Undirected Graphs

Graph: Set of vertices connected pairwise by edges.
Undirected Graphs

Why study graph algorithms?

- Thousands of practical applications.
- Hundreds of graph algorithms known.
- Interesting and broadly useful abstraction.
- Challenging branch of computer science and discrete math.
### Graph Applications

<table>
<thead>
<tr>
<th>Graph</th>
<th>Vertex</th>
<th>Edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>communication</td>
<td>Telephone, computer</td>
<td>fiber optic cable</td>
</tr>
<tr>
<td>circuit</td>
<td>gate, register, processor</td>
<td>wire</td>
</tr>
<tr>
<td>mechanical</td>
<td>joint</td>
<td>rod, beam, spring</td>
</tr>
<tr>
<td>financial</td>
<td>stock, currency</td>
<td>transactions</td>
</tr>
<tr>
<td>transportation</td>
<td>street intersection, airport</td>
<td>highway, airway route</td>
</tr>
<tr>
<td>internet</td>
<td>class C network</td>
<td>connection</td>
</tr>
<tr>
<td>game</td>
<td>board position</td>
<td>legal move</td>
</tr>
<tr>
<td>social relationship</td>
<td>person, actor</td>
<td>friendship, movie cast</td>
</tr>
<tr>
<td>chemical compound</td>
<td>molecule</td>
<td>bond</td>
</tr>
</tbody>
</table>
Graph Terminology

- **Path:**
  - Sequence of vertices connected by edges.

- **Cycle**
  - Path whose first and last vertices are the same.

- Two vertices are connected if there is a path between them.
Some graph-processing problems

- Path:
  - Is there a path between s and t?

- Shortest path.
  - What is the shortest path between s and t?

- Cycle.
  - Is there a cycle in the graph?

- Euler tour.
  - Is there a cycle that uses each edge exactly once?

- Hamilton tour.
  - Is there a cycle that uses each vertex exactly once.

- Connectivity.
  - Is there a way to connect all of the vertices?
Some graph-processing problems

- **MST.**
  - What is the best way to connect all of the vertices?
- **Biconnectivity.**
  - Is there a vertex whose removal disconnects the graph?
- **Planarity.**
  - Can you draw the graph in the plane with no crossing edges
- **Graph isomorphism.**
  - Do two adjacency lists represent the same graph?

Challenge. Which of these problems are easy? difficult? intractable?
Graph representation

Graph drawing

• Provides intuition about the structure of the graph.
Graph representation

- Vertex representation:
  - use integers between 0 and \( V - 1 \).
- Applications: convert between names and integers with symbol table.

No self loop, No parallel edges
Graph Class

public class Graph{
    Graph(int V)  //create an empty graph with V
    void addEdge(int v, int w)  //add an edge v-w
    Iterable<Integer>adj(int v) //vertices adjacent to v
    int V()     //number of vertices
    int E()     //number of edges
    String toString()  //string representation
}

Set-of-edges graph representation

- Maintain a list of the edges (linked list or array).
Adjacency-matrix graph representation

- Maintain a two-dimensional V-by-V boolean array;
- for each edge v–w in graph:
  - \( \text{adj}[v][w] = \text{adj}[w][v] = \text{true} \).
Adjacency-list graph representation

- Maintain vertex-indexed array of lists.

![Diagram of an adjacency-list graph representation with vertex-indexed array of lists showing connections between vertices 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. The array `adj[]` is shown with connections for each vertex representing adjacent vertices.]
Graph representation

- In practice. Use adjacency-lists representation.
  - Algorithms based on iterating over vertices adjacent to v.
- Real-world graphs tend to be sparse.

Two graphs (V = 50)

sparse (E = 200)  dense (E = 1000)

huge number of vertices, small average vertex degree
Graph representation

Comparisons of three different representations:

<table>
<thead>
<tr>
<th>representation</th>
<th>space</th>
<th>add edge</th>
<th>edge between v and w?</th>
<th>iterate over vertices adjacent to v?</th>
</tr>
</thead>
<tbody>
<tr>
<td>list of edges</td>
<td>E</td>
<td>1</td>
<td>E</td>
<td>E</td>
</tr>
<tr>
<td>adjacency matrix</td>
<td>$V^2$</td>
<td>$1^*$</td>
<td>1</td>
<td>$V$</td>
</tr>
<tr>
<td>adjacency lists</td>
<td>$E + V$</td>
<td>1</td>
<td>degree(v)</td>
<td>degree(v)</td>
</tr>
</tbody>
</table>

* disallows parallel edges
Adjacency-list graph representation:
Java implementation

```java
public class Graph{
    private final int V;
    private Bag<Integer>[] adj;
    public Graph(int V) {
        this.V = V;
        adj = (Bag<Integer>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Integer>();
    }
    public void addEdge(int v, int w) {
        adj[v].add(w);
        adj[w].add(v);
    }
    public Iterable<Integer> adj(int v) {
        return adj[v];
    }
}
```
Graph Algorithms: Depth First Search

• Trémaux maze exploration Algorithm
  • Unroll a ball of string behind you.
  • Mark each visited intersection and each visited passage.
  • Retrace steps when no unvisited options
Maze Exploration
Depth First Search

**Goal.** Systematically search through a graph. **Idea.** Mimic maze exploration.

**DFS (to visit a vertex v)**
- Mark v as visited.
- Recursively visit all unmarked vertices w adjacent to v.

**Typical applications:**
- Find all vertices connected to a given source vertex.
- Find a path between two vertices.
DFS Demo

To visit a vertex $v$:
- Mark vertex $v$ as visited.
- Recursively visit all unmarked vertices adjacent to $v$. 
DFS Demo

To visit a vertex $v$:
Mark vertex $v$ as visited.
Recursively visit all unmarked vertices adjacent to $v$. 

<table>
<thead>
<tr>
<th>V</th>
<th>marked[]</th>
<th>edgeTo[v]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
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<tr>
<td>4</td>
<td></td>
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<tr>
<td>5</td>
<td></td>
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<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
DFS Demo

To visit a vertex \( v \):
Mark vertex \( v \) as visited.
Recursively visit all unmarked vertices adjacent to \( v \).
public class DepthFirstPaths {
    private boolean[] marked;
    private int[] edgeTo;
    private int s;
    public DepthFirstSearch(Graph G, int s) {
        ...
        dfs(G, s);
    }
    private void dfs(Graph G, int v) {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w]) {
                dfs(G, w);
                edgeTo[w] = v;
            }
    }
}
Breadth-first search (BFS)

- BFS starts at a vertex and explores the neighbor vertices first, before moving to the next level neighbors.

Repeat until queue is empty:
- Remove vertex \( v \) from queue.
- Add to queue all unmarked vertices adjacent to \( v \) and mark them.
Breadth-first search (BFS)
Breadth-first search

Depth-first search: Put unvisited vertices on a stack.
Breadth-first search: Put unvisited vertices on a queue.
Shortest path: Find path from s to t that uses fewest number of edges.

BFS (from source vertex s)
Put s onto a FIFO queue, and mark s as visited.
Repeat until the queue is empty:
  remove the least recently added vertex v
  add each of v's unvisited neighbors to the queue, and mark them as visited.

Intuition: BFS examines vertices in increasing distance from s.
Breadth-first search

```java
public class BreadthFirstPaths {
    private boolean[] marked;
    private int[] edgeTo;
    ...
    private void bfs(Graph G, int s) {
        Queue<Integer> q = new Queue<Integer>();
        q.enqueue(s);
        marked[s] = true;
        while (!q.isEmpty()) {
            int v = q.dequeue();
            for (int w : G.adj(v)) {
                if (!marked[w]) {
                    q.enqueue(w);
                    marked[w] = true;
                    edgeTo[w] = v;
                }
            }
        }
    }
}
```
BFS Application: Kevin Bacon Number

- Kevin Bacon graph
  - Include one vertex for each performer and one for each movie.
  - Connect a movie to all performers that appear in that movie.
  - Compute shortest path from \( s = \text{Kevin Bacon} \).
Connected components

Goal:

- Partition vertices into connected components.

Connected components
Initialize all vertices $v$ as unmarked.
For each unmarked vertex $v$, run DFS to identify all vertices discovered as part of the same component.