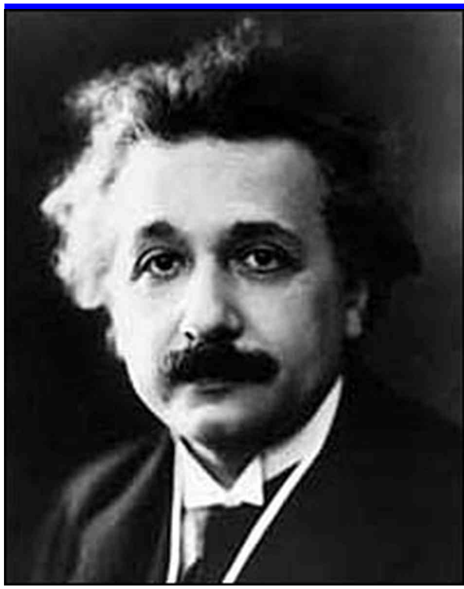
CMSC 330: Organization of Programming Languages

Lambda Calculus

100 years ago

- Albert Einstein proposed special theory of relativity in **1905**
  - In the paper *On the Electrodynamics of Moving Bodies*

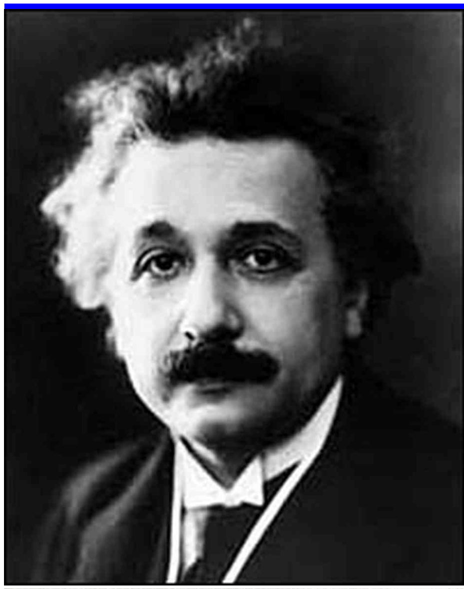
Prioritätsstreit, “priority dispute”

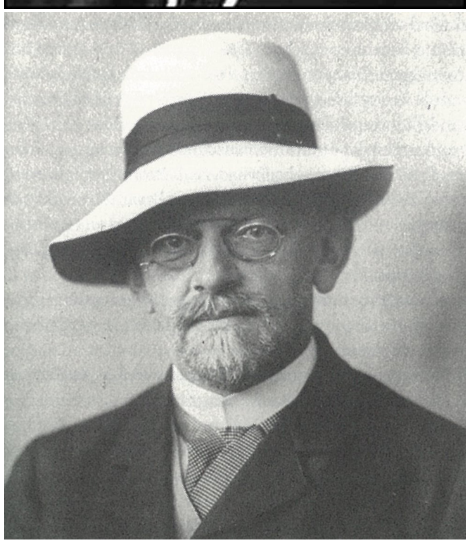


General Theory of Relativity

- Einstein's field equations presented in Berlin: **Nov 25, 1915**
- **Published: Dec 2,1915**

Prioritätsstreit, “priority dispute”



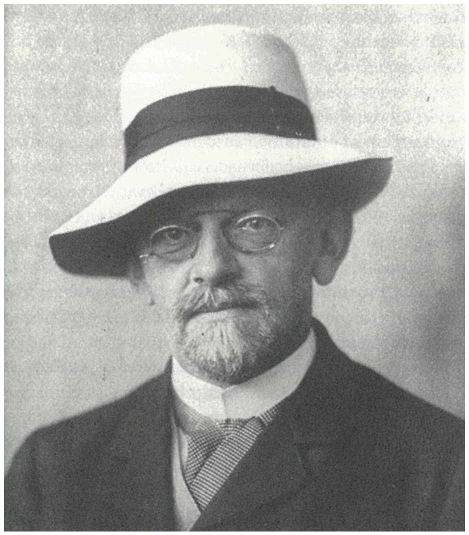


General Theory of Relativity

- Einstein's field equations presented in Berlin: **Nov 25, 1915**
- **Published: Dec 2,1915**

- **David Hilbert**'s equations presented in Gottingen: **Nov 20, 1915**
- **Published: March 6, 1916**

Entscheidungsproblem “decision problem”



Is there an algorithm to determine if a statement is true in all models of a theory?

Entscheidungsproblem “decision problem“

Algorithm, formalised

Alonzo Church: Lambda calculus

An unsolvable problem of elementary number theory, *Bulletin the American Mathematical Society*, May 1935

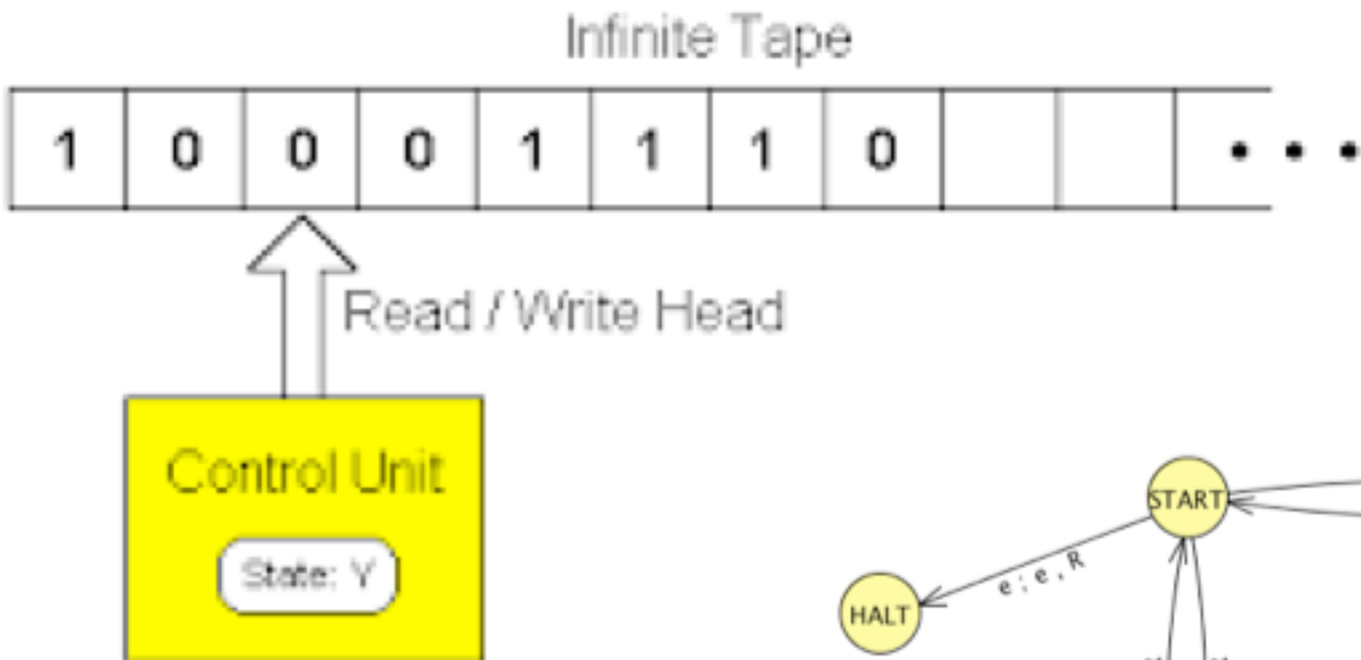
Kurt Gödel: Recursive functions

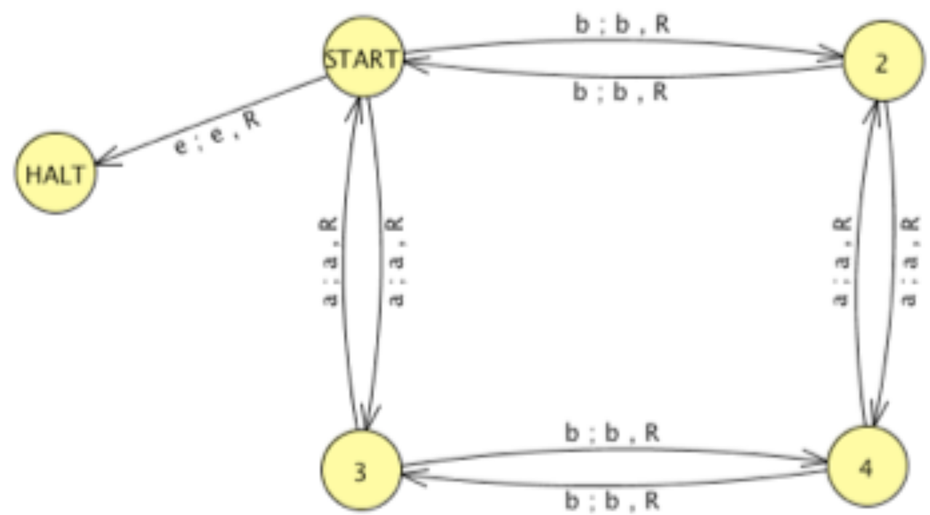
Stephen Kleene, General recursive functions of natural numbers, *Bulletin the American Mathematical Society*, July 1935

Alan M. Turing: Turing machines

On computable numbers, with an application to the *Entscheidungsproblem*, *Proceedings of the London Mathematical Society*, received 25 May 1936

Turing Machine





Turing Completeness

- Turing machines are the most powerful description of computation possible
  - They define the Turing-computable functions
- A programming language is Turing complete if
  - It can map every Turing machine to a program
  - A program can be written to emulate a Turing machine
  - It is a superset of a known Turing-complete language
- Most powerful programming language possible
  - Since Turing machine is most powerful automaton

Programming Language Expressiveness

- So what language features are needed to express all computable functions?
  - What's a minimal language that is Turing Complete?
- Observe: some features exist just for convenience
  - Multi-argument functions   foo (a, b, c)
    - Use currying or tuples
  - Loops   while (a < b) …
    - Use recursion
  - Side effects   a := 1
    - Use functional programming pass "heap" as an argument to each function, return it when with function's result

Mini C

You only have:

- If statement
- Plus 1
- Minus 1
- functions

Sum n = 1+2+3+4+5…n in Mini C

```
int add1(int n){return n+1;}
int sub1(int n){return n-1;}
int add(int a,int b){
    if(b == 0) return a;
    else return add( add1(a),sub1(b));
}
int sum(int n){
    if(n == 1) return 1;
    else return add(n, sum(sub1(n)));
}
int main(){
    printf("%d\n",sum(5));
}
```

Lambda Calculus (λ-calculus)

- Proposed in 1930s by
  - Alonzo Church
    (born in Washingon DC!)
- Formal system
  - Designed to investigate functions & recursion
  - For exploration of foundations of mathematics
- Now used as
  - Tool for investigating computability
  - Basis of functional programming languages
    - Lisp, Scheme, ML, OCaml, Haskell…

Lambda Calculus Syntax

- A lambda calculus expression is defined as

e ::= x	**variable**
| λx.e	**abstraction** (fun def)
| e e	**application** (fun call)

  - This grammar describes ASTs; not for parsing (ambiguous!)
  - Lambda expressions also known as lambda **terms**

- λx.e is like `(fun x -> e)` in OCaml

That's it! Nothing but higher-order functions

Why Study Lambda Calculus?

- It is a "core" language
  - Very small but still Turing complete
- But with it can explore general ideas
  - Language features, semantics, proof systems, algorithms, …
- Plus, higher-order, anonymous functions (aka *lambdas*) are now very popular!
  - C++ (C++11), PHP (PHP 5.3.0), C# (C# v2.0), Delphi (since 2009), Objective C, Java 8, Swift, Python, Ruby (Procs), … (and functional languages like OCaml, Haskell, F#, …)

Three Conventions

- Scope of λ extends as far right as possible
  - Subject to scope delimited by parentheses
  - λx. λy.x y is same as λx.(λy.(x y))
- Function application is left-associative
  - x y z is (x y) z
  - Same rule as OCaml
- As a convenience, we use the following "syntactic sugar" for local declarations
  - let x = e1 in e2 is short for (λx.e2) e1

OCaml Lambda Calc Interpreter

- e ::= x
  | λx.e
  | e e

```
type id = string
type exp = Var of id
| Lam of id * exp
| App of exp * exp
```

y	`Var "y"`
λx.x	`Lam ("x", Var "x")`
λx.λy.x y	`Lam ("x",(Lam("y",App (Var "x", Var "y"))))`
(λx.λy.x y) λx.x x	`App (Lam("x",Lam("y",App(Var"x",Var"y"))), Lam ("x", App (Var "x", Var "x")))`

Quiz #1

`λx.(y z)` and **`λx.y z`** are equivalent

A. True
B. False

Quiz #1

`λx.(y z)` and **`λx.y z`** are equivalent

A. **True**
B. False

Quiz #2

What is this term's AST?

λx.x x

```
type id = string
type exp =
        Var of id
      | Lam of id * exp
      | App of exp * exp
```

A. `App (Lam (“x”, Var “x”), Var “x”)`
B. `Lam (Var “x”, Var “x”, Var “x”)`
C. `Lam (“x”, App (Var “x”,Var “x”))`
D. `App (Lam (“x”, App (“x”, “x”)))`

Quiz #2

What is this term's AST?

λx.x x

```
type id = string
type exp =
      Var of id
    | Lam of id * exp
    | App of exp * exp
```

A. `App (Lam ("x", Var "x"), Var "x")`
B. `Lam (Var "x", Var "x", Var "x")`
C. `Lam ("x", App (Var "x",Var "x"))`
D. `App (Lam ("x", App ("x", "x")))`

Quiz #3

This term is equivalent to which of the following?

`λx.x a b`

A. `(λx.x) (a b)`
B. `(((λx.x) a) b)`
C. `λx.(x (a b))`
D. `(λx.((x a) b))`

Quiz #3

This term is equivalent to which of the following?

`λx.x a b`

A. **`(λx.x) (a b)`**
B. **`(((λx.x) a) b)`**
C. **`λx.(x (a b))`**
D. **`(λx.((x a) b))`**

Lambda Calculus Semantics

- Evaluation: All that's involved are function calls
  (λx.e1) e2
  - Evaluate e1 with x replaced by e2
- This application is called beta-reduction
  - (λx.e1) e2 → e1[x:=e2]
    - e1[x:=e2] is e1 with occurrences of x replaced by e2
    - This operation is called *substitution*
      - **Replace** formals with actuals
      - Instead of using environment to map formals to actuals
  - We allow reductions to occur *anywhere* in a term
    - Order reductions are applied does not affect final value!
- When a term cannot be reduced further it is in beta normal form

Beta Reduction Example

- (λx.λz.x z) y
  - → (λx.(λz.(x z))) y   // since λ extends to right
  - → (λx.(λz.(x z))) y   // apply (λx.e1) e2 → e1[x:=e2]
    // where e1 = λz.(x z), e2 = y
  - → λz.(y z)   // final result

Parameters
- Formal
- Actual

- Equivalent OCaml code
  - (fun x -> (fun z -> (x z))) y → fun z -> (y z)

Beta Reduction Examples

- $(\lambda x.x)\ z \to z$
- $(\lambda x.y)\ z \to y$
- $(\lambda x.x\ y)\ z \to z\ y$
  - A function that applies its argument to y

Beta Reduction Examples (cont.)

- $(\lambda x.x\ y)\ (\lambda z.z) \rightarrow\ (\lambda z.z)\ y \rightarrow y$
- $(\lambda x.\lambda y.x\ y)\ z \rightarrow\ \lambda y.z\ y$
  - A curried function of two arguments
  - Applies its first argument to its second
- $(\lambda x.\lambda y.x\ y)\ (\lambda z.zz)\ x \rightarrow (\lambda y.(\lambda z.zz)y)x \rightarrow (\lambda z.zz)x \rightarrow xx$

Beta Reduction Examples (cont.)

$(\lambda x.x\ (\lambda y.y))\ (u\ r) \rightarrow$

$(\lambda x.(\lambda w.\ x\ w))\ (y\ z) \rightarrow$

Beta Reduction Examples (cont.)

$(\lambda x.x\ (\lambda y.y))\ (u\ r) \rightarrow (u\ r)\ (\lambda y.y)$

$(\lambda x.(\lambda w.\ x\ w))\ (y\ z) \rightarrow (\lambda w.\ (y\ z)\ w)$

Quiz #4

`(λx.y) z` can be beta-reduced to

A. **`y`**
B. **`y z`**
C. **`z`**
D. cannot be reduced

Quiz #4

`(λx.y) z` can be beta-reduced to

A. `y`
B. `y z`
C. `z`
D. cannot be reduced

Quiz #5

Which of the following reduces to $\lambda z.\ z$?

a) $(\lambda y.\ \lambda z.\ x)\ z$
b) $(\lambda z.\ \lambda x.\ z)\ y$
c) $(\lambda y.\ y)\ (\lambda x.\ \lambda z.\ z)\ w$
d) $(\lambda y.\ \lambda x.\ z)\ z\ (\lambda z.\ z)$

Quiz #5

Which of the following reduces to $\lambda z.\ z$?

a) $(\lambda y.\ \lambda z.\ x)\ z$
b) $(\lambda z.\ \lambda x.\ z)\ y$
c) **$(\lambda y.\ y)\ (\lambda x.\ \lambda z.\ z)\ w$**
d) $(\lambda y.\ \lambda x.\ z)\ z\ (\lambda z.\ z)$

Static Scoping & Alpha Conversion

- Lambda calculus uses static scoping
- Consider the following
  - (λx.x (λx.x)) z $\rightarrow$?
    - The rightmost "x" refers to the second binding
  - This is a function that
    - Takes its argument and applies it to the identity function
- This function is "the same" as (λx.x (λy.y))
  - Renaming bound variables consistently preserves meaning
    - This is called alpha-renaming or alpha conversion
  - Ex. λx.x = λy.y = λz.z λy.λx.y = λz.λx.z

Quiz #6

Which of the following expressions is alpha equivalent to (alpha-converts from)

$$(\lambda x.\ \lambda y.\ x\ y)\ y$$

a) $\lambda y.\ y\ y$

b) $\lambda z.\ y\ z$

c) $(\lambda x.\ \lambda z.\ x\ z)\ y$

d) $(\lambda x.\ \lambda y.\ x\ y)\ z$

Quiz #6

Which of the following expressions is alpha equivalent to (alpha-converts from)

$$(\lambda x.\ \lambda y.\ x\ y)\ y$$

a) $\lambda y.\ y\ y$

b) $\lambda z.\ y\ z$

c) $(\lambda x.\ \lambda z.\ x\ z)\ y$

d) $(\lambda x.\ \lambda y.\ x\ y)\ z$

Defining Substitution

- Use recursion on structure of terms
  - x[x:=e] = e   // Replace x by e
  - y[x:=e] = y   // y is different than x, so no effect
  - (e1 e2)[x:=e] = (e1[x:=e]) (e2[x:=e])

    // Substitute both parts of application
  - (λx.e')[x:=e] = λx.e'
    - In λx.e' , the x is a parameter, and thus a local variable that is different from other x' s. Implements static scoping.
    - So the substitution has no effect in this case, since the x being substituted for is different from the parameter x that is in e'
  - (λy.e')[x:=e] = ?
    - The parameter y does not share the same name as x, the variable being substituted for
    - Is λy.(e' [x:=e]) correct? No…

Variable capture

- How about the following?
  - (λx.λy.x y) y → ?
  - When we replace y inside, we don't want it to be captured by the inner binding of y, as this violates static scoping
  - I.e., (λx.λy.x y) y ≠ λy.y y

- Solution
  - (λx.λy.x y) is "the same" as (λx.λz.x z)
    - Due to alpha conversion
  - So alpha-convert (λx.λy.x y) y to (λx.λz.x z) y first
    - Now (λx.λz.x z) y → λz.y z

Completing the Definition of Substitution

- Recall: we need to define $(\lambda y.e')[x:=e]$
  - We want to avoid capturing (free) occurrences of y in e
  - Solution: alpha-conversion!
    - Change y to a variable w that does not appear in e' or e
      (Such a w is called fresh)
    - Replace all occurrences of y in e' by w.
    - Then replace all occurrences of x in e' by e!
- Formally:

  $(\lambda y.e')[x:=e] = \lambda w.((e'\ [y:=w])\ [x:=e])$ (w is fresh)

Beta-Reduction, Again

- Whenever we do a step of beta reduction
  - (λx.e1) e2 → e1[x:=e2]
  - We must alpha-convert variables as necessary
  - Sometimes performed implicitly (w/o showing conversion)

- Examples
  - (λx.λy.x y) y = (λx.λz.x z) y → λz.y z // y → z
  - (λx.x (λx.x)) z = (λy.y (λx.x)) z → z (λx.x) // x → y

OCaml Implementation: Substitution

```
(* substitute e for y in m--   m[y:=e]     *)
let rec subst m y e =
  match m with
      Var x ->
        if y = x then e (* substitute *)
        else m          (* don't subst *)
    | App (e1,e2) ->
        App (subst e1 y e, subst e2 y e)
    | Lam (x,e0) -> …
```

OCaml Impl: Substitution (cont'd)

```
(* substitute e for y in m--   m[y:=e]     *)
let rec subst m y e = match m with …
    | Lam (x,e0) ->
      if y = x then m
      else if not (List.mem x (fvs e)) then
        Lam (x, subst e0 y e)
      else
        let z = newvar() in (* fresh *)
        let e0' = subst e0 x (Var z) in
        Lam (z,subst e0' y e)
```

Shadowing blocks substitution

Safe: no capture possible

Might capture; need to α-convert

OCaml Impl: Reduction

```
let rec reduce e =
  match e with
      App (Lam (x,e), e2) -> subst e x e2
    | App (e1,e2) ->
      let e1' = reduce e1 in
      if e1' != e1 then App(e1',e2)
      else App (e1,reduce e2)
    | Lam (x,e) -> Lam (x, reduce e)
    | _ -> e
```

Straight β rule

Reduce lhs of app

Reduce rhs of app

Reduce function body

nothing to do

Quiz #7

Beta-reducing the following term produces what result?

$$(\lambda x.x\ \lambda y.y\ x)\ y$$

A. $y\ (\lambda z.z\ y)$
B. $z\ (\lambda y.y\ z)$
C. $y\ (\lambda y.y\ y)$
D. $y\ y$

Quiz #7

Beta-reducing the following term produces what result?

$$(\lambda x.x\ \lambda y.y\ x)\ y$$

A. **y (λz.z y)**
B. z (λy.y z)
C. y (λy.y y)
D. y y

Quiz #8

Beta reducing the following term produces what result?

$$\lambda x.(\lambda y.\ y\ y)\ w\ z$$

a) $\lambda x.\ w\ w\ z$

b) $\lambda x.\ w\ z$

c) $w\ z$

d) Does not reduce

Quiz #8

Beta reducing the following term produces what result?

$$\lambda x.(\lambda y.\ y\ y)\ w\ z$$

a) λx. w w z

b) λx. w z

c) w z

d) Does not reduce