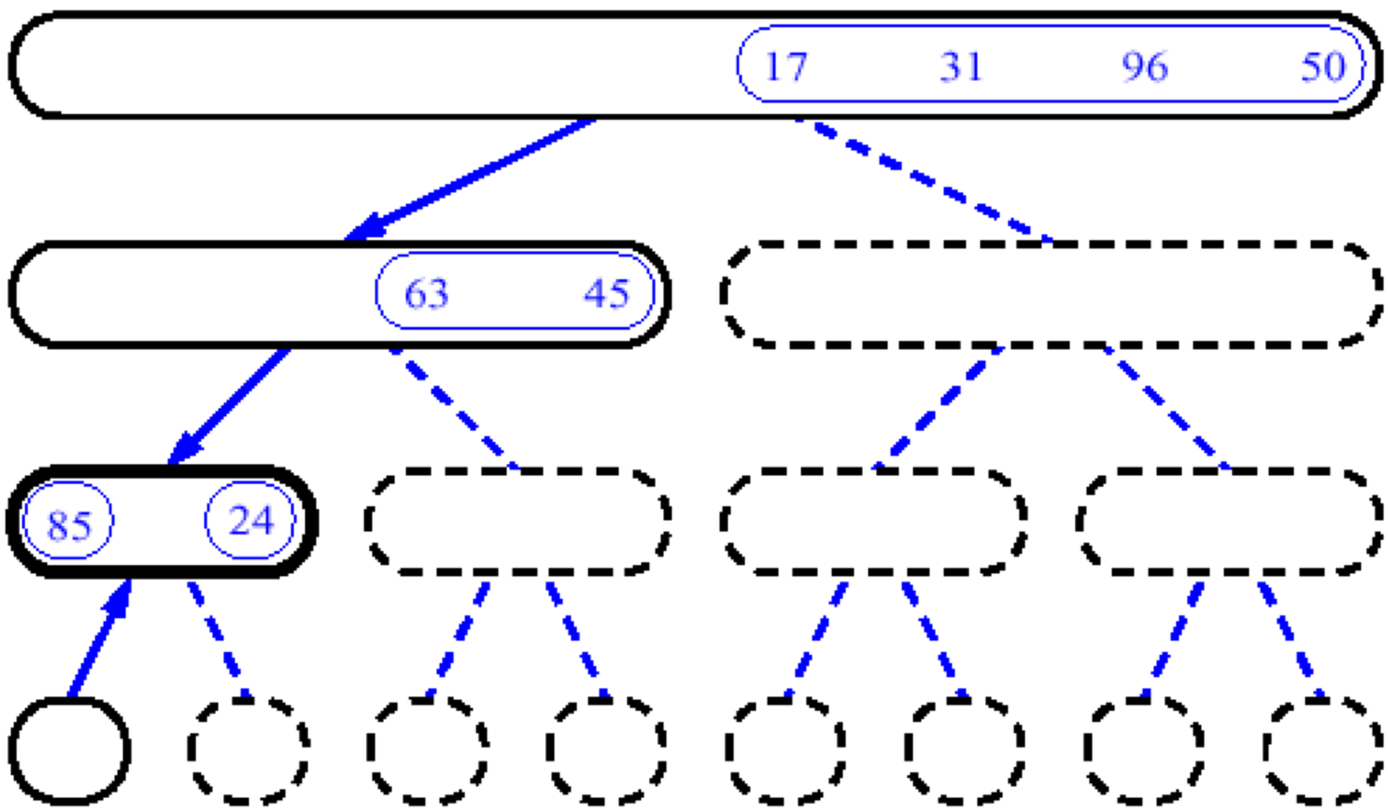
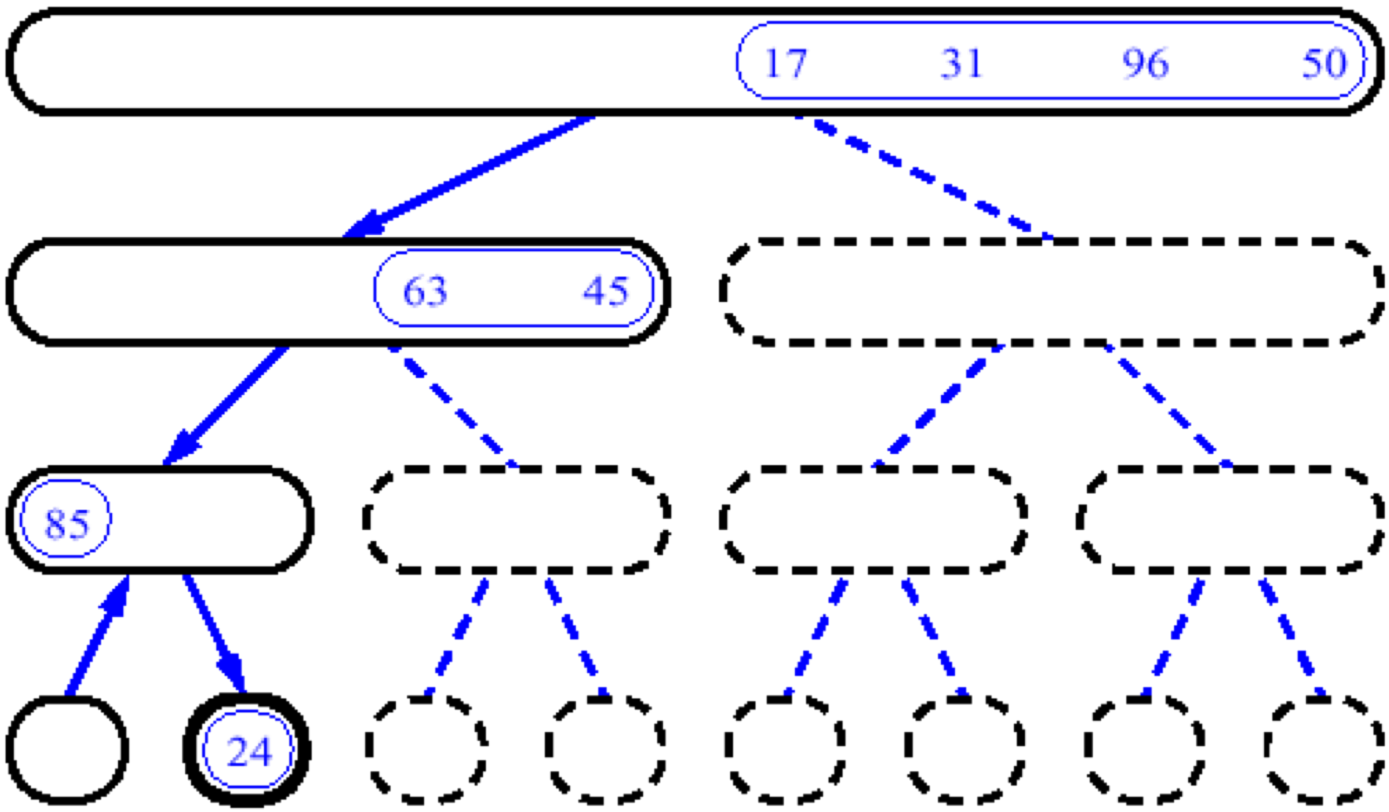
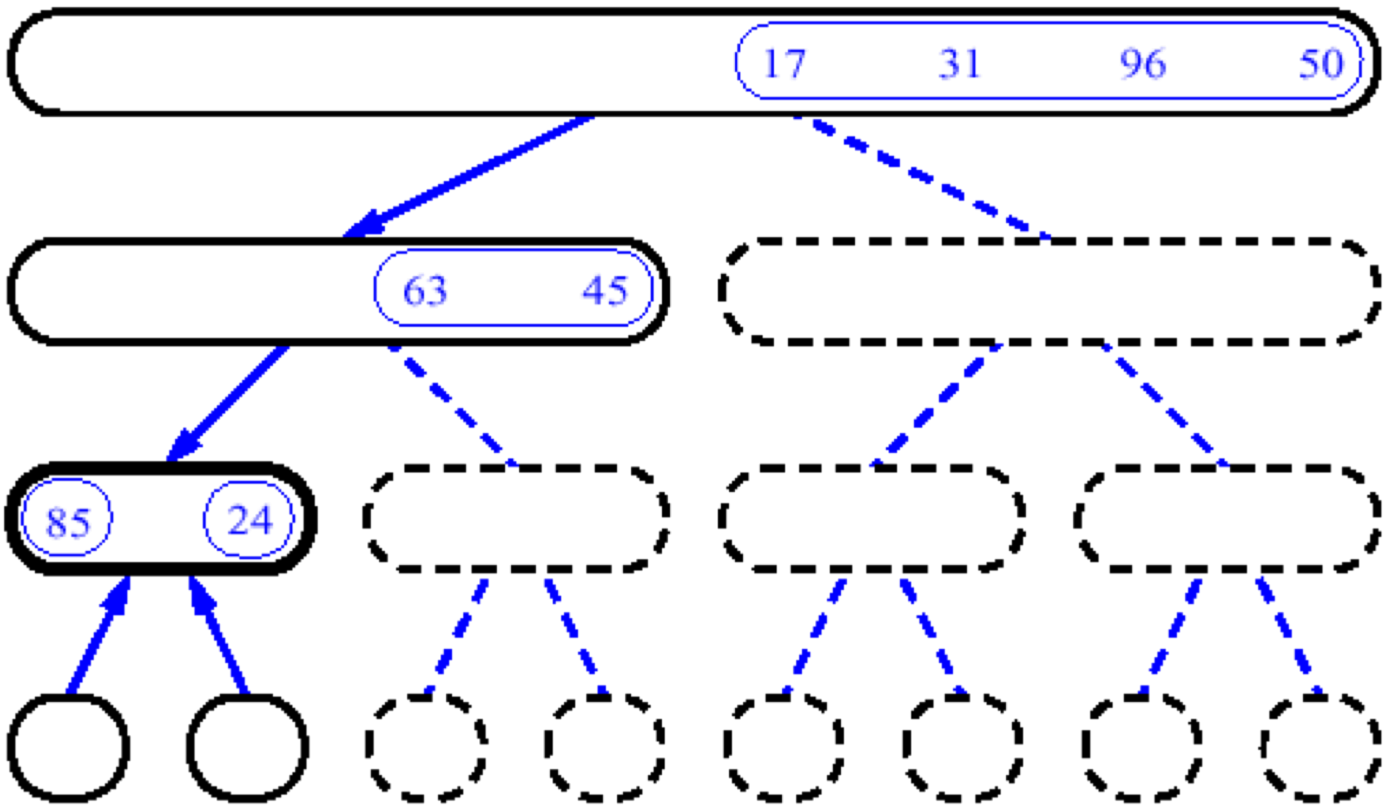


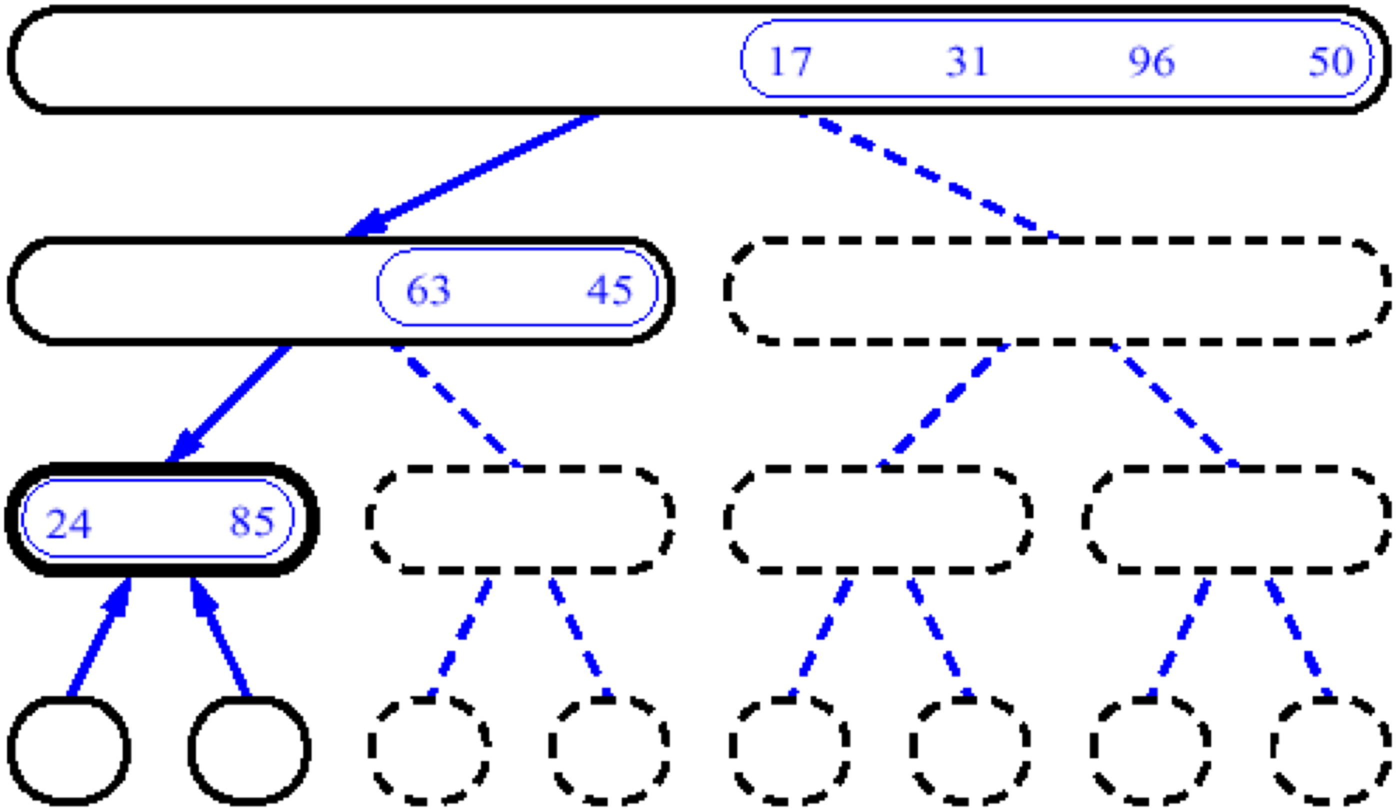
# Merge sort (Example)



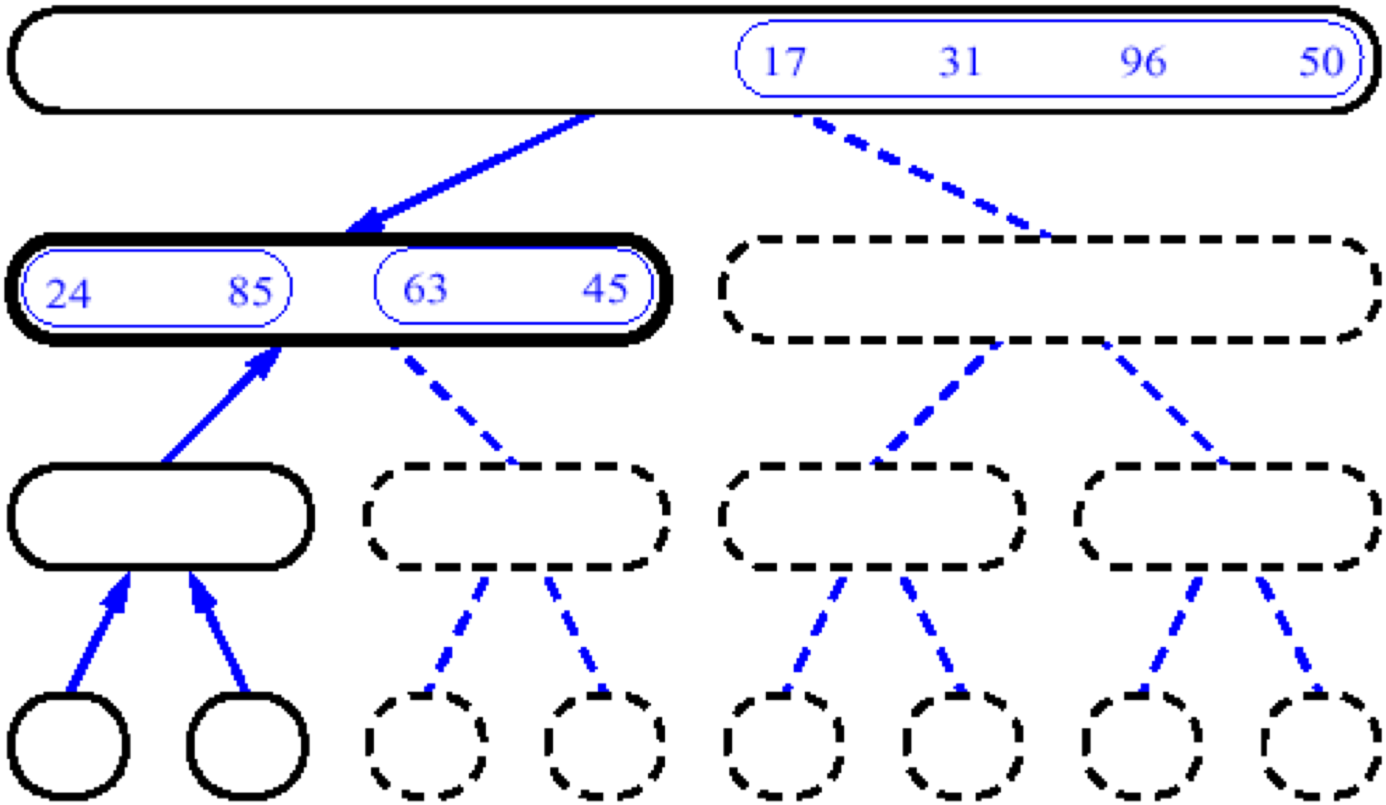
# Merge sort (Example)



# Merge sort (Example)



# Merge sort (Example)



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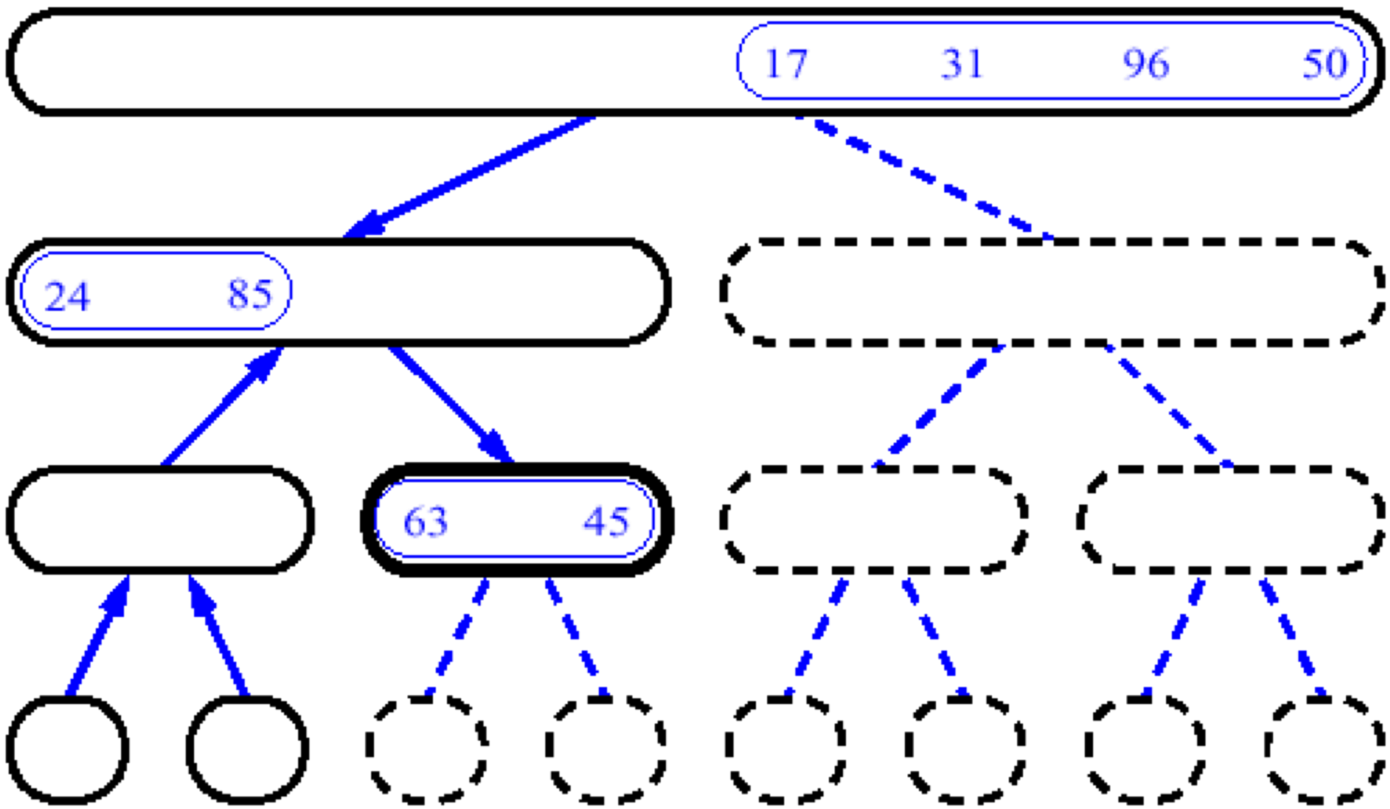
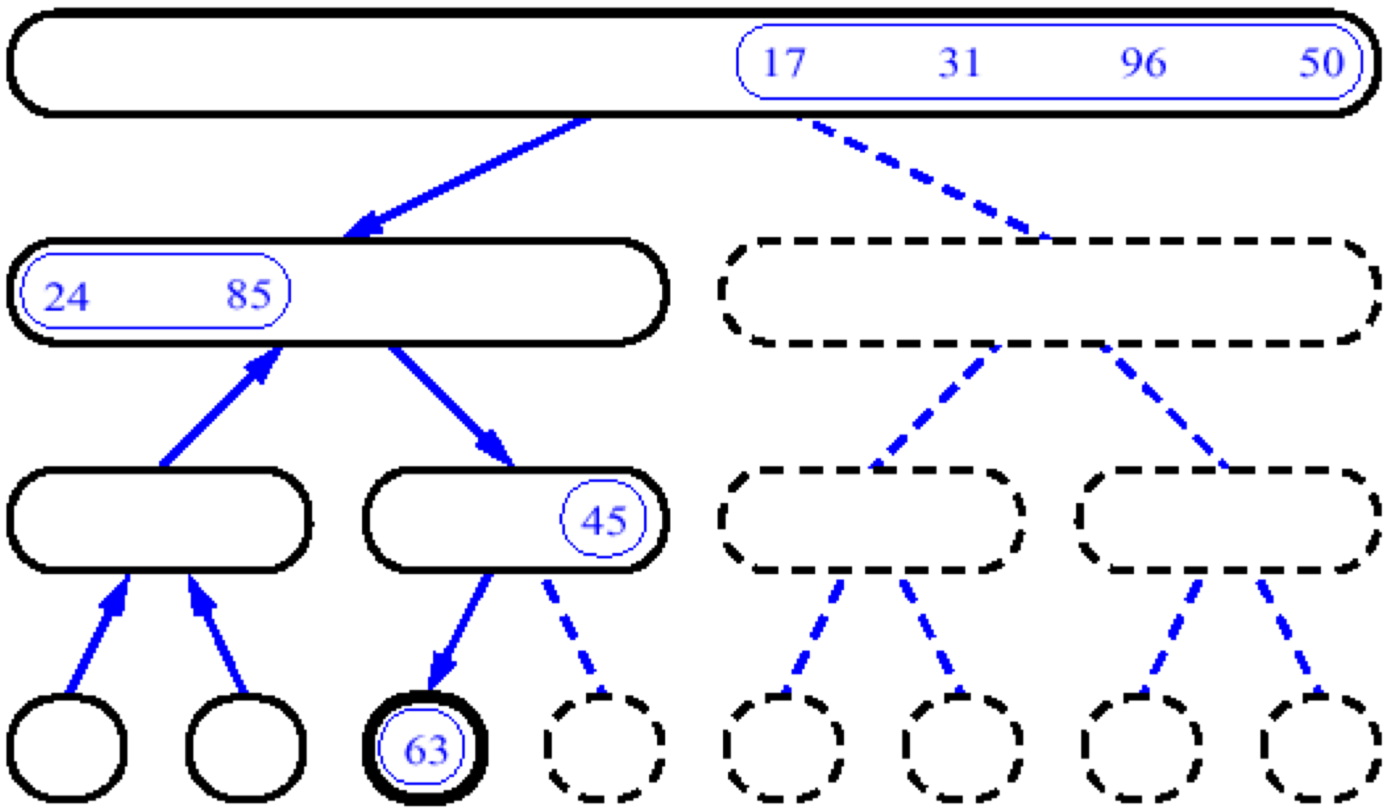
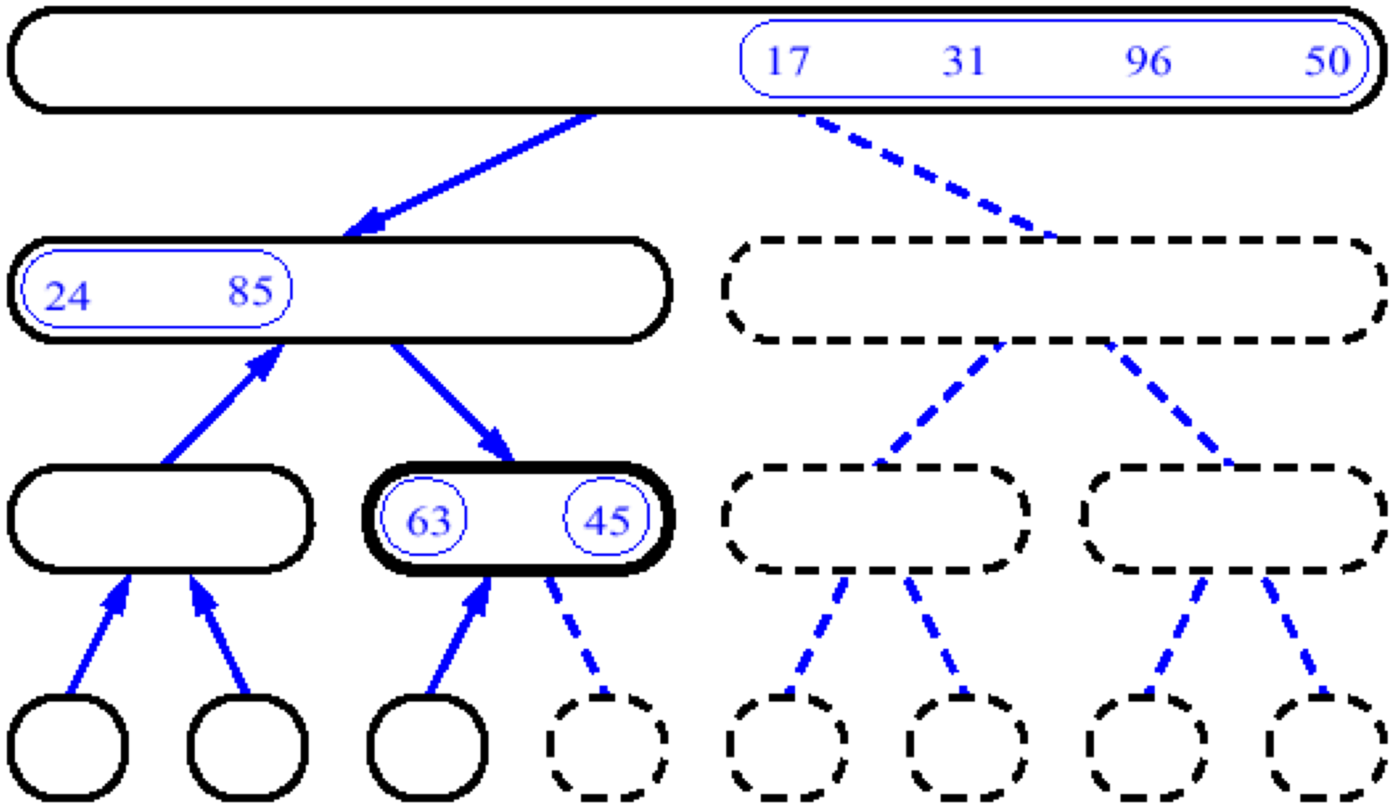


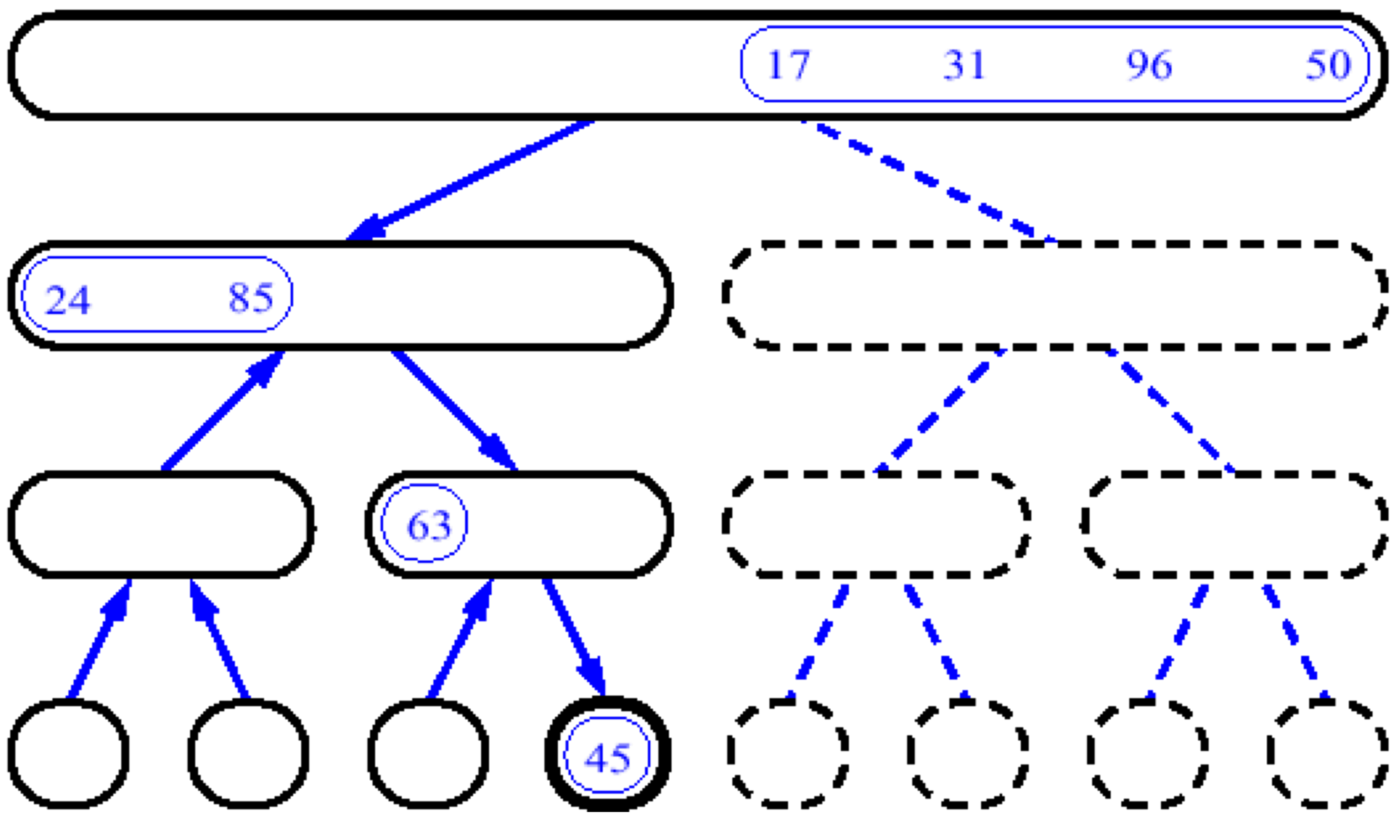
# Merge sort (Example)



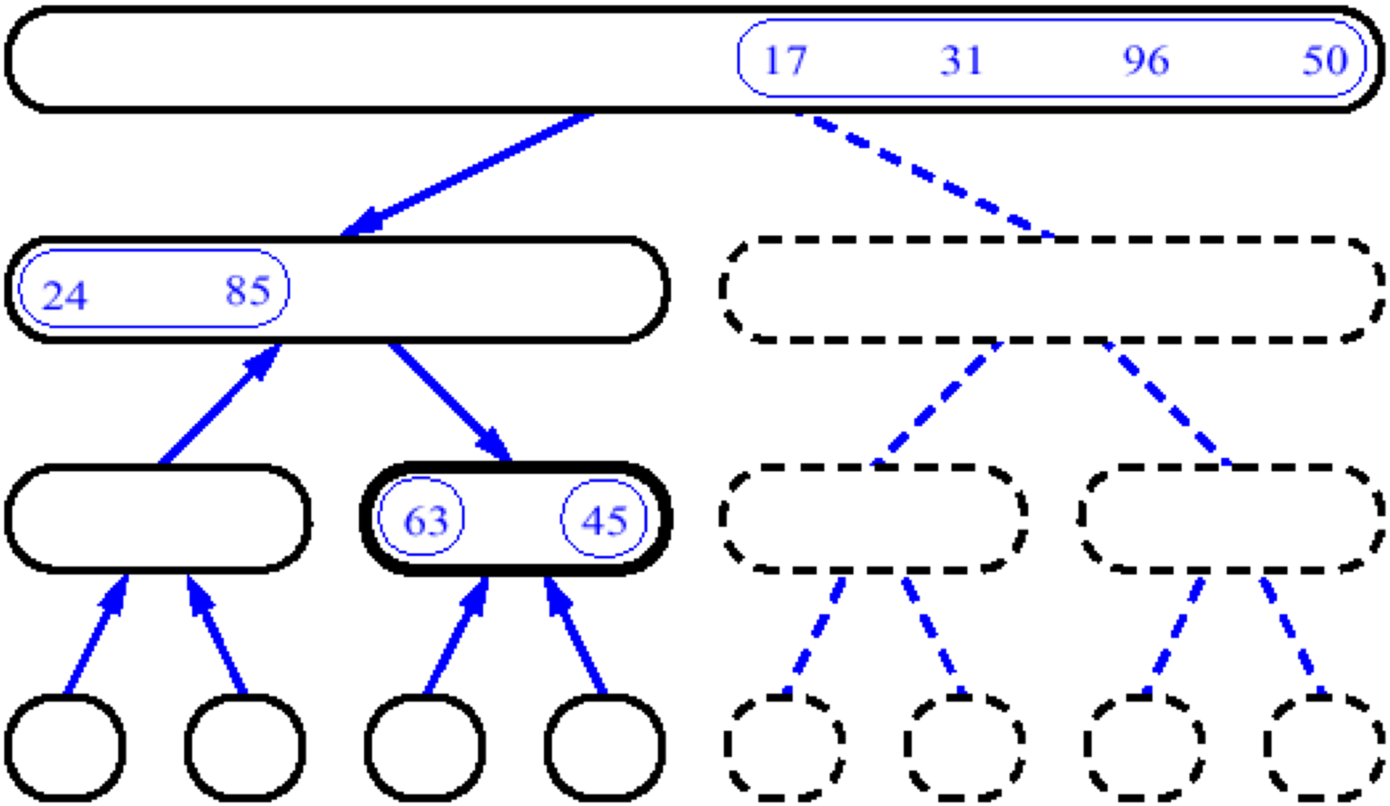
# Merge sort (Example)



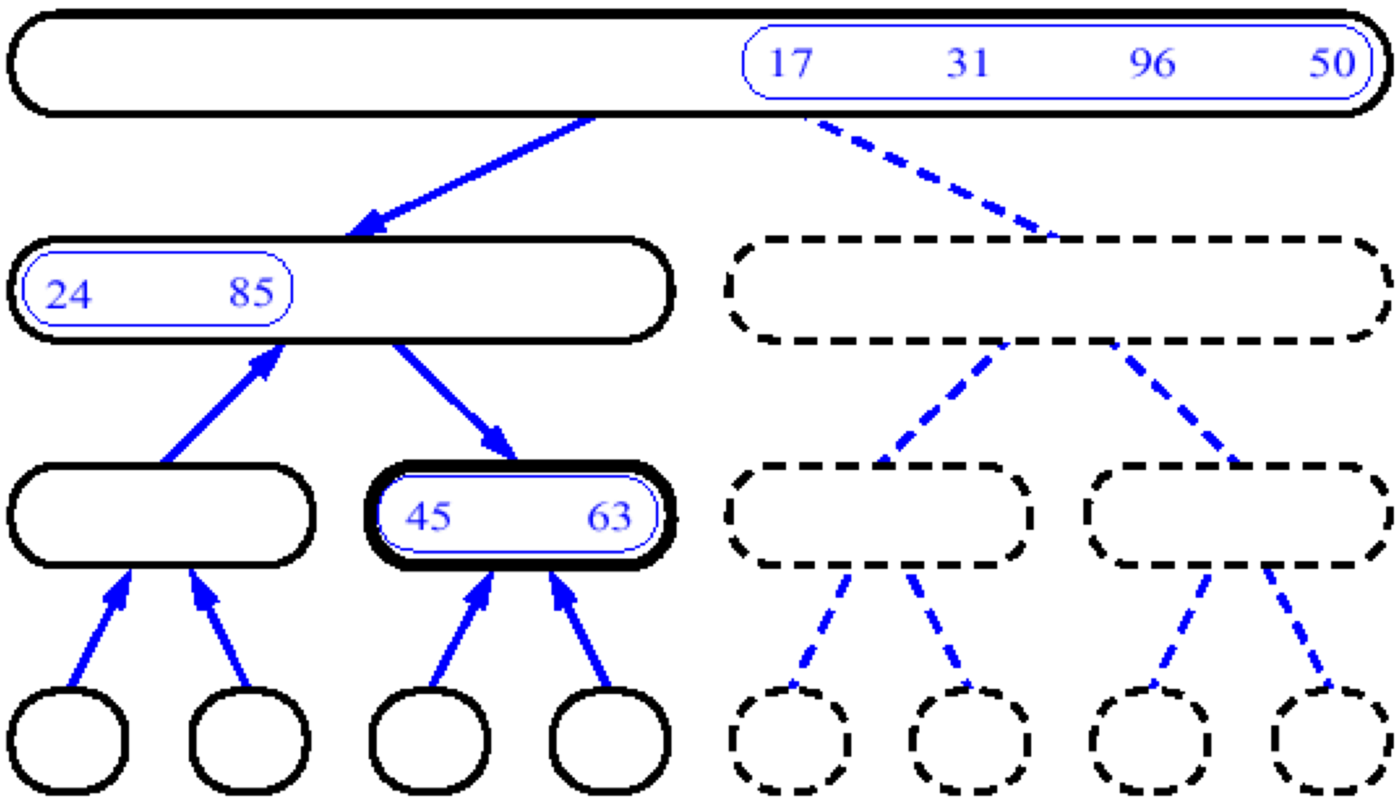
# Merge sort (Example)



# Merge sort (Example)



# Merge sort (Example)



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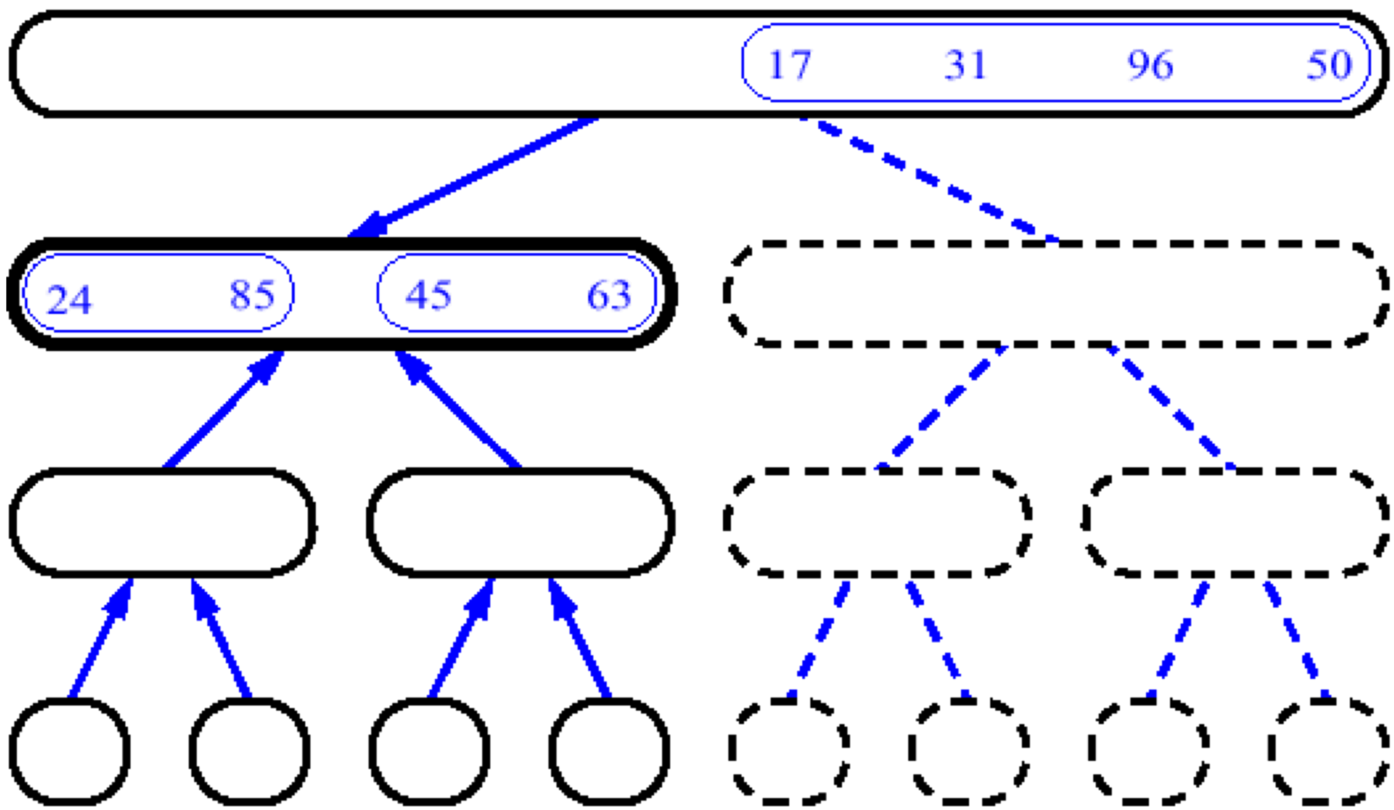

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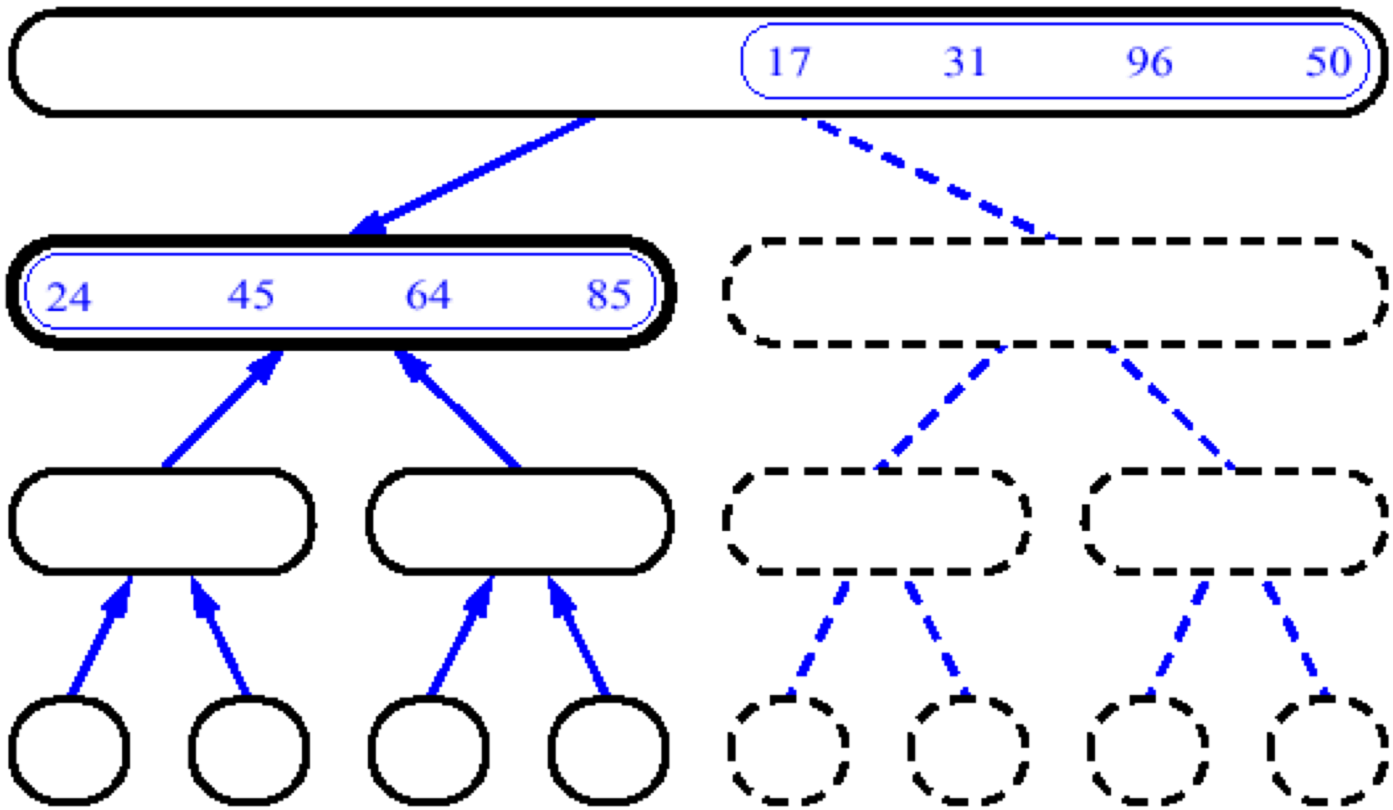

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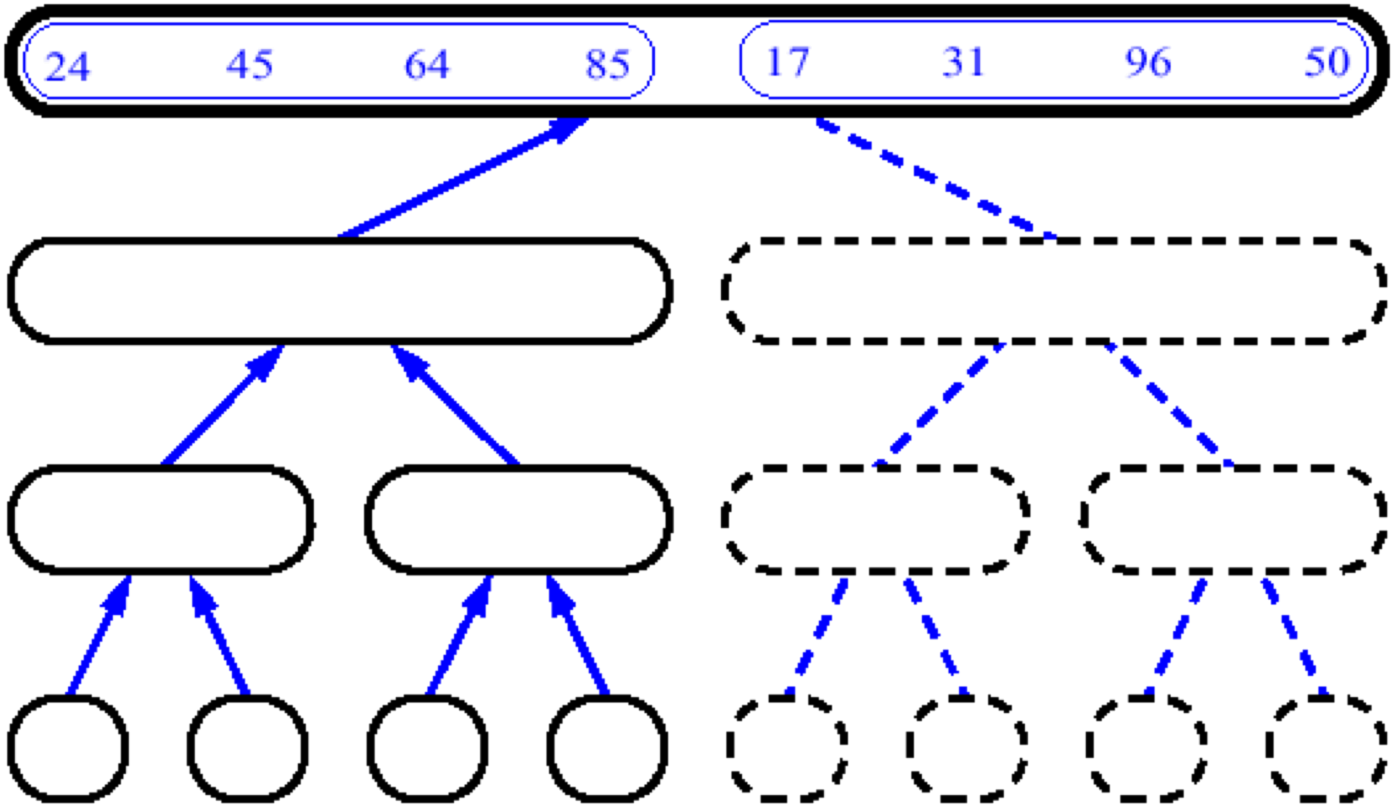

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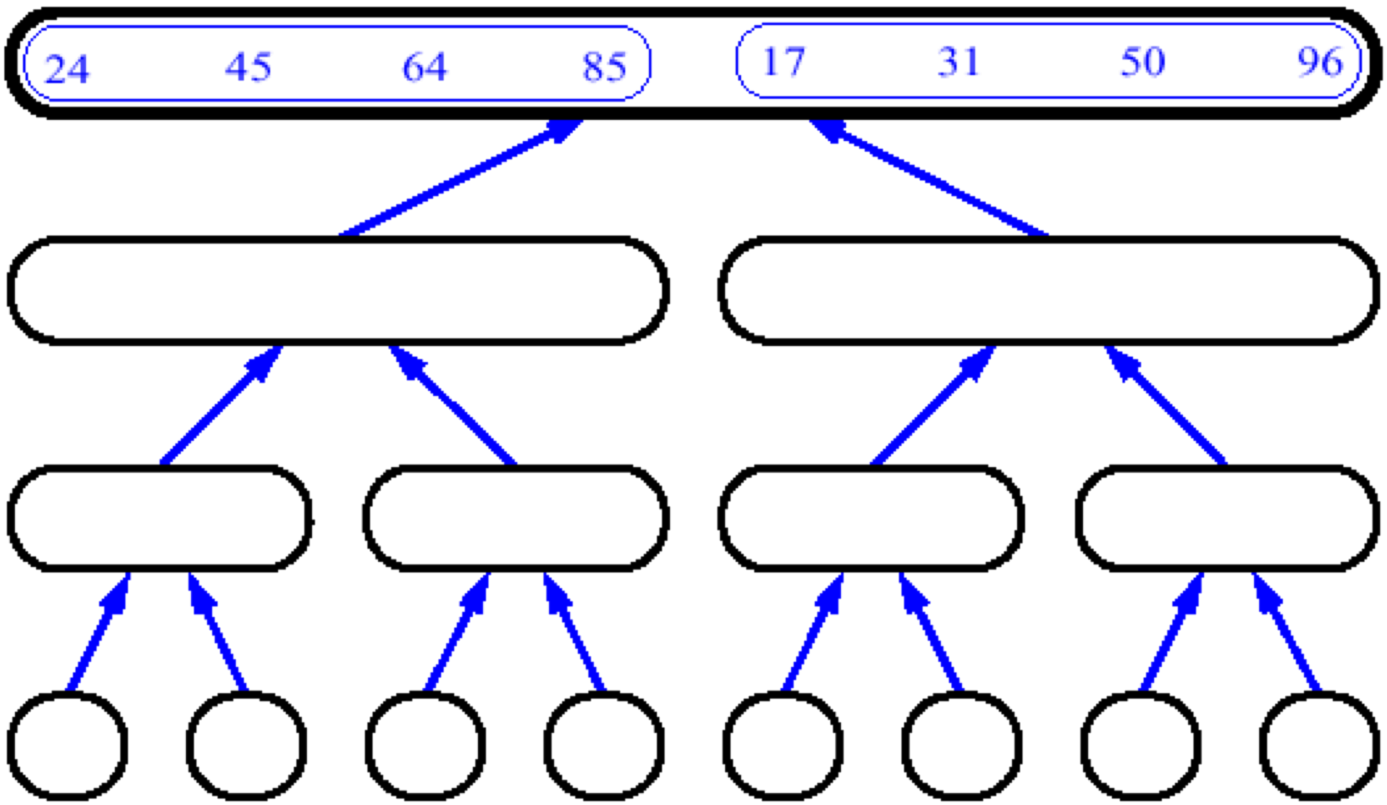

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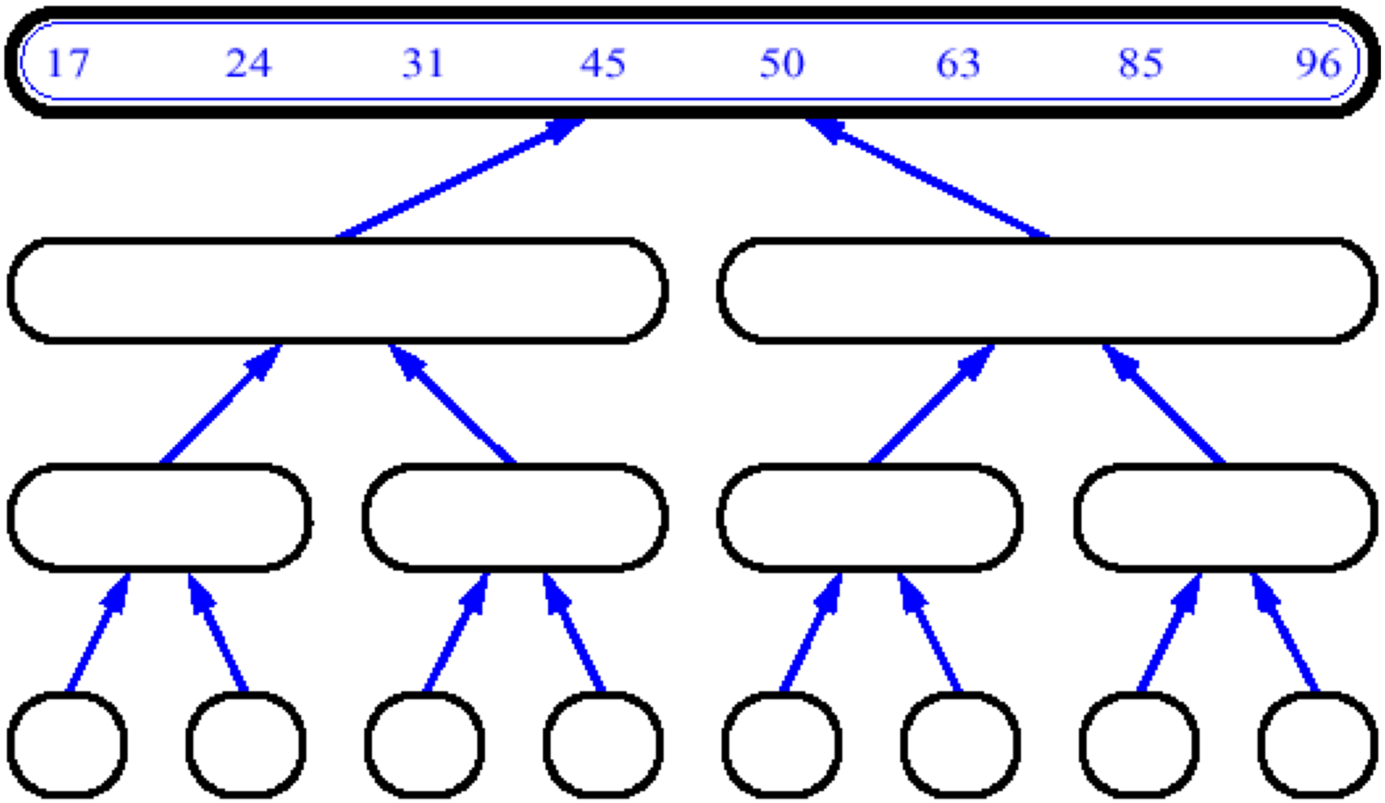


# Analysis of merge sort

| | |
|---|---|
| $T(n)$ | **MERGE-SORT** $A[1 \,..\, n]$ |
| $\Theta(1)$ | 1. If $n = 1$, done. |
| $2T(n/2)$ | 2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$. |
| $\Theta(n)$ | ***3.*** ***"Merge"*** the 2 sorted lists |

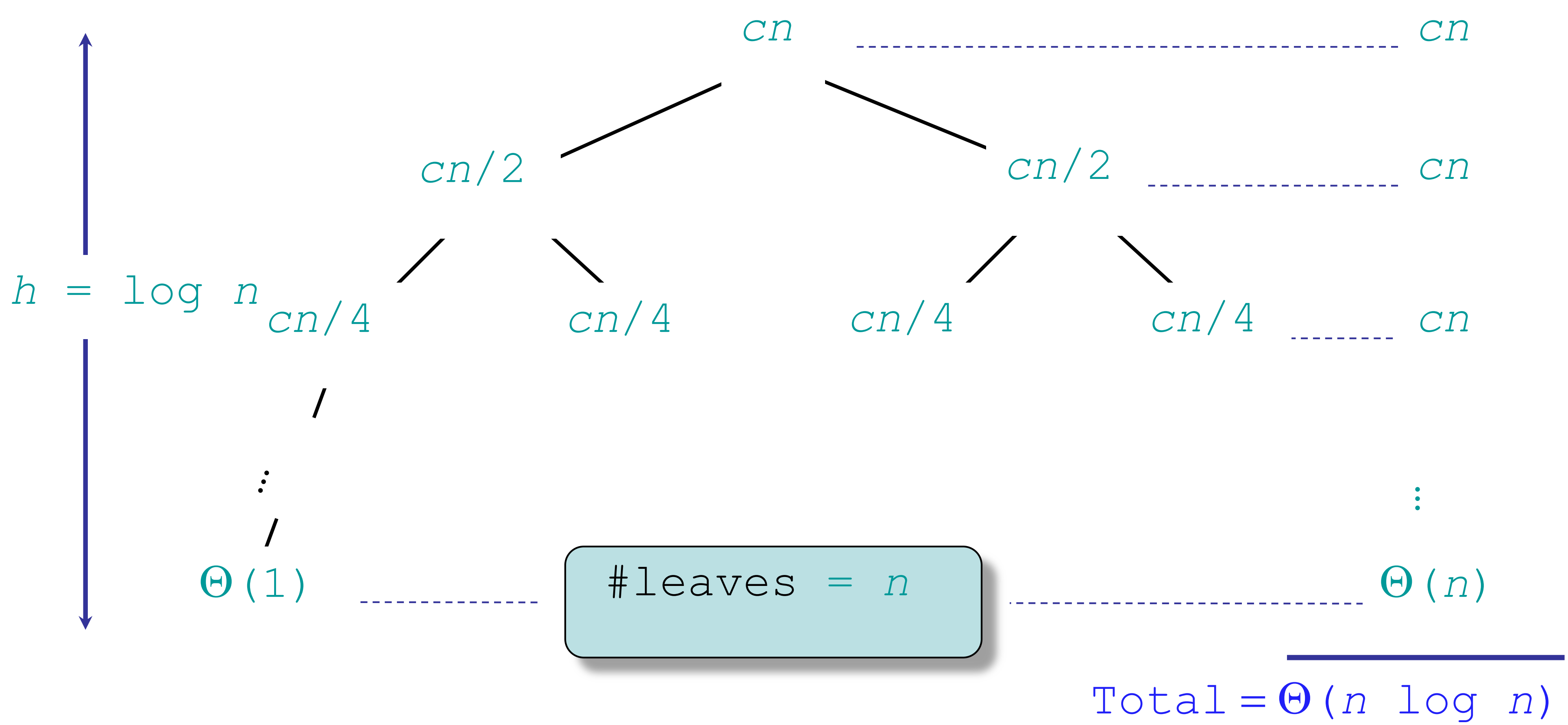
# Analyzing merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

$$T(n) = \Theta(n \lg n) \qquad (\mathrm{n} > 1)$$

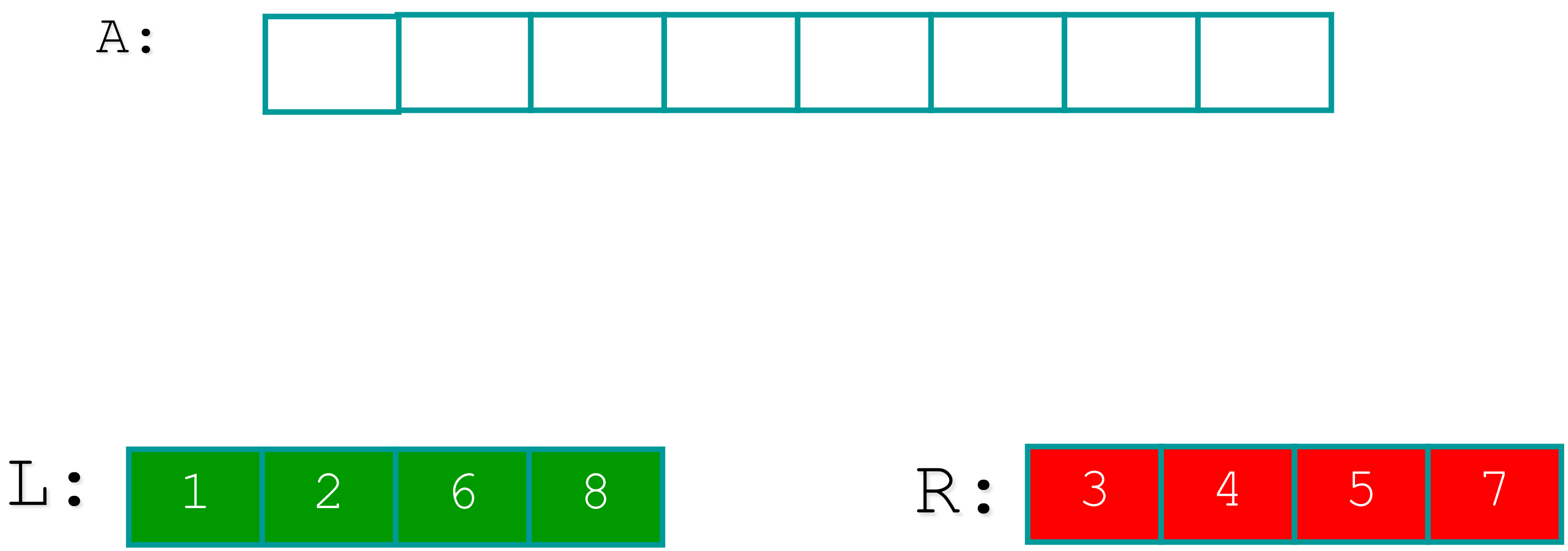
# Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



# Memory Requirement

Needs additional *n* locations because it is difficult to merge two sorted sets in place



# Merge Sort Conclusion

- Merge Sort: $O(n \log n)$
  - asymptotically beats insertion sort in the worst case
  - In practice, merge sort beats insertion sort for $n > 30$ or so
- Space requirement:
  - $O(n)$, not in-place

# HEAPSORT

# Heapsort

- Merge sort time is O(*n* log *n*) but still requires, temporarily, *n* extra storage locations
- *Heapsort* does not require any additional storage
- As its name implies, heapsort uses a heap to store the array

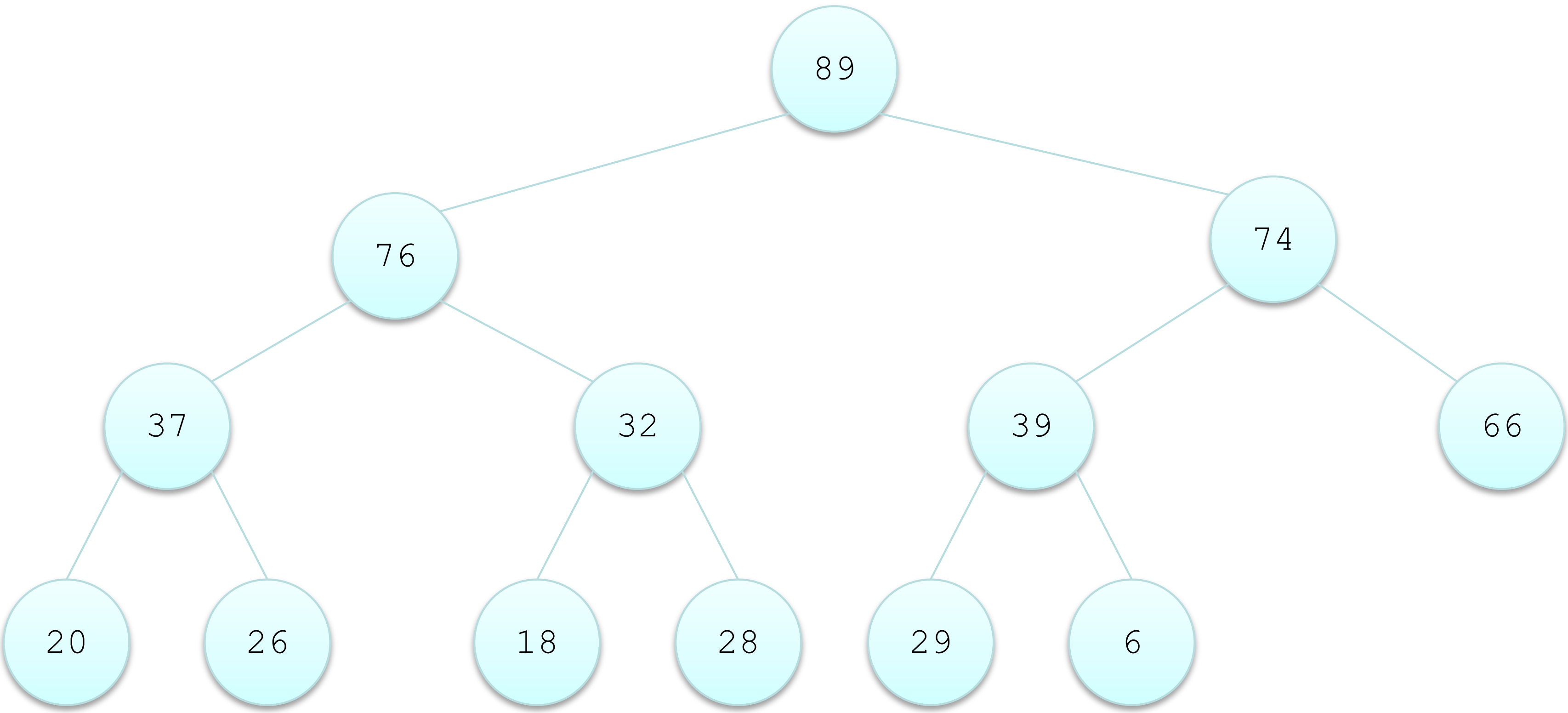
# Heapsort Algorithm

- When used as a priority queue, a heap maintains a smallest value at the top
- The following algorithm
  - places an array's data into a heap,
  - then removes each heap item (O($n$ log $n$)) and moves it back into the array
- This version of the algorithm requires $n$ extra storage locations

**Heapsort Algorithm: First Version**

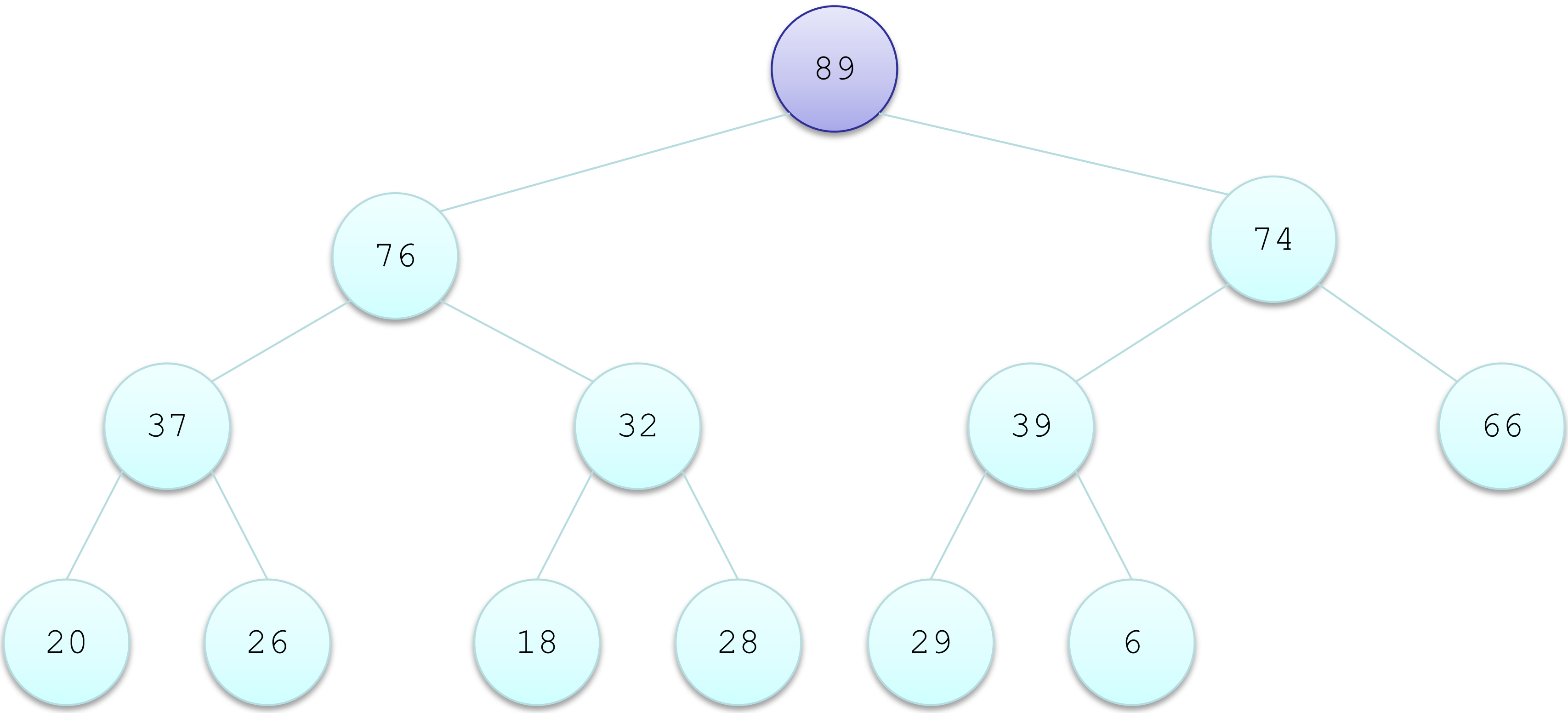
1. **Insert each value from the array to be sorted into a priority queue (heap).**
2. **Set i to 0**
3. **while the priority queue is not empty**
4. **Remove an item from the queue and insert it back into the array at position i**
5. **Increment i**

# Trace of Heapsort

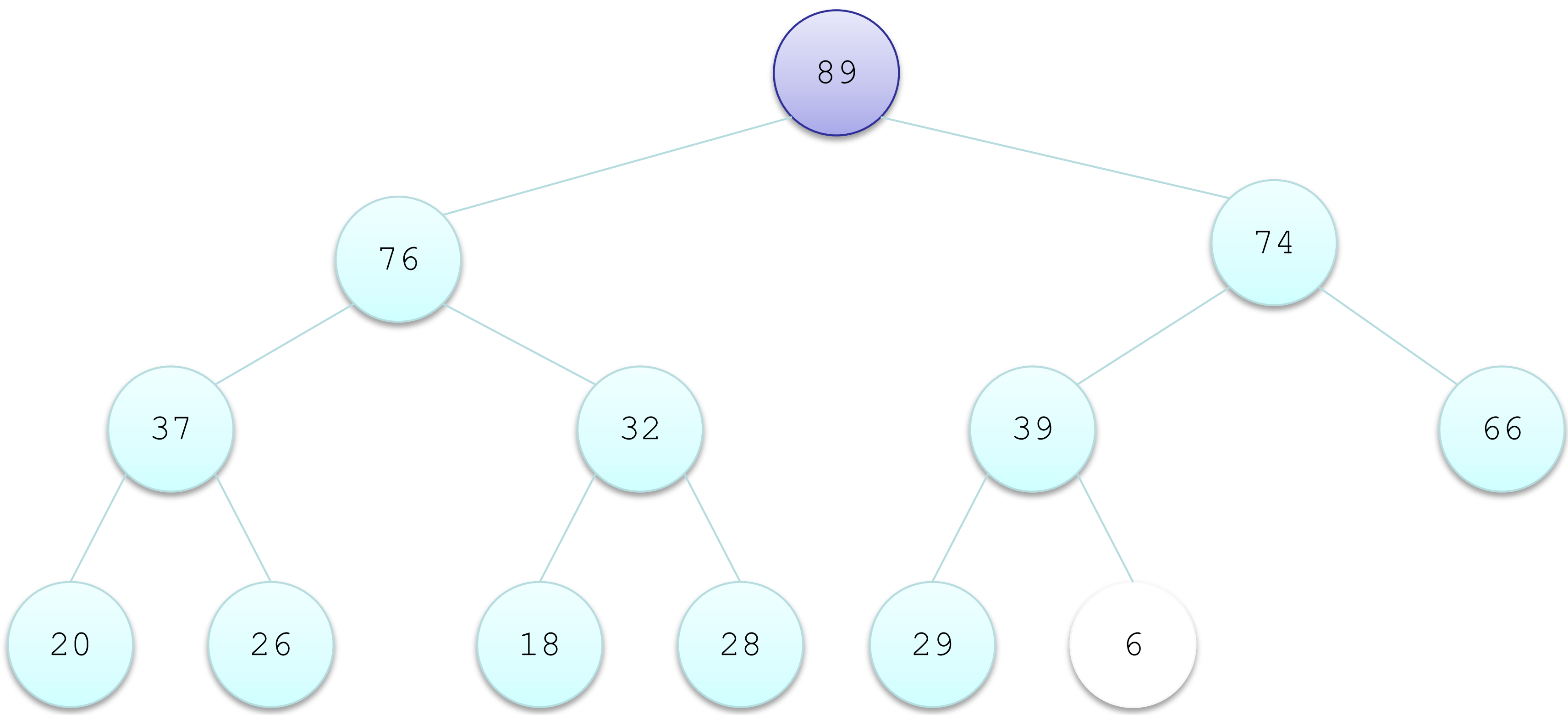



| [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [9] | [10] | [11] | [12] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 89 | 76 | 74 | 37 | 32 | 39 | 66 | 20 | 26 | 18 | 28 | 29 | 6 |

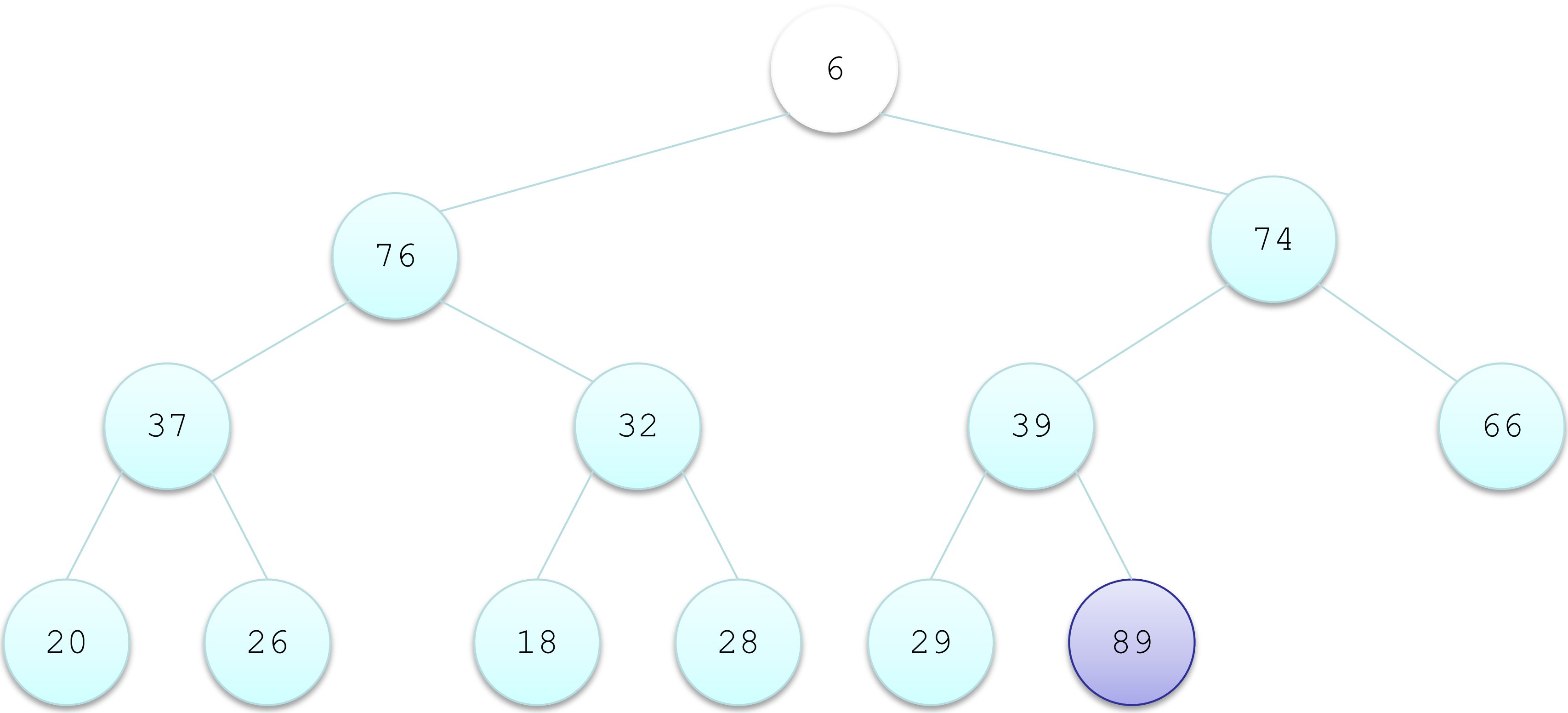
# Trace of Heapsort (cont.)



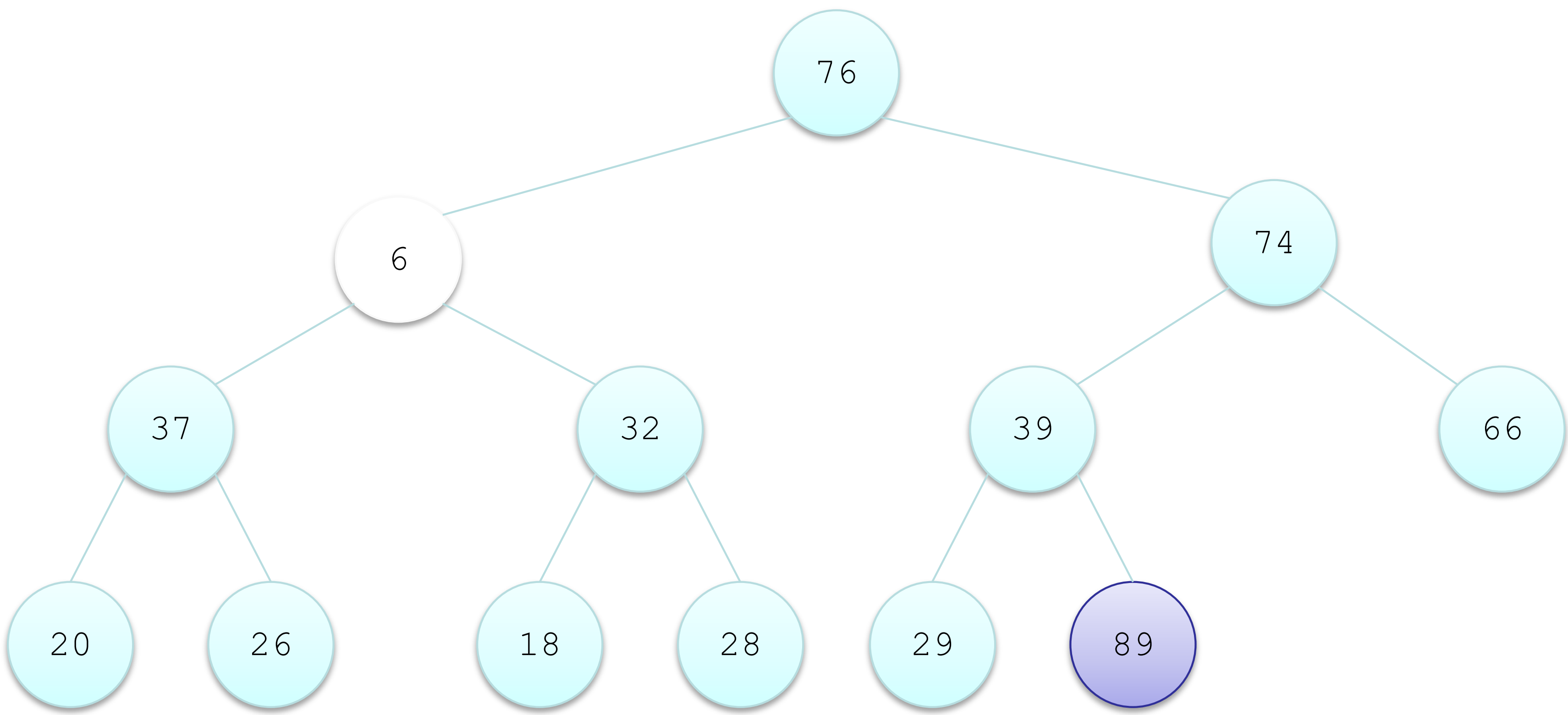
# Trace of Heapsort (cont.)



# Trace of Heapsort (cont.)



# **Trace of Heapsort** (cont.)



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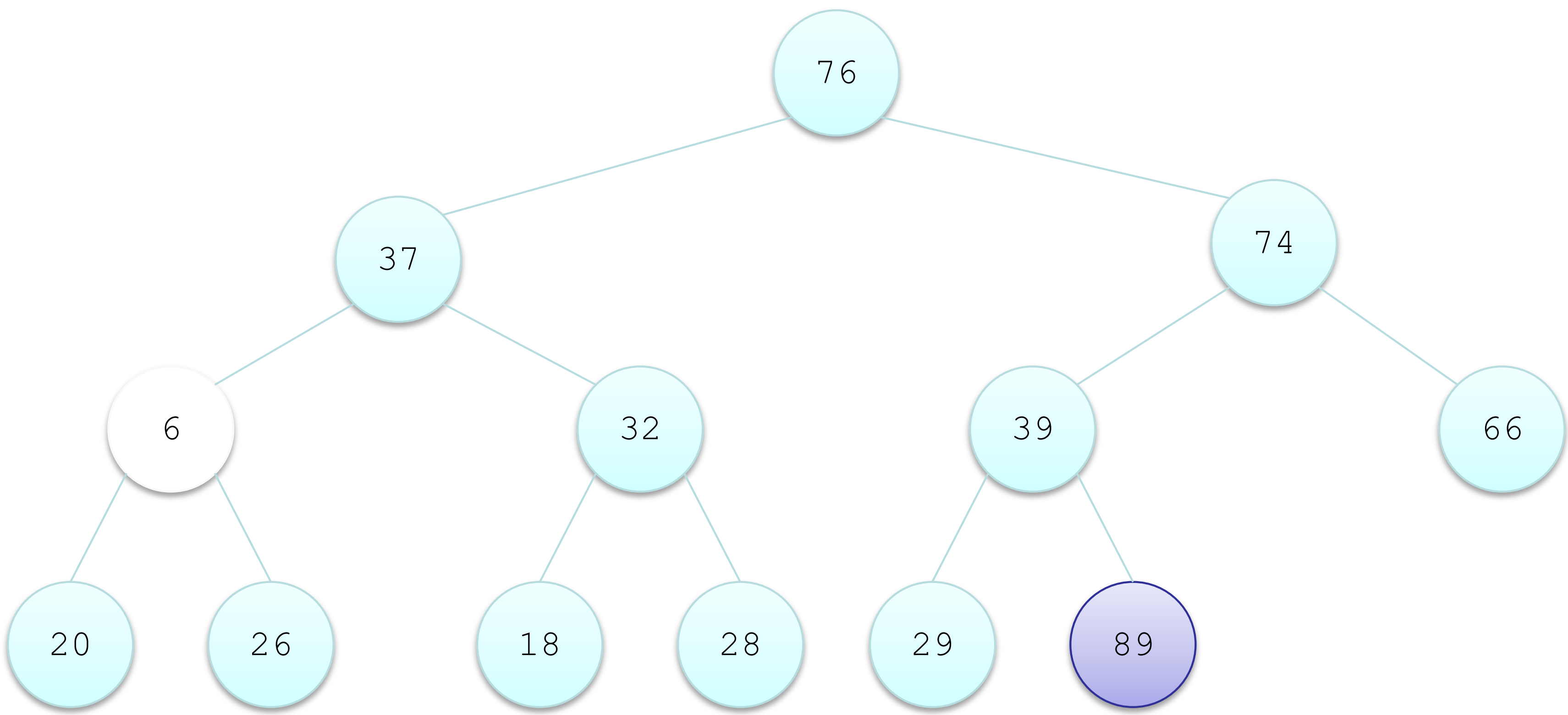
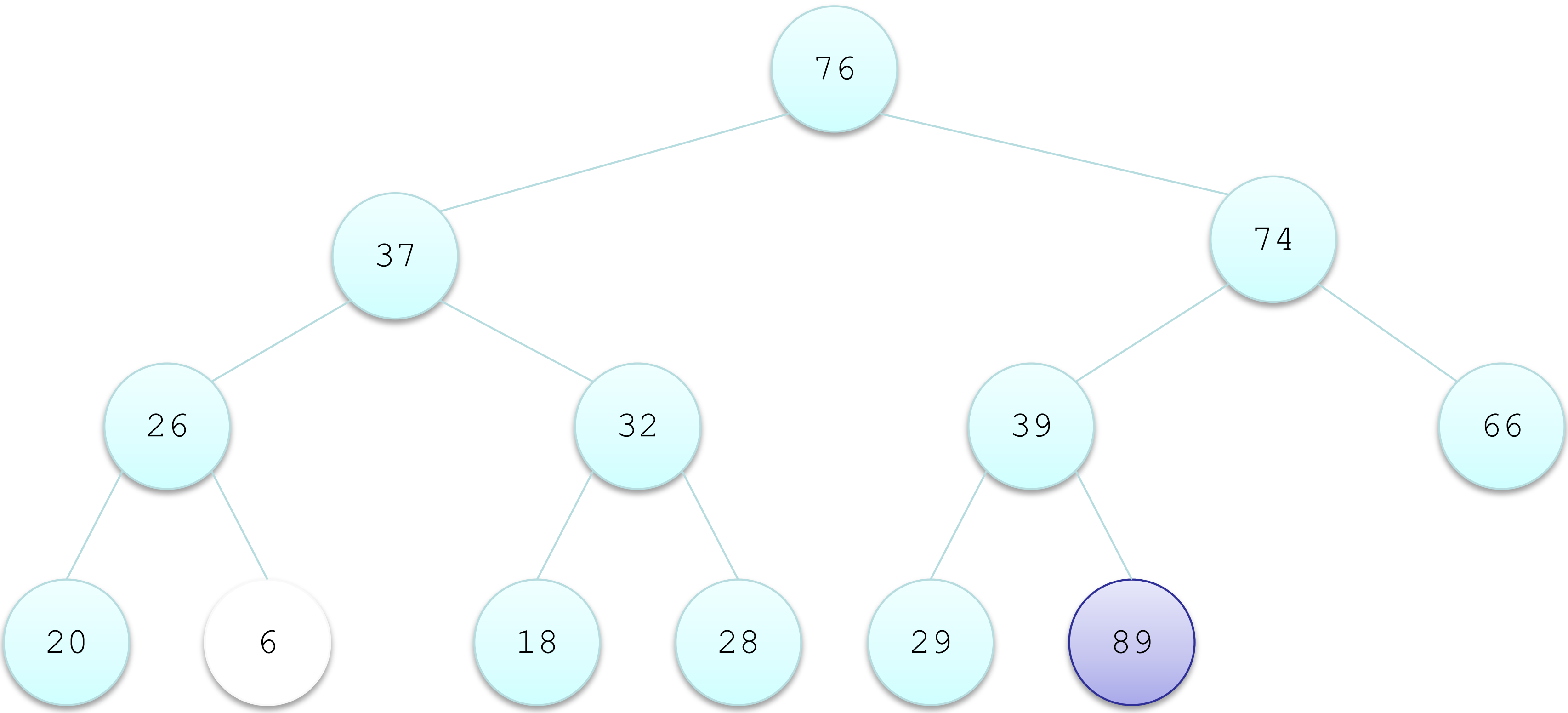
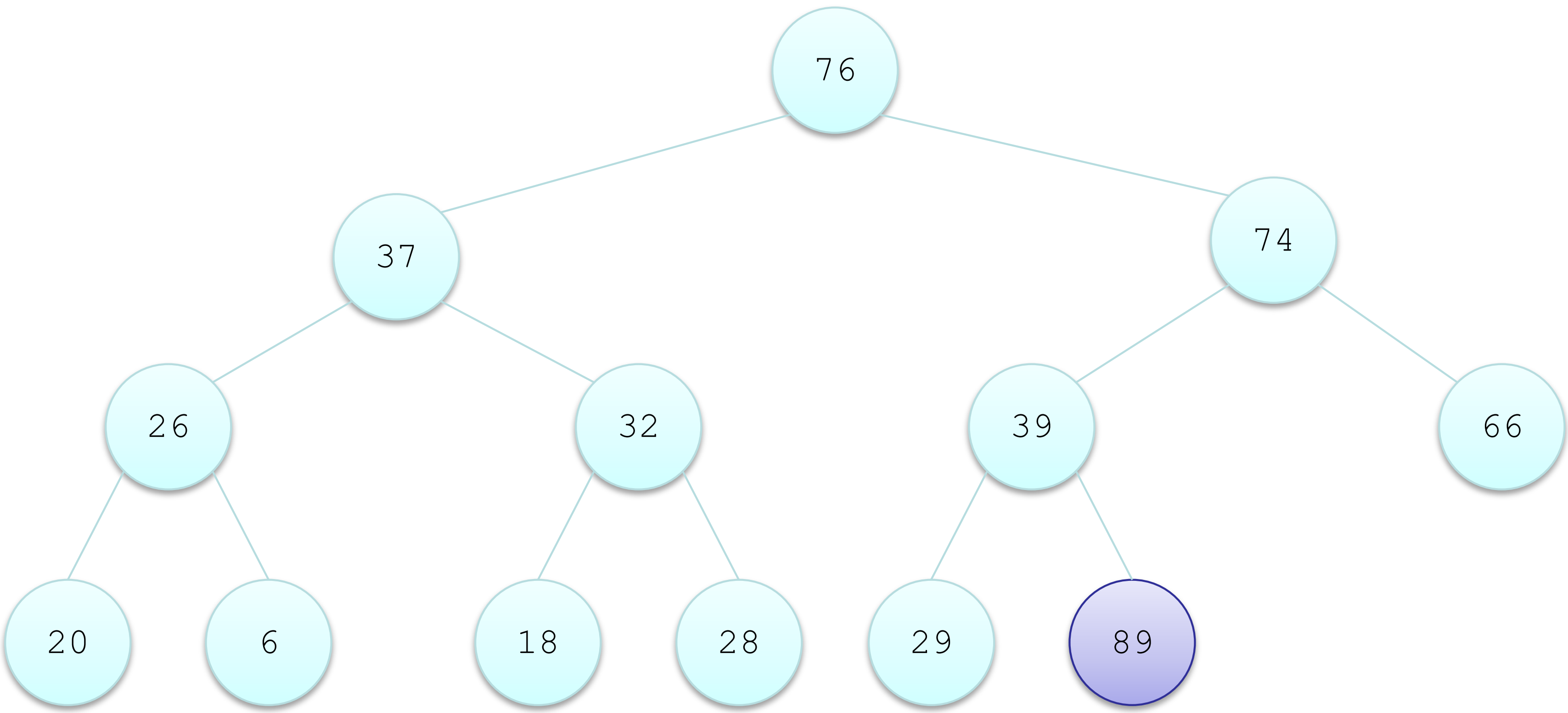


# Trace of Heapsort (cont.)



# Trace of Heapsort (cont.)

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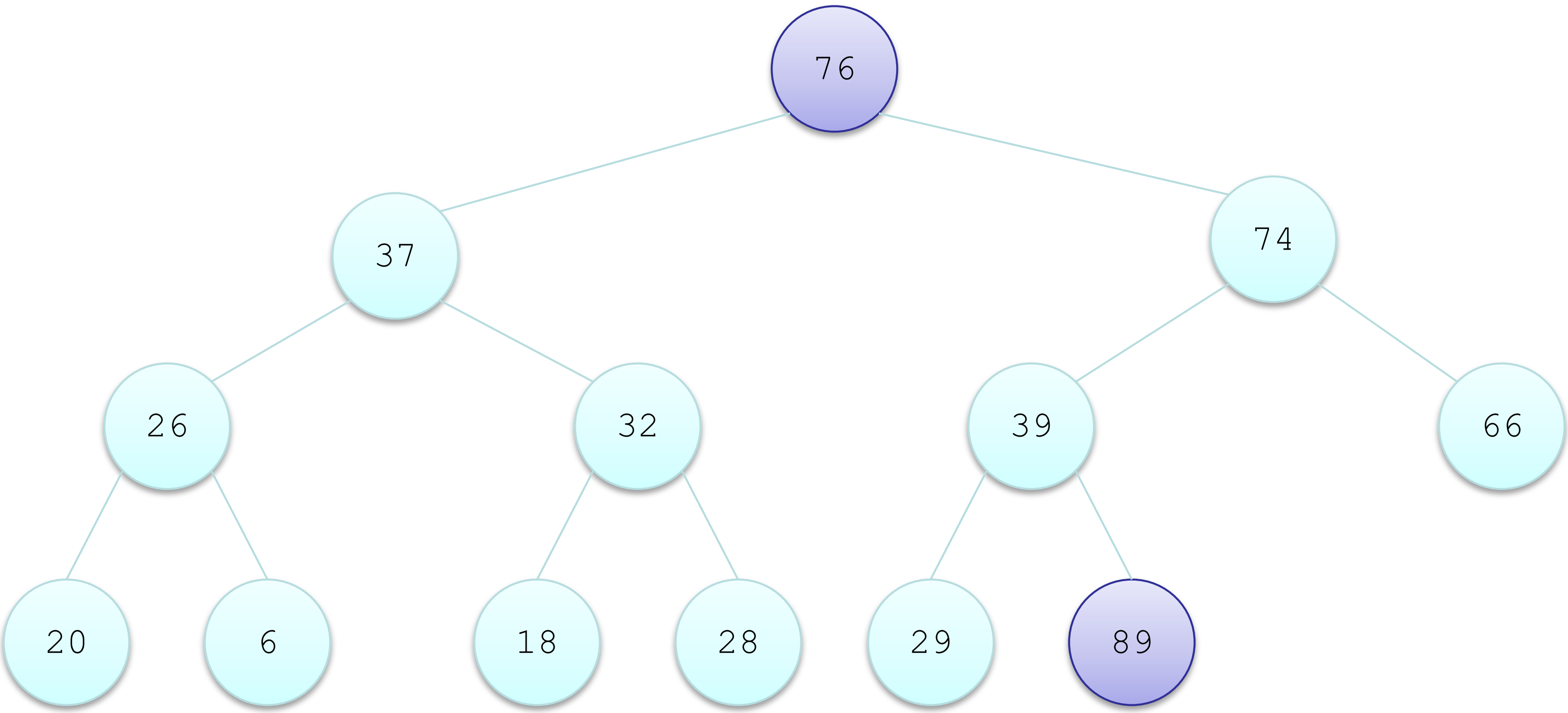
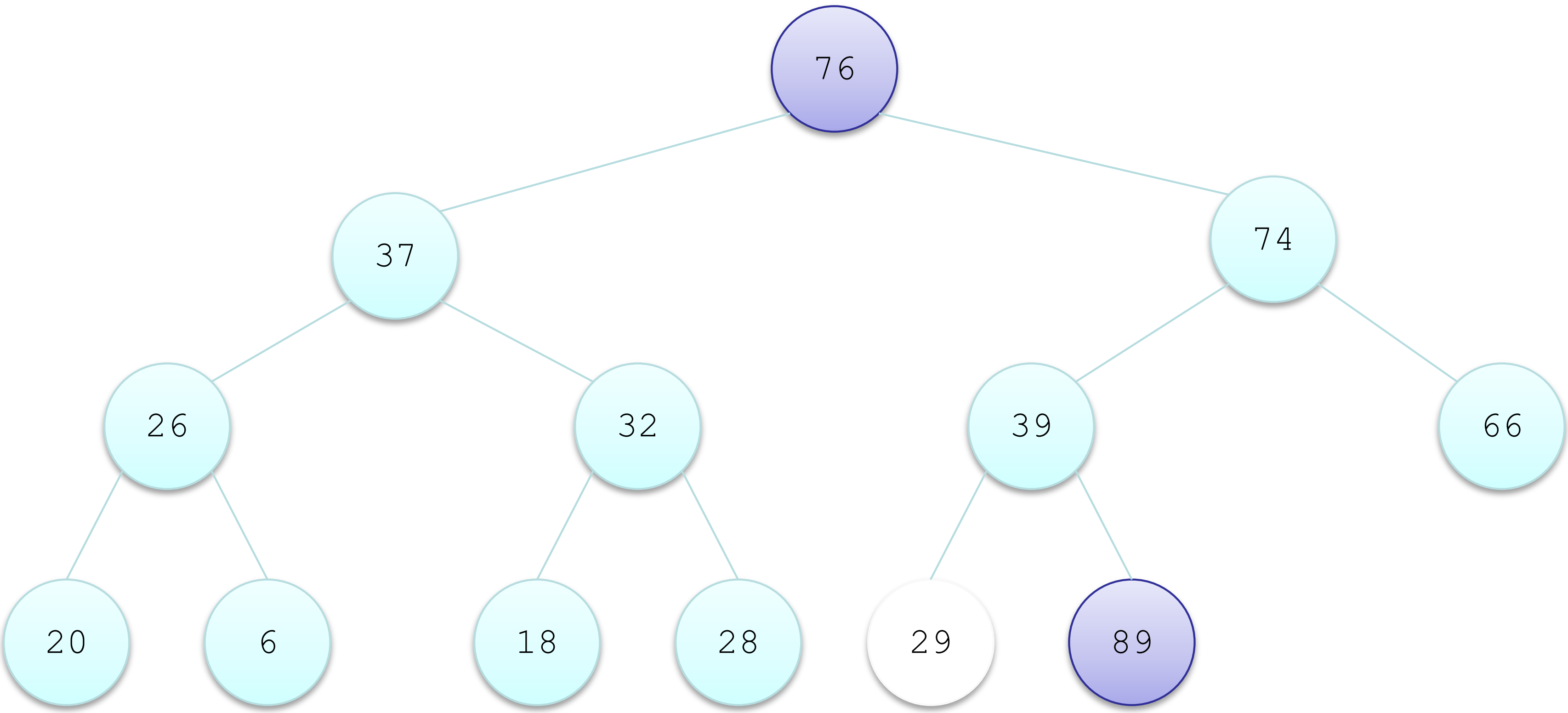
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 76 | 37 | 74 | 26 | 32 | 39 | 66 | 20 | 6 | 18 | 28 | 29 | 89 |

# Trace of Heapsort (cont.)



# Trace of Heapsort (cont.)



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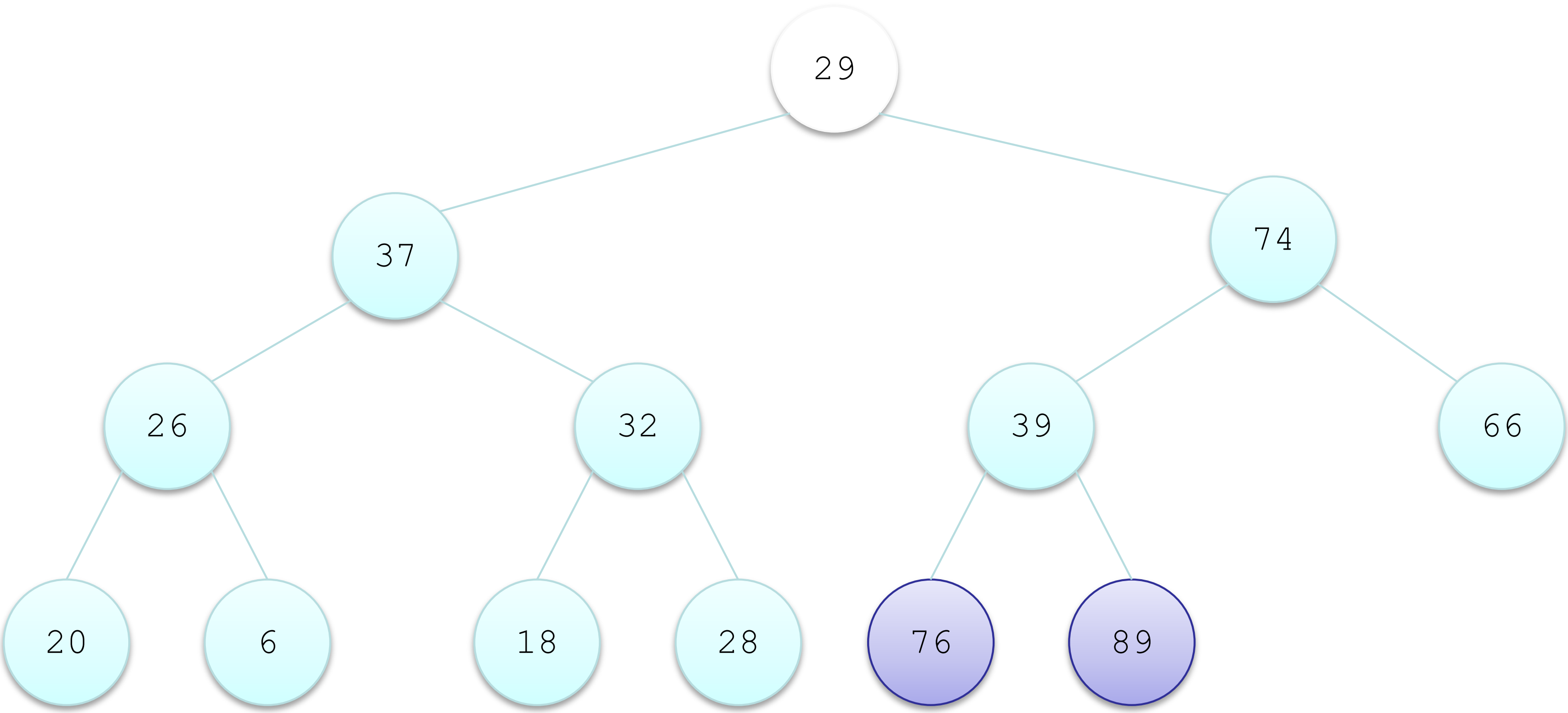
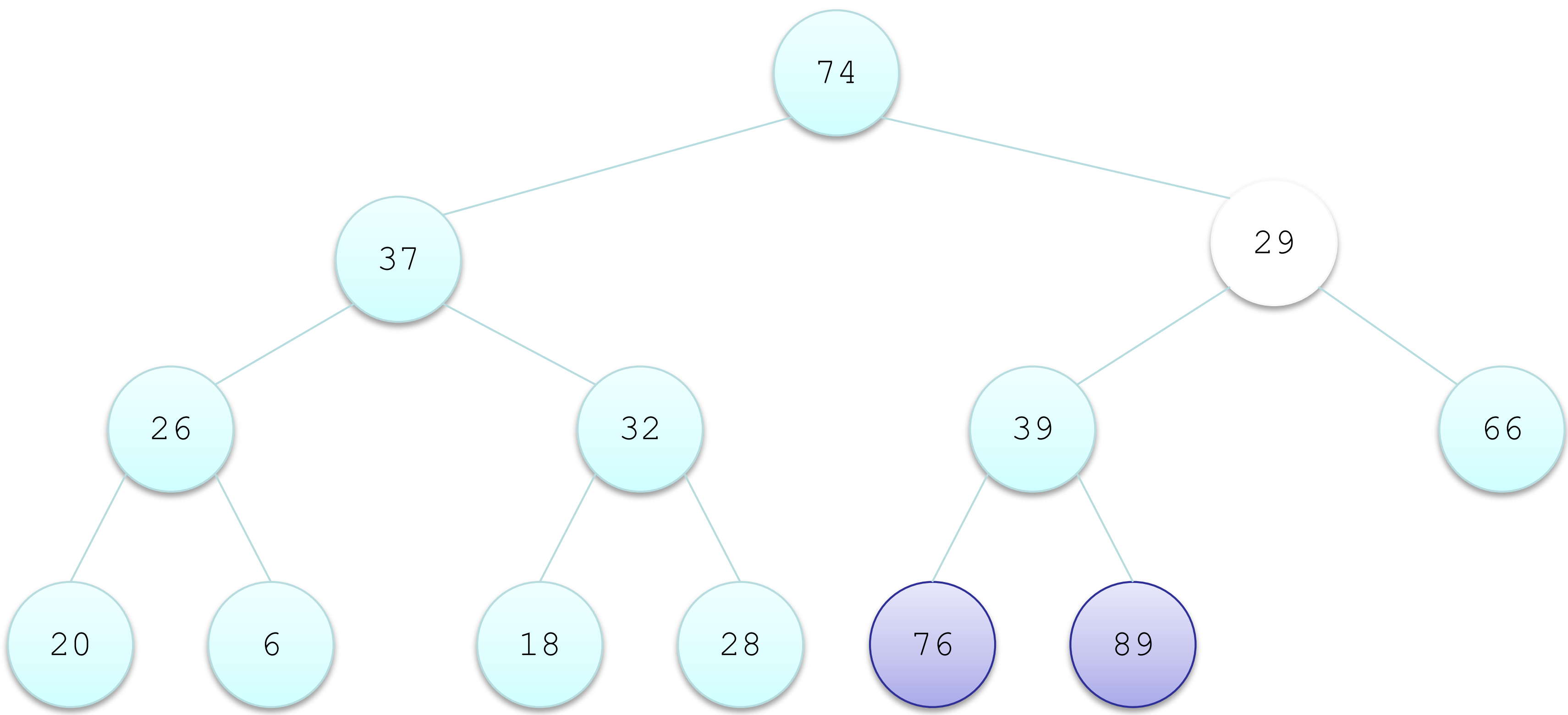
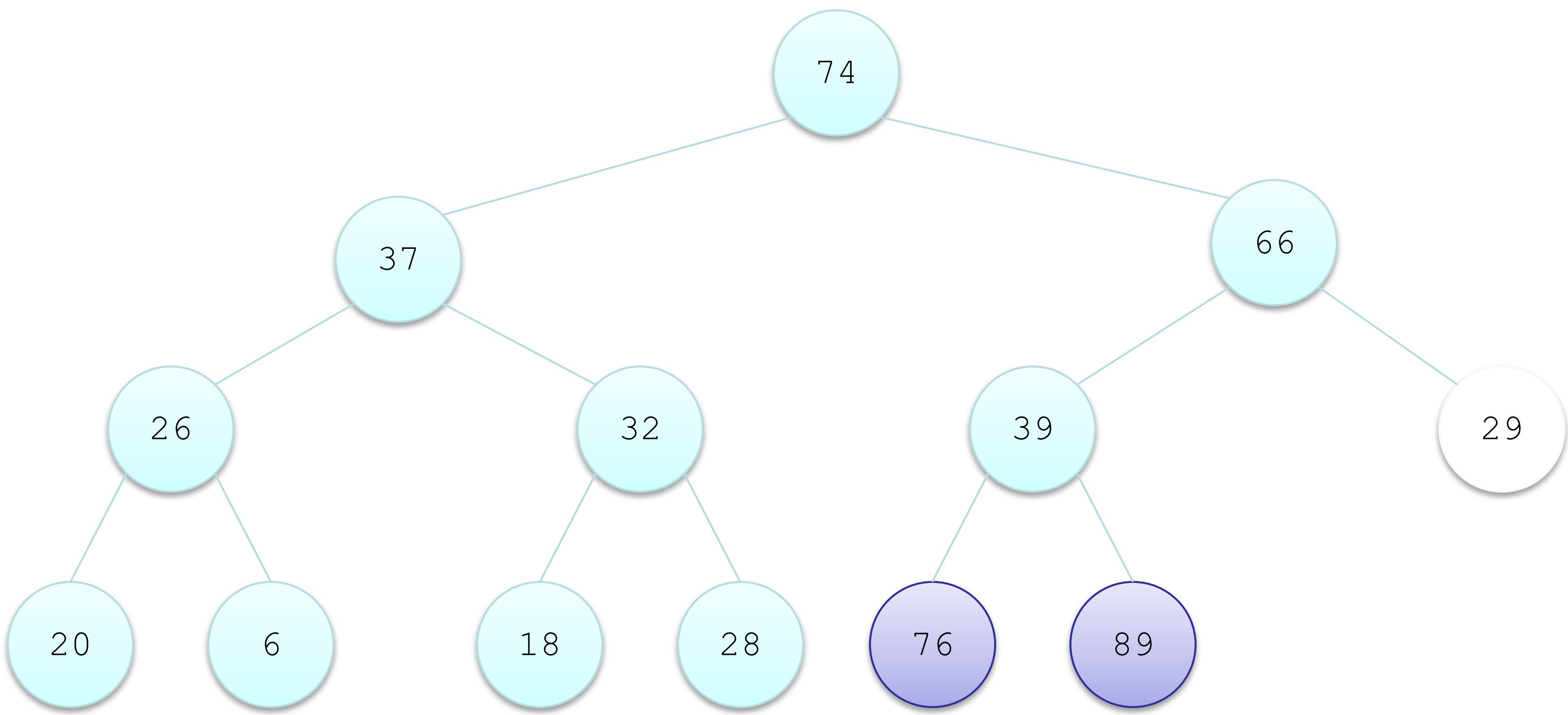


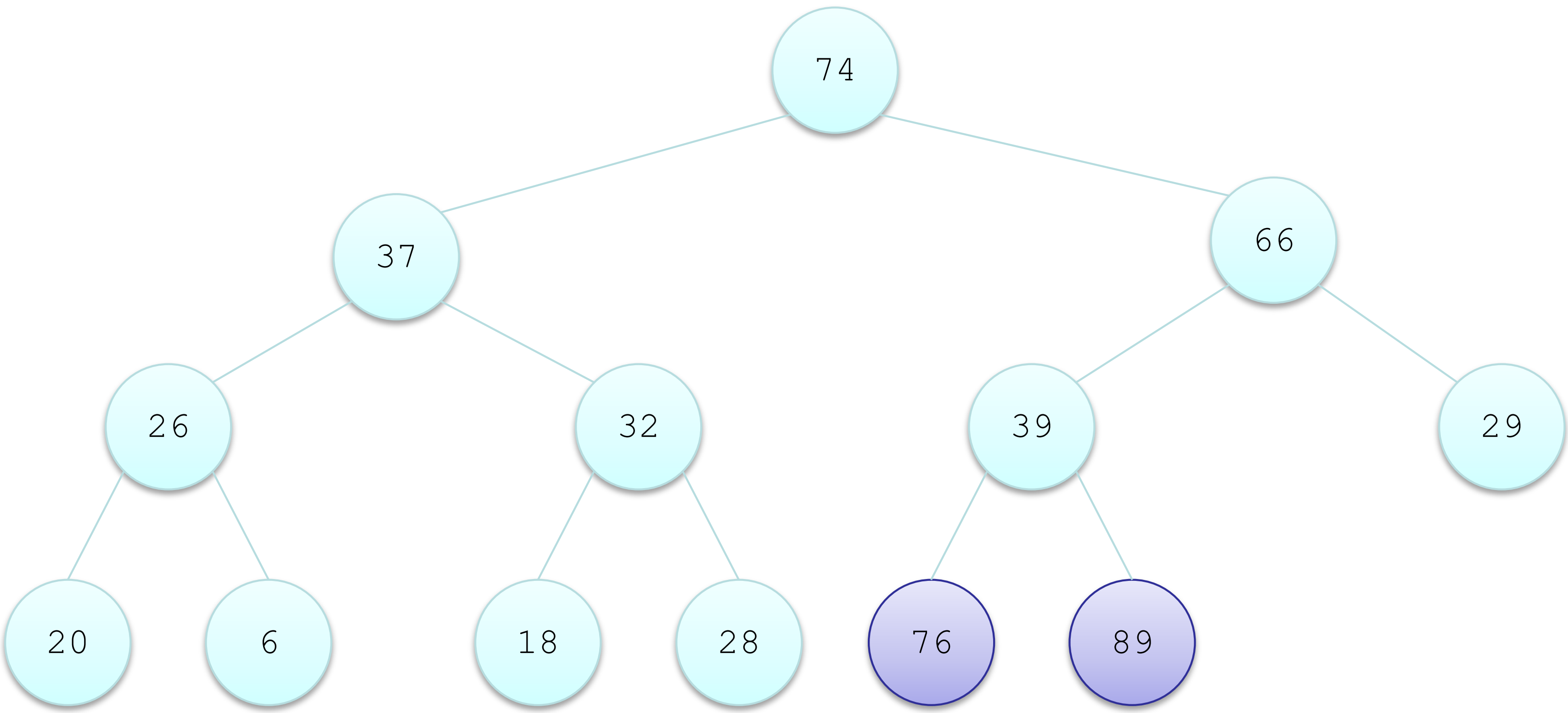
# Trace of Heapsort (cont.)



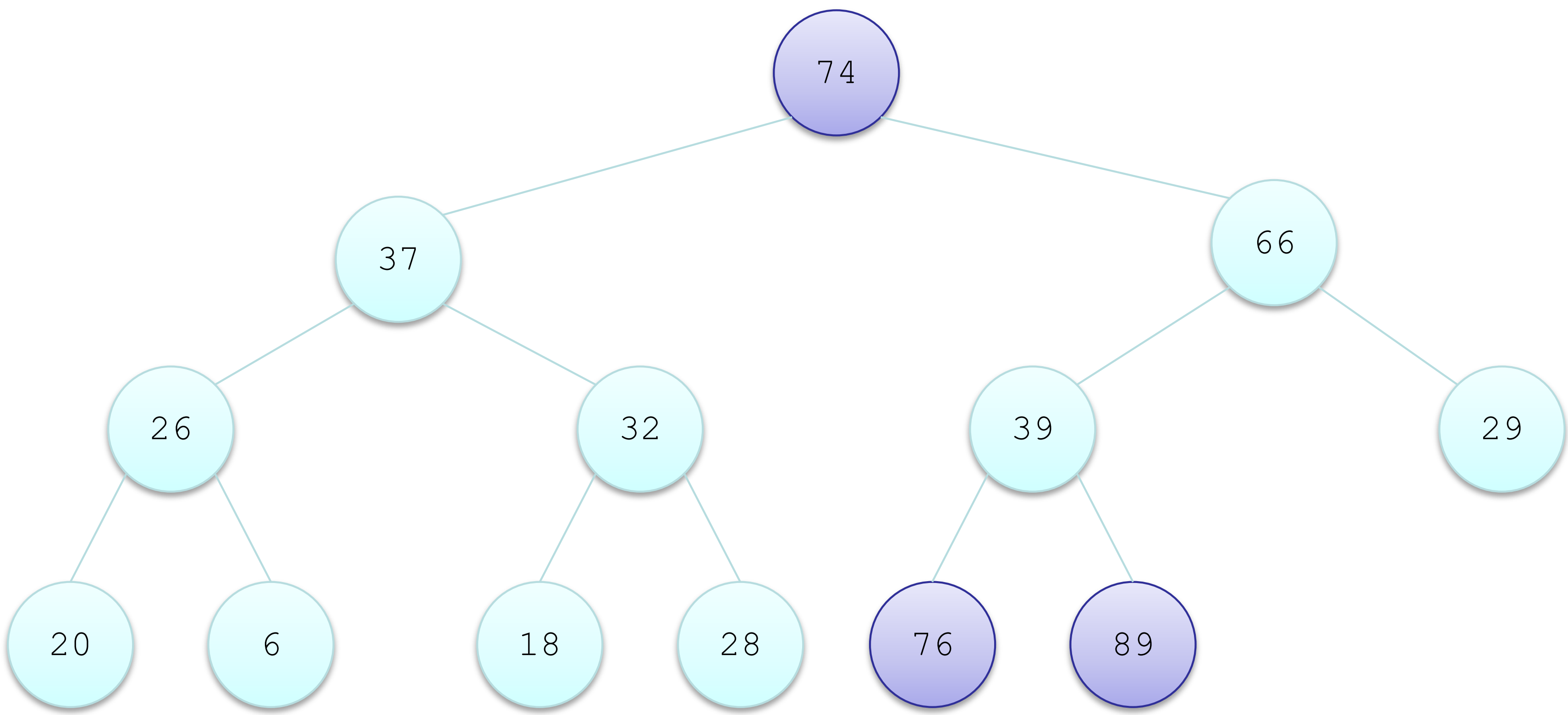
# Trace of Heapsort (cont.)



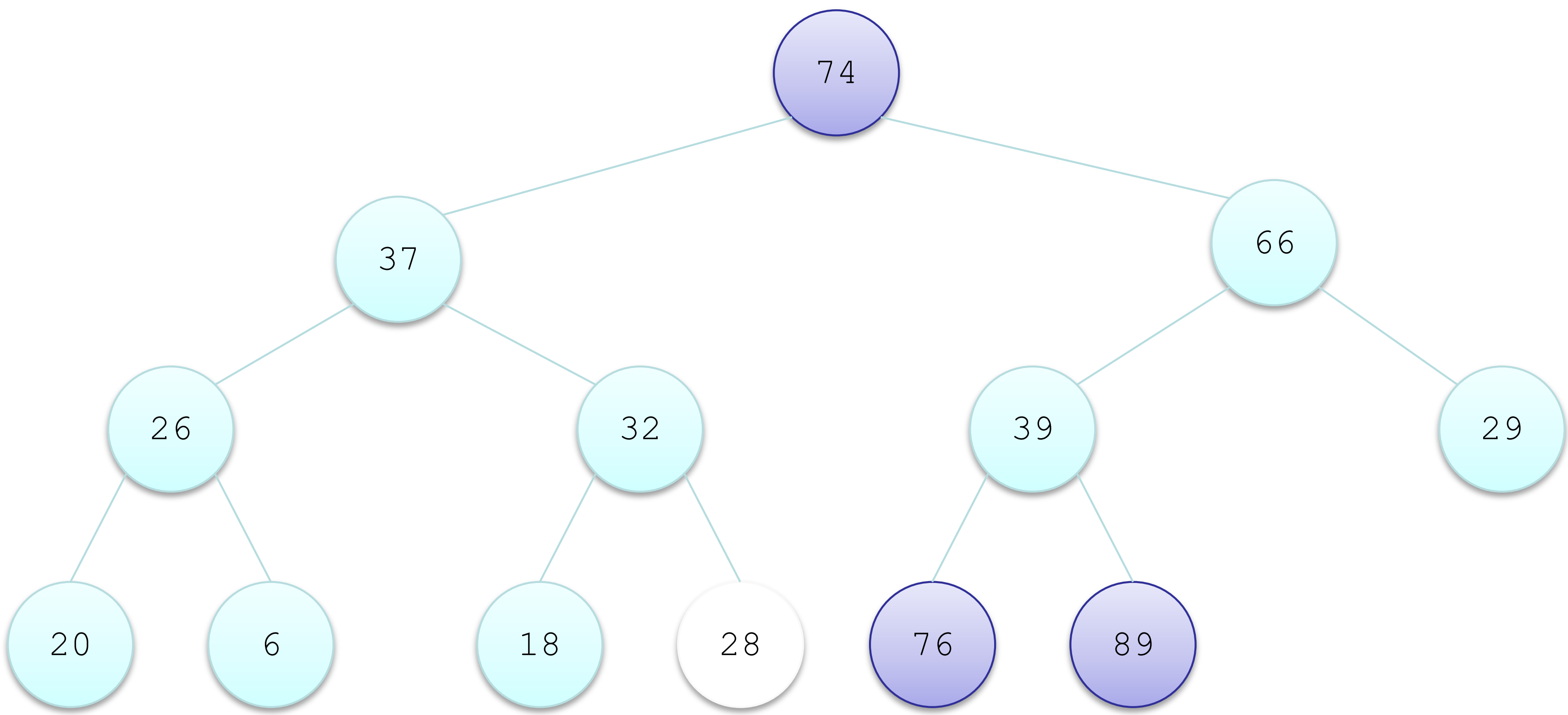
# **Trace of Heapsort** (cont.)



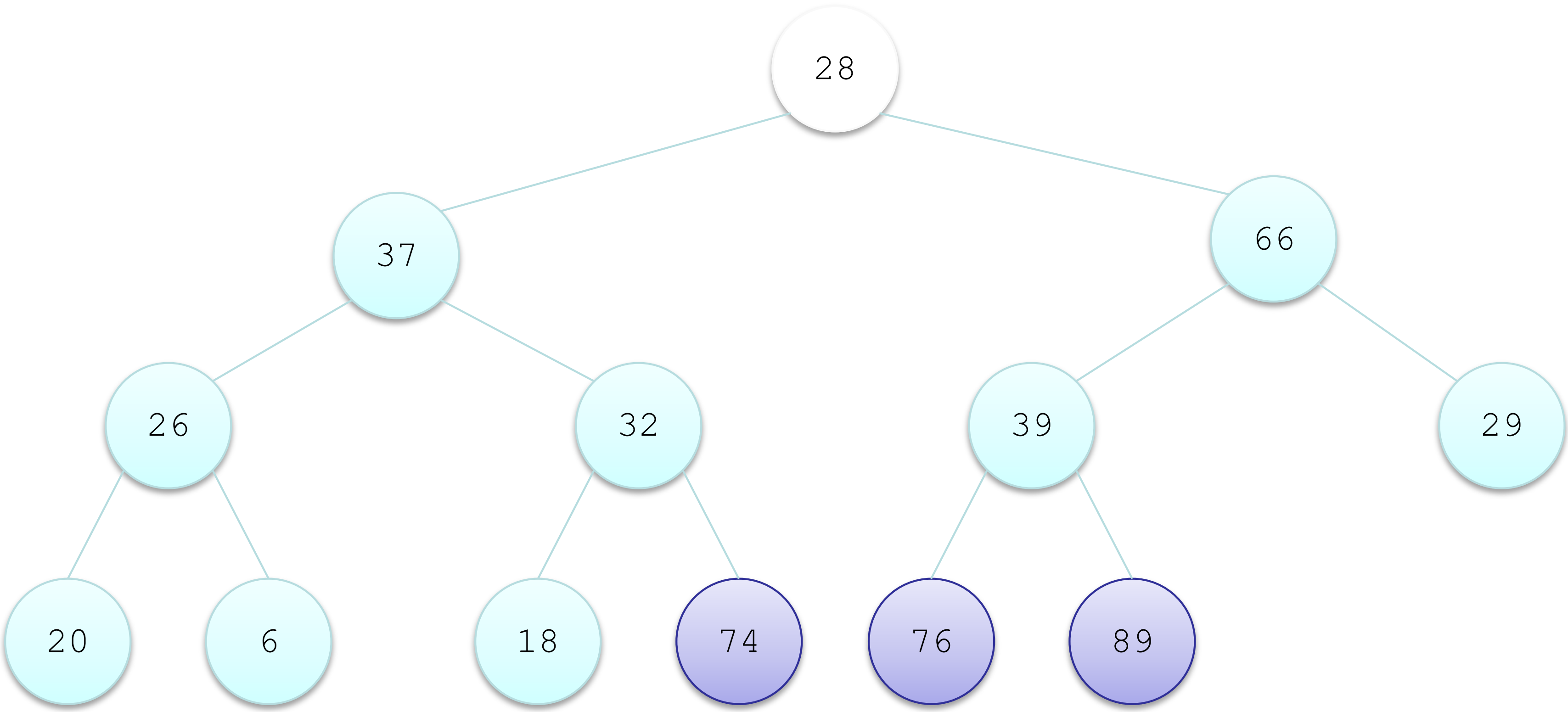
# Trace of Heapsort (cont.)



# **Trace of Heapsort** (cont.)

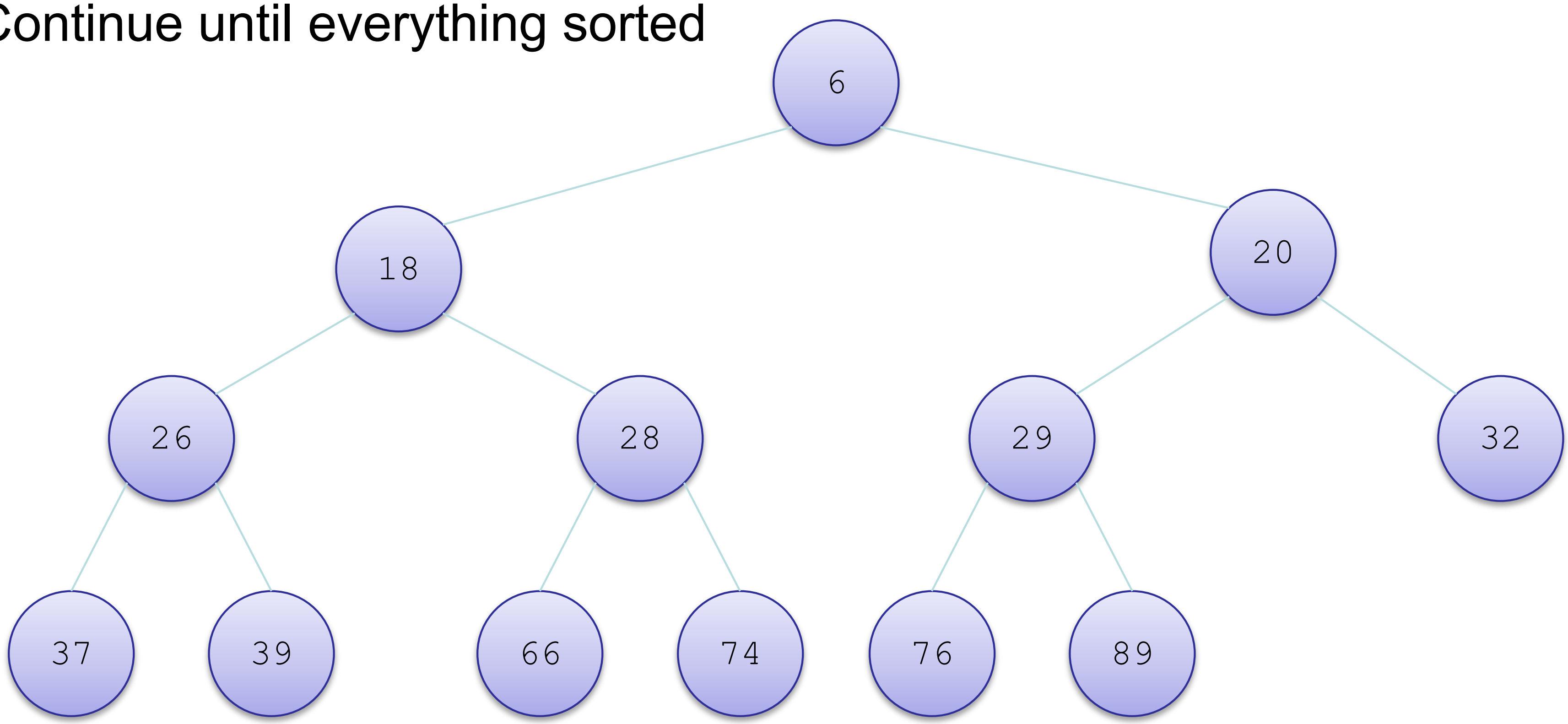


# Trace of Heapsort (cont.)



# Trace of Heapsort (cont.)

Continue until everything sorted



# Revising the Heapsort Algorithm

- If we implement the heap as an array
  - each element removed will be placed at the end of the array, and
  - the heap part of the array decreases by one element

| [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [9] | [10] | [11] | [12] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
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|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 74 | 37 | 66 | 26 | 32 | 39 | 29 | 20 | 6 | 18 | 28 | 76 | 89 |

⋮

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|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 18 | 20 | 26 | 28 | 29 | 32 | 37 | 39 | 66 | 74 | 76 | 89 |

# Algorithm for In-Place Heapsort

1. Build a heap by rearranging the elements in an unsorted array
2. While the heap is not empty
3. Remove the first item from the heap by swapping it with the last item in the heap and restoring the heap property

# Analysis of Heapsort

- Because a heap is a complete binary tree, it has log *n* levels
- Building a heap of size *n* requires finding the correct location for an item in a heap with log *n* levels
- Each insert (or remove) is $O(\log n)$
- With *n* items, building a heap is $O(n \log n)$
- No extra storage is needed

# QUICKSORT

# Quicksort

- Developed in 1962
- Quicksort selects a specific value called a pivot and rearranges the array into two parts (called *partioning*)
  - all the elements in the left subarray are less than or equal to the pivot
  - all the elements in the right subarray are larger than the pivot
  - The pivot is placed between the two subarrays
- The process is repeated until the array is sorted

# Merge sort vs Quick Sort

Merge sort

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 13 | 89 | 46 | 22 | 57 | 76 | 98 | 34 | 66 | 83 |

split

| | | | | |
|---|---|---|---|---|
| 13 | 89 | 46 | 22 | 57 |

| | | | | |
|---|---|---|---|---|
| 76 | 98 | 34 | 66 | 83 |

sort recursively

| | | | | |
|---|---|---|---|---|
| 13 | 22 | 46 | 57 | 89 |

| | | | | |
|---|---|---|---|---|
| 76 | 98 | 34 | 66 | 83 |

merge

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 13 | 22 | 34 | 46 | 57 | 66 | 76 | 83 | 89 | 98 |

# Merge sort vs Quick Sort

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 13 | 89 | 46 | 22 | 57 | 76 | 98 | 34 | 66 | 83 |

Split (smart, extra work here)

| | | | | |
|---|---|---|---|---|
| 13 | 46 | 22 | 57 | 34 |

| | | | | |
|---|---|---|---|---|
| 89 | 76 | 98 | 66 | 83 |

sort recursively

| | | | | |
|---|---|---|---|---|
| 13 | 22 | 34 | 46 | 57 |

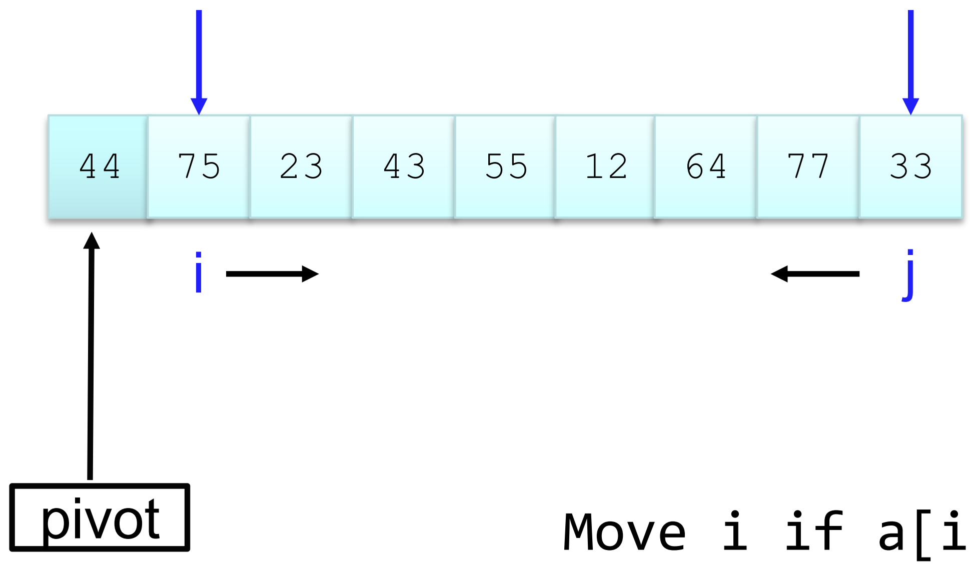
| | | | | |
|---|---|---|---|---|
| 66 | 76 | 83 | 89 | 98 |

Merge is not necessary

# Trace of Quicksort

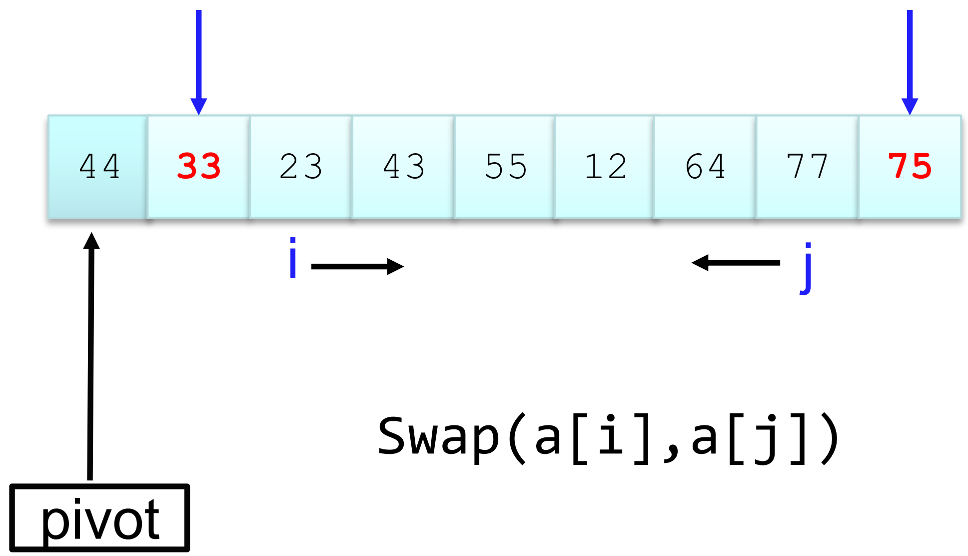
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 44 | 75 | 23 | 43 | 55 | 12 | 64 | 77 | 33 |

# **Trace of Quicksort** (cont.)



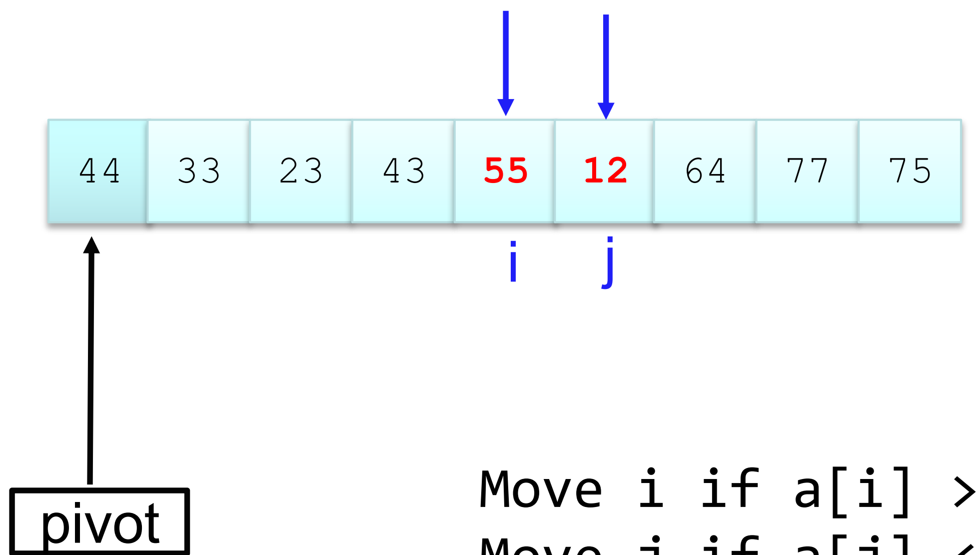

```
Move i if a[i] > pivot
Move j if a[j] < pivot
```

# Trace of Quicksort (cont.)



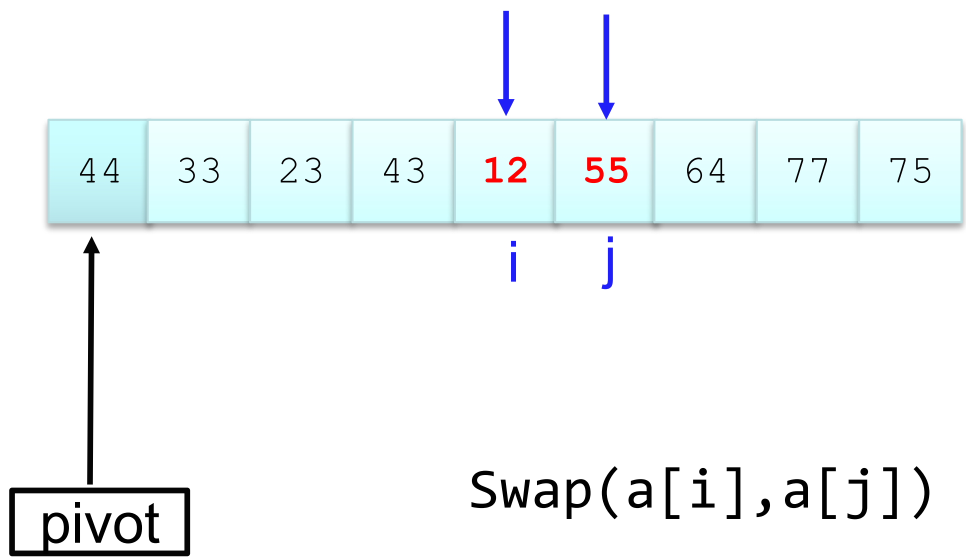

`Swap(a[i],a[j])`

# **Trace of Quicksort** (cont.)



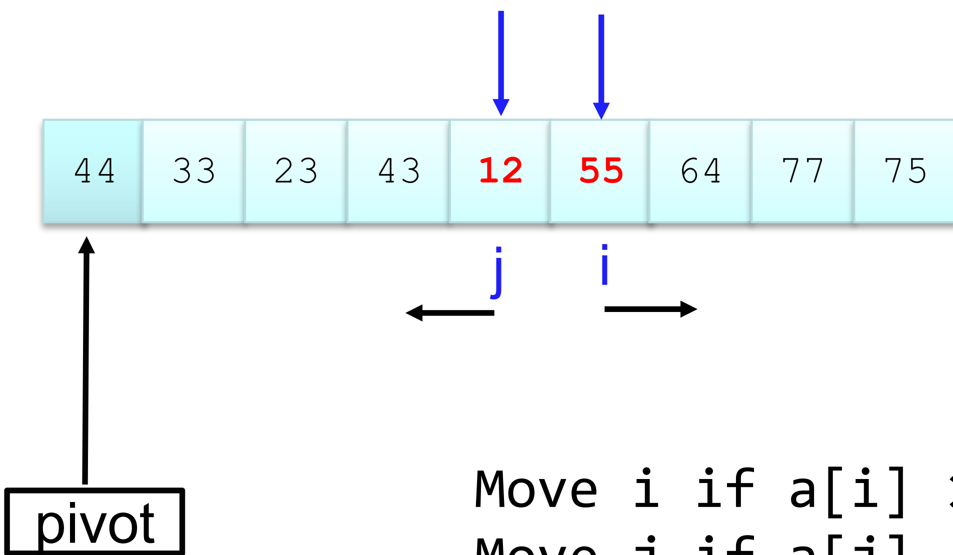

```
Move i if a[i] > pivot
Move j if a[j] < pivot
```

# Trace of Quicksort (cont.)



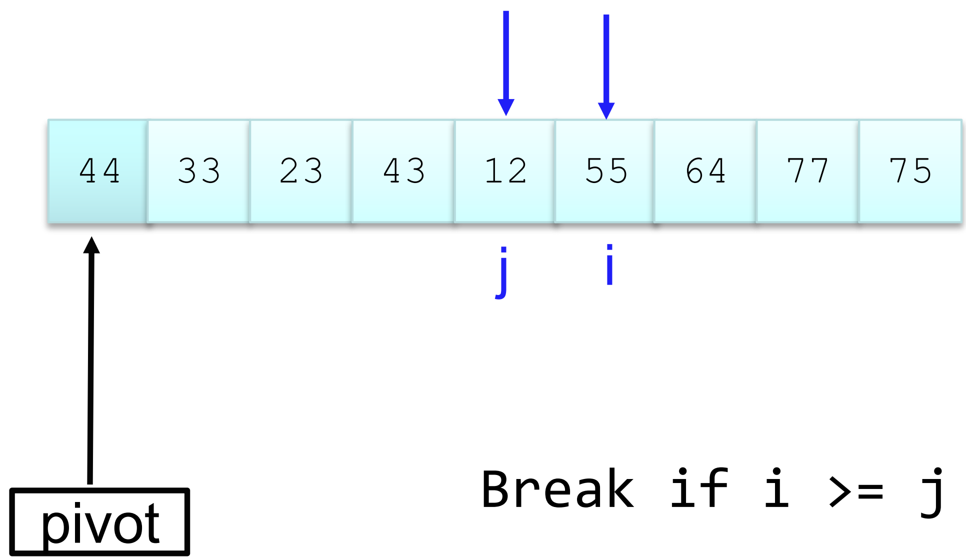

`Swap(a[i],a[j])`

# Trace of Quicksort (cont.)



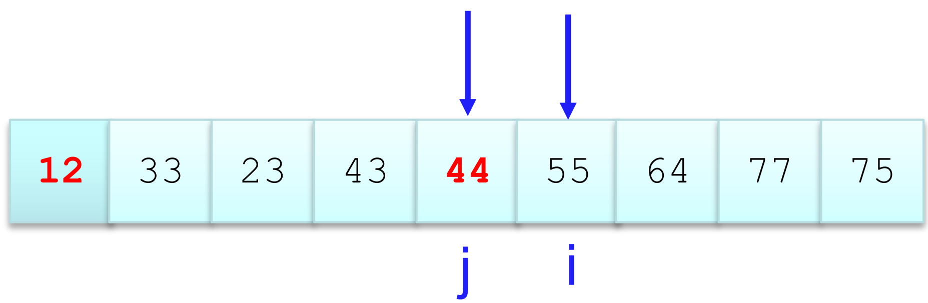

```
Move i if a[i] > pivot
Move j if a[j] < pivot
```

# Trace of Quicksort (cont.)




`Break if i >= j`

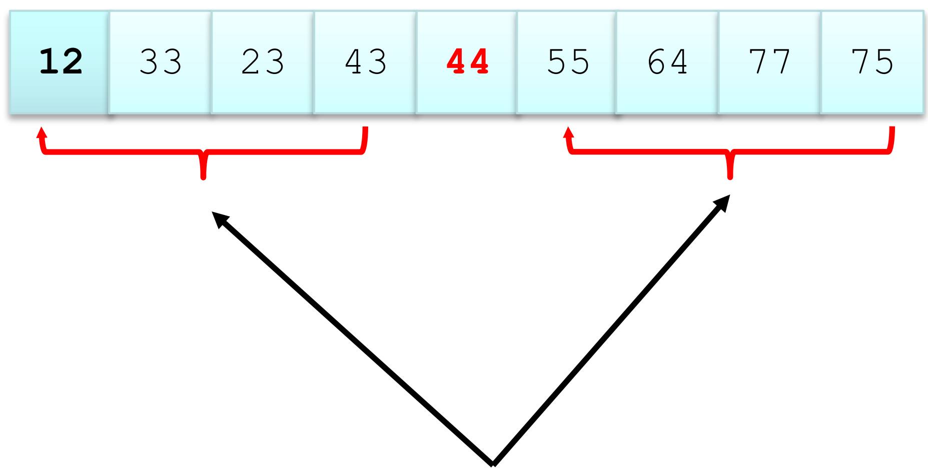
# **Trace of Quicksort** (cont.)




```
Swap(pivot,a[j]
```

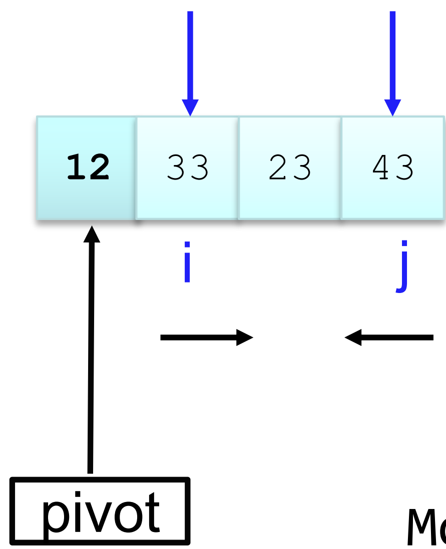
One iteration is done

# **Trace of Quicksort** (cont.)



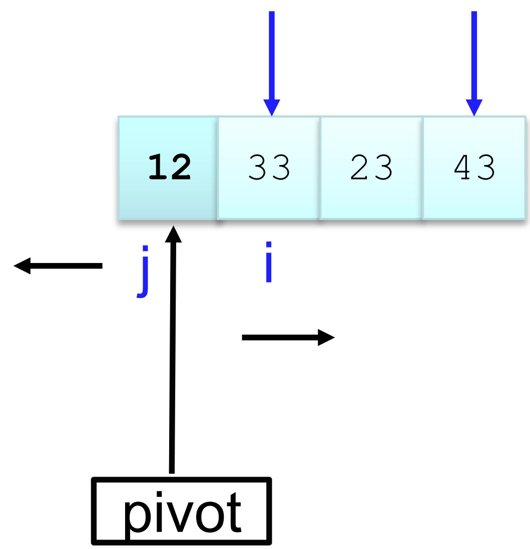

Recursively sort first and second subarray

# Trace of Quicksort (cont.)



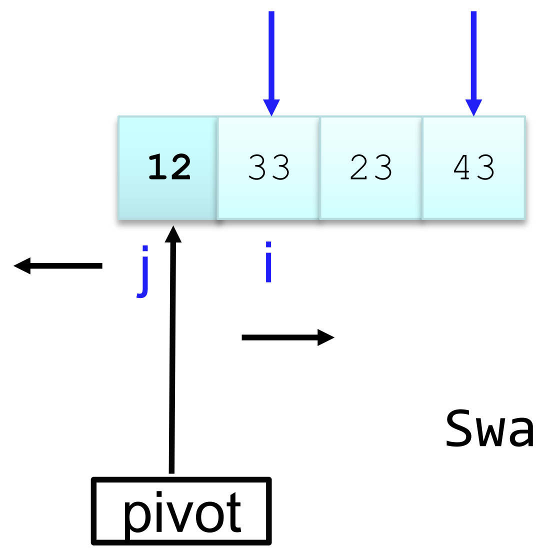

```
Move i if a[i] > pivot
Move j if a[j] < pivot
```

# **Trace of Quicksort** (cont.)




Break if i >= j

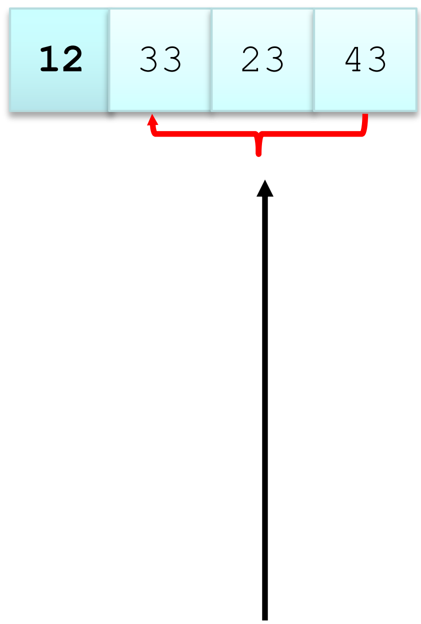
# **Trace of Quicksort** (cont.)




`Swap(pivot,a[j]]`

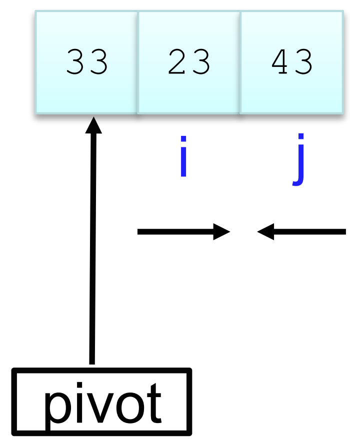
Second iteration for left half is done

# **Trace of Quicksort** (cont.)



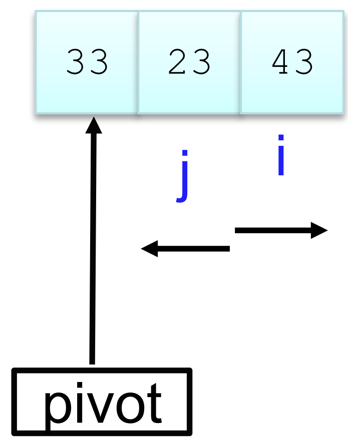

Recursively sort second subarray

# Trace of Quicksort (cont.)



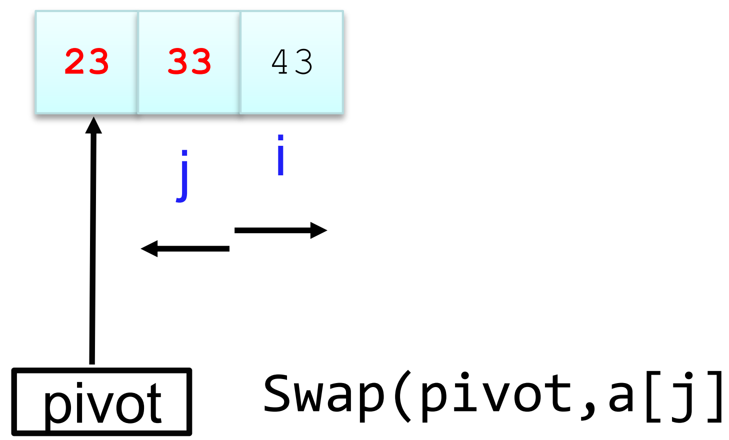

```
Move i if a[i] > pivot
Move j if a[j] < pivot
```

# **Trace of Quicksort** (cont.)



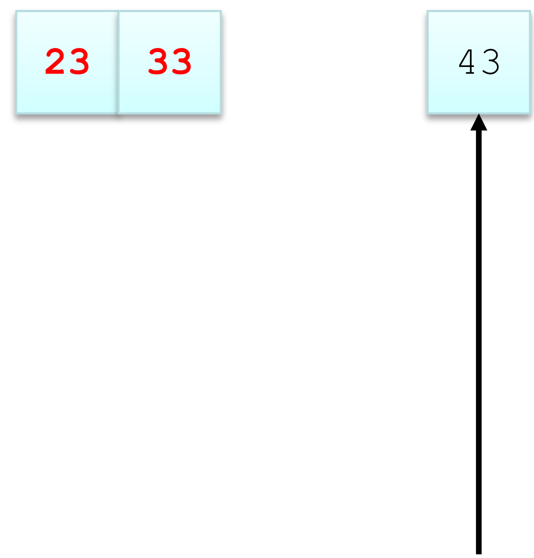

`Break if i >= j`

# **Trace of Quicksort** (cont.)



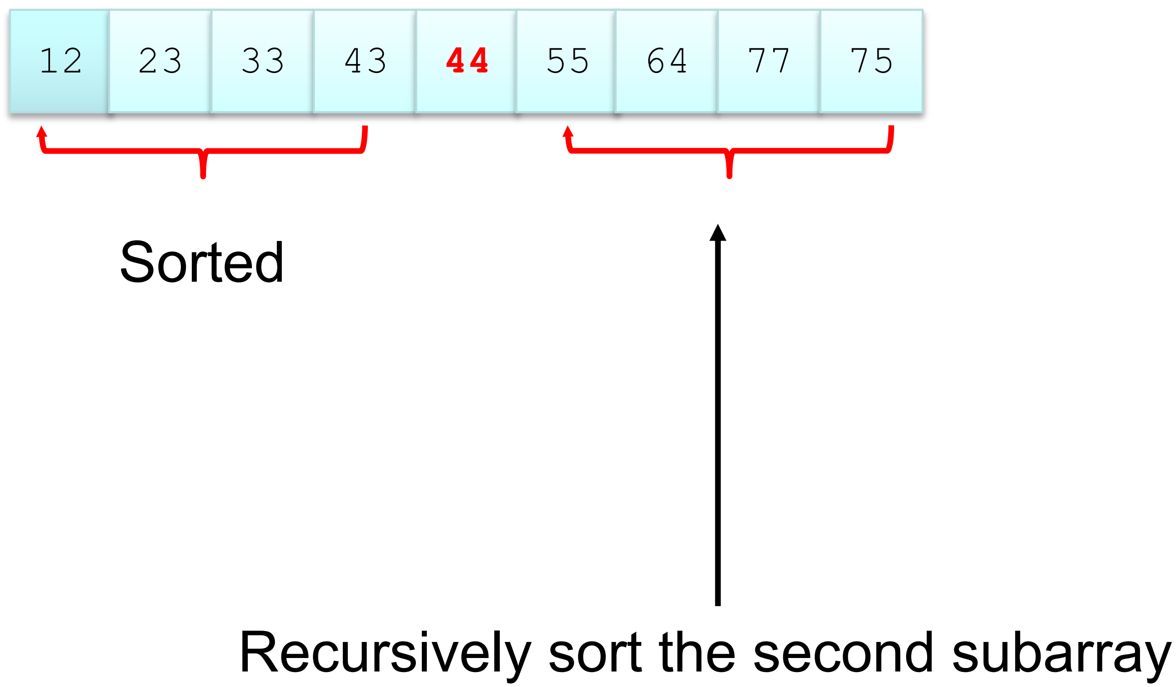

Another iteration is done

# **Trace of Quicksort** (cont.)




Subarray to sort

# Trace of Quicksort (cont.)



# Quick Sort Algorithm

```
/* quicksort the subarray from a[lo] to a[hi] */

void sort(Comparable[] a, int lo, int hi) {
   if (hi <= lo) return;
   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

# Partition

```
// partition the subarray a[lo..hi] so that a[lo..j-1] <= a[j] <=
a[j+1..hi]
// and return the index j.
int partition(Comparable[] a, int lo, int hi) {
   int i = lo;
   int j = hi + 1;
   Comparable v = a[lo];
   while (true) {
       // find item on lo to swap
       while (less(a[++i], v))
           if (i == hi) break;
      /* find item on hi to swap */
      while (less(v, a[--j]))
        if (j == lo) break;
      // check if pointers cross
      if (i >= j) break;
      exch(a, i, j);
    }
    // put partitioning item v at a[j]
    exch(a, lo, j);
  // now, a[lo .. j-1] <= a[j] <= a[j+1 .. hi]
  return j;
}
```

# Analysis of Quicksort

- If the pivot value is a random value selected from the current subarray,
  - then statistically half of the items in the subarray will be less than the pivot and half will be greater
- If both subarrays have the same number of elements (best case), there will be log *n* levels of recursion
- At each recursion level, the partitioning process involves moving every element to its correct position—*n* moves
- Quicksort is O(*n* log *n*), just like merge sort

# **Analysis of Quicksort** (cont.)

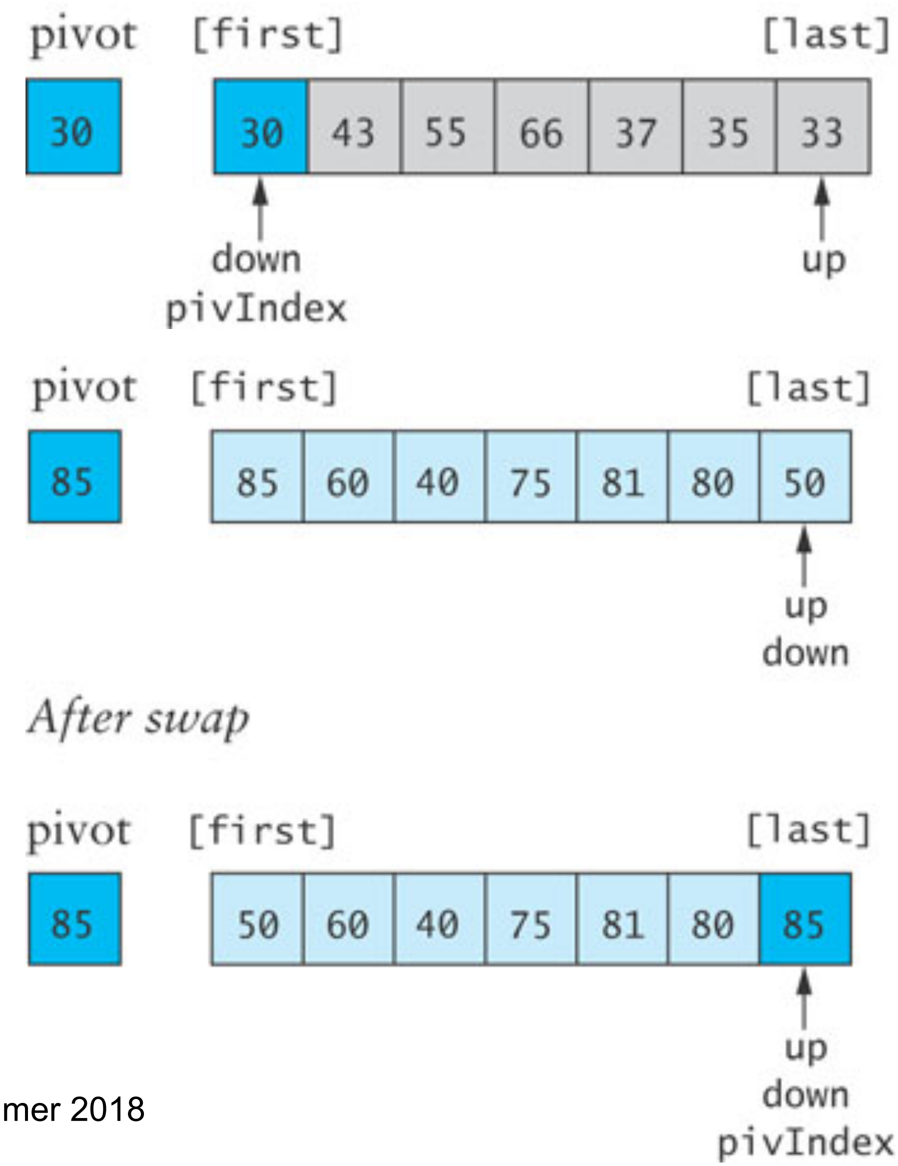
- The array split may not be the best case, i.e. 50-50
- An exact analysis is difficult (and beyond the scope of this class), but, the running time will be bounded by a constant x *n* log *n*

# Analysis of Quicksort (cont.)

- A quicksort will give very poor behavior if, each time the array is partitioned, a subarray is empty.
- In that case, the sort will be $O(n^2)$

- Under these circumstances, the overhead of recursive calls and the extra run-time stack storage required by these calls makes this version of quicksort a poor performer relative to the quadratic sorts.

# Code for `partition` when `Pivot` is the largest or smallest value



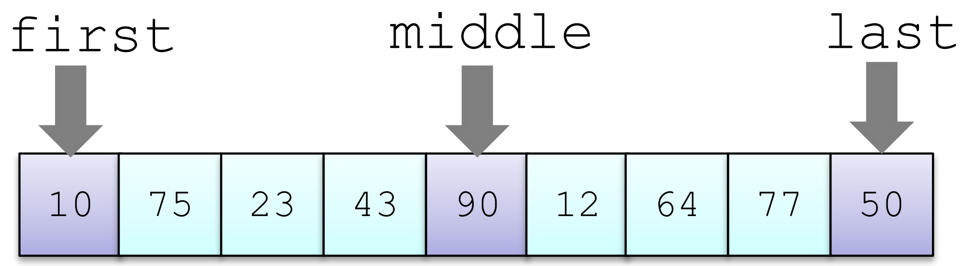
# Revised Partition Algorithm

- Quicksort is $O(n^2)$ when each split yields one empty subarray, which is the case when the array is presorted
- A better solution is to pick the pivot value in a way that is less likely to lead to a bad split
  - Use three references: `first, middle, last`
  - Select the median of the these items as the pivot

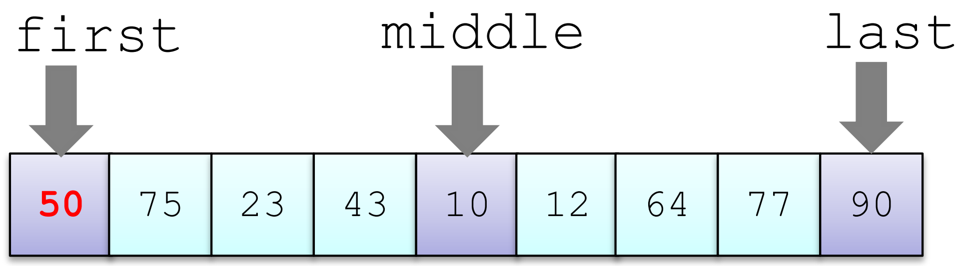
# Trace of Revised Partitioning

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 10 | 75 | 23 | 43 | 90 | 12 | 64 | 77 | 50 |

# Trace of Revised Partitioning (cont.)



# Trace of Revised Partitioning (cont.)




Make the middle number pivot

# Sorting Algorithm Comparison

| Name | Best | Averag e | Worst | Memory | Stable |
|---|---|---|---|---|---|
| Bubble Sort | $n$ | $n^2$ | $n^2$ | 1 | yes |
| Selection Sort | $n^2$ | $n^2$ | $n^2$ | 1 | no |
| Insertion Sort | n | $n^2$ | $n^2$ | 1 | yes |
| Merge Sort | *nlogn* | *nlogn* | *nlogn* | $n$ | yes |
| Quick Sort | *nlogn* | *nlogn* | $\mathrm{n}^2$ | *log n* | no |
| Heap Sort | *nlogn* | *nlogn* | *nlogn* | 1 | no |